

Supplement 1: Toolkit Functions

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask “If I know one quantity, can I then determine the other?” This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function

Function: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say “the output is a function of the input.”

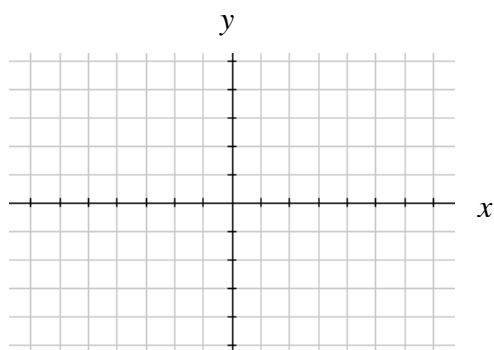
Function Notation

The notation $\text{output} = f(\text{input})$ defines a function named f . This would be read “output is f of input”

Graphs as Functions

Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has y on the vertical axis and x on the horizontal axis, and we say y is a function of x , or $y = f(x)$ when the function is named f .



Basic Toolkit Functions

In this class, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our “toolkit of functions” – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use x as the input variable and $f(x)$ as the output variable.

Toolkit Functions		
Linear		
Constant:	$f(x) = c$, where c is a constant (number)	
Identity:	$f(x) = x$	
Absolute Value:	$f(x) = x $	
Power		
Quadratic:	$f(x) = x^2$	
Cubic:	$f(x) = x^3$	
Reciprocal:	$f(x) = \frac{1}{x}$	
Reciprocal squared:	$f(x) = \frac{1}{x^2}$	
Square root:	$f(x) = \sqrt[2]{x} = \sqrt{x}$	
Cube root:	$f(x) = \sqrt[3]{x}$	

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this course. In order to successfully follow along later in the course, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

It should be noted that not every important equation can be written as $y = f(x)$. An example of this is the equation of a circle:

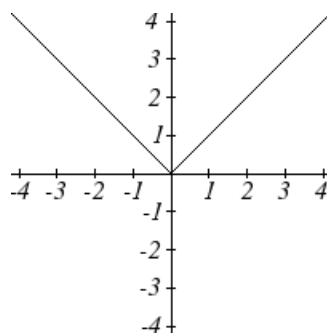
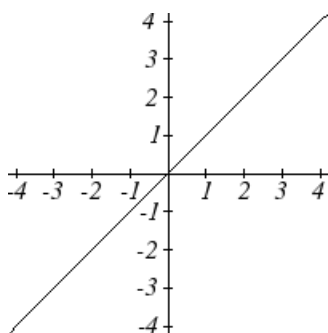
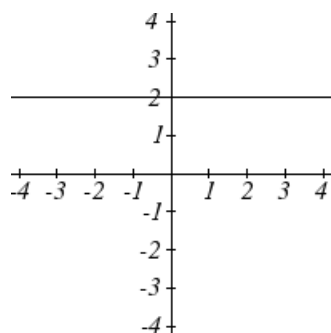
A circle with radius r with center (h, k) has equation $r^2 = (x - h)^2 + (y - k)^2$

We cannot, however solve for y as a function of x in a single equation. Circles will be further explored at a later time, and we won't go into more detail here.

Graphs of the Toolkit Functions

Constant Function: $f(x) = 2$ Identity: $f(x) = x$

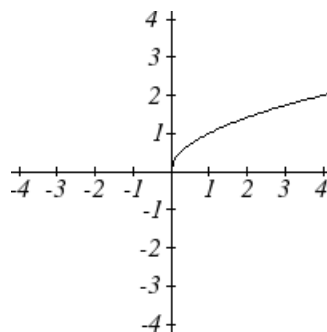
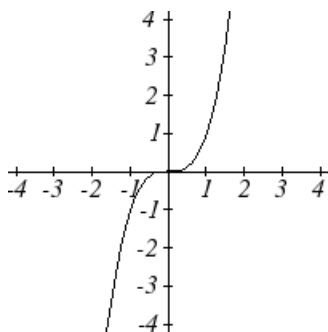
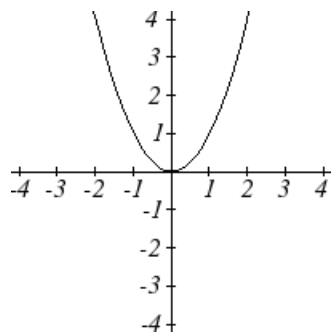
Absolute Value: $f(x) = |x|$



