Discussion Notes

Lei & Yoon-G

February 22, 2020

Problem .

Given an array with n positive integer $A = [a_0, a_1, ..., a_{n-1}]$ and a positive integer s, find the minimum length of A's sub-array that satisfies the summation of all its elements is equal to or larger than s.

Solution

(I) **Pseudo-code**

Find_Min(A,n,s)

Require: $n, s > 0$, and $A = [a_0, a_1, ..., a_{n-1}]$, where $\forall 0 \le i \le n-1, a_i > 0$.

Ensure: the minimum length of the sub-array A', which satisfies the summation of all the elements in A' is equal to or larger than s.

```
i ⇐ 0, j ⇐ 0, sum ⇐ a_0, Min ⇐ MaxInt
if sum ≥ s then
  Min = 1
end if
while i < n and j < n do
  if sum ≥ s then
    if j − i + 1 < Min then
      Min ⇐ j − i + 1
    end if
    sum ⇐ sum − a_i
    i ⇐ i + 1
  else
    j ⇐ j + 1
    sum ⇐ sum + a_j
  end if
end while
return Min
```

(II) **State the Loop Invariant**

Define M_i, which is initialized as MaxInt, and

$$M_i = \min\{(r-i+1)|\sum_{k=i}^{r} a_k \geq s\}$$

At the beginning of each loop, when $i = i'$ and $j = j'$, we have

(a) $sum = \sum_{k=i'}^{j'} a_k$

(b) if $sum \geq s \Rightarrow \mathrm{M}_i' = j' - i' + 1$

(c) $Min = \min\{M_k | 1 \leq k \leq i'\}$

(III) **Prove the Initiation of the Loop Invariant**

At the beginning of the first loop, $i = j = sum = 0, Min = MaxInt$(if $a_0 < s$) or 1 (if $a_0 \geq s$).

$$\sum_{k=0}^{0} a_k = a_0 = sum \Rightarrow Loop\ Invariant(a)$$

If $sum = a_0 \geq s$, then obviously $M_0 = 1 = j' - i' + 1$ and if $sum = a_0 < s$, then M_0 is still MaxInt $\Rightarrow Loop\ Invariant(b)$

If $sum = a_0 \geq s$, then according to Loop Invariant (b) now we have $M_0 = 1 = Min$, and if $sum = a_0 < s$, then similarly we have $M_0 = MaxInt = Min \Rightarrow Loop\ Invariant(c)$

(IV) **Prove the Maintenance of the Loop Invariant**

Assume the Loop Invariant holds at the beginning of the loop when $i = i'$ and $j = j'$, then we're going to prove that the loop invariant will still hold at the beginning of the next loop.

Case I: if $sum \geq s$

then at the beginning of the next loop, the new value of i, j will be $i' + 1, j'$, and the new value of sum will be

$$sum - a_{i'} = \sum_{k=i'}^{j'} a_k - a_{i'} = \sum_{k=i'+1}^{j'} a_k \Rightarrow Loop\ Invariant(a)$$

Then we prove: if the new $sum = \sum_{k=i'+1}^{j'} a_k \geq s$, then $M_{i'+1} = j' - (i'+1) + 1$ by contradiction.

Assume $\exists$ another sub-array $S_A = [a_{i'+1}, ..., a_x]$, where $M_{i'+1} = x - (i'+1) + 1, i'+1 \leq x \leq n-1$ and $x \neq j'$.

If $x > j$, then the length of $x - (i'+1) + 1 > j' - (i'+1) + 1$, which leads to contradiction to the definition of M_i'.

If $x < j$, then

$$\sum_{k=i'}^{x} a_k = a_{i'} + \sum_{k=i'+1}^{x} a_k \geq s$$

$$\Rightarrow M_{i'} \leq x - i' + 1 < j' - i' + 1$$

which is contradicted to the Assumption that $M_{i'} = j' - i' + 1$.

Accordingly, we can prove that in case I, if the new value $sum \geq s$, then $M_{i'+1} = j - (i' + 1) + 1 \Rightarrow Loop\ Invariant(b)$

If $M_{i'} = j' - i' + 1 < Min$, then according to the pseudo-code, the new value of Min will be $M_{i'}$, which proves the Loop Invariant (c).

Case II: if $sum < s$

Similarly we can prove Loop Invariant (a).

Then we will prove Loop Invariant (b) by Contradiction.

Assume when $\sum_{k=i'}^{j'+1} a_k \geq s, \exists$ another sub-array $S_A = [a_i', ..., a_x]$, such that $M_i' = x - i' + 1$, where $i' + 1 \leq x \leq n - 1$ and $x \neq j' + 1$.

If $i' \leq x \leq j'$, then

$$\sum_{k=i'}^{j'} a_k = \sum_{k=i'}^{x} a_k + \sum_{k=x+1}^{j'} a_k \geq \sum_{k=i'}^{x} a_k \geq s$$

which is contradicted to the condition that $\sum_{k=i'}^{j'} a_k < s$.

If $j' + 1 < x \leq n - 1$, then the length of $S_A = x - i' + 1 > (j' + 1) - i' + 1$, which is contradicted to the definition of M_i'.

Accordingly, we can prove that in case II, if the new summation $sum < s$, then $M_i' = (j' + 1) - i' + 1$.

(V) **Prove the Loop will Terminate**

The loop will terminate when $i \geq n$ or $j \geq n$. The initial value of i and j are both 0, and in each loop either i or j will increment, i.e. $i + j$ will increment. After $2n - 1$ loops, $i + j = 2n - 1$. According to the Drawer Principle, either i or j will be equal to or greater than n, then we can prove the loop will terminate after finite times.

(VI) **Prove the Return Value is Proper when Termination**

The loop will terminate when $i \geq n$ or $j \geq n$, and since i, j increment from 0, and will not increase by more than 1 at one loop, then the terminate condition is equivalent to $i = n$ or $j = n$.

If the loop terminates at the beginning of the loop when $i = n$, then according to the Loop Invariant(c), now we have

$$Min = \min\{M_k | 0 \leq k \leq n-1\} = min\{r - l + 1 | 0 \leq l \leq r \leq n-1, s.t. \sum_{k=i'}^{j'} a_k\} \geq s\}$$

which is the result we require.

If the loop terminates at the beginning of the loop when $j = n$, then we know that for $\forall x \geq i', \sum_{k=x}^{n-1} a_k \leq \sum_{k=i'}^{n-1} a_k < s \Rightarrow \forall x \geq i', M_x = MaxInt$. Therefore we cannot find any M_x for $x \geq i'$, such that M_x could be smaller than the current Min. Then the current

$$Min = \min\{M_k | 0 \leq k \leq n-1\} = min\{r - l + 1 | 0 \leq l \leq r \leq i', s.t. \sum_{k=i'}^{j'} a_k \geq s\}$$

$$= min\{r - l + 1 | 0 \leq l \leq r \leq n-1, s.t. \sum_{k=i'}^{j'} a_k \geq s\}$$

which is the result we require.