Quadratic: $f(x) = x^2$

Cubic: $f(x) = x^3$

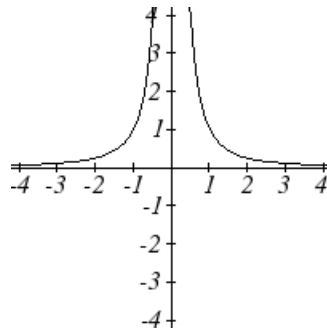
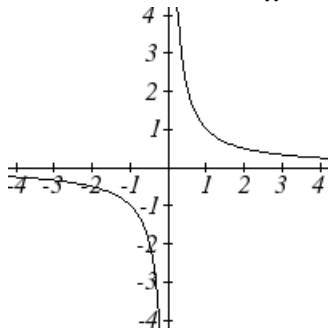
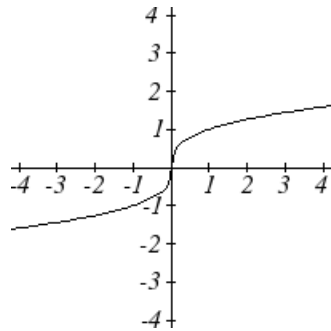
Square root: $f(x) = \sqrt{x}$



Cube root: $f(x) = \sqrt[3]{x}$

Reciprocal: $f(x) = \frac{1}{x}$

Reciprocal squared: $f(x) = \frac{1}{x^2}$



Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, c	1.7	2.5	5.5	8.2	13.7
Height, h	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

Domain: The set of possible input values to a function

Range: The set of possible output values of a function

Domain and Range Notations

Inequalities can be used to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$, $0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, $p = \$0.44, \$0.61, \$0.78, \text{ or } \0.95 . These numbers represent a set of specific values: $\{0.44, 0.61, 0.78, 0.95\}$

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable x that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x \mid 10 \leq x < 30\}$. The curly brackets $\{\}$ are read as “the set of”, and

the vertical bar $|$ is read as “such that”, so altogether we would read $\{x | 10 \leq x < 30\}$ as “the set of x -values such that 10 is less than or equal to x and x is less than 30.”

When describing ranges in set-builder notation, we could similarly write something like $\{f(x) | 0 < f(x) < 100\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h | 0 < h \leq 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t | t \text{ is a real number}\}$, or $\{t | t \in \mathbb{R}\}$, read as “the set of t -values such that t is an element of the set of real numbers.”

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for “strictly less than,” and square brackets are used for “less than or equal to.” Since infinity is not a number, we can’t include it in the interval, so we always use curved parentheses with ∞ and $-\infty$. The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	Set Builder Notation	Interval notation
$5 < h \leq 10$	$\{h 5 < h \leq 10\}$	$(5, 10]$
$5 \leq h < 10$	$\{h 5 \leq h < 10\}$	$[5, 10)$
$5 < h < 10$	$\{h 5 < h < 10\}$	$(5, 10)$
$h < 10$	$\{h h < 10\}$	$(-\infty, 10)$
$h \geq 10$	$\{h h \geq 10\}$	$[10, \infty)$
all real numbers	$\{h h \in \mathbb{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word “or”. In interval notation, we use the union symbol, \cup , to combine two unconnected intervals together.

Example 3

Describe the intervals of values shown on the line graph below using set builder and interval notations.



To describe the values, x , that lie in the intervals shown above we would say, “ x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$

In set builder notation: $\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$

In interval notation: $[1, 3] \cup (5, \infty)$

Remember when writing or reading interval notation:

Using a square bracket $[$ means the start value is included in the set

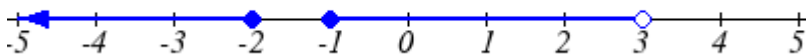
Using a square bracket $]$ means the end value is included in the set

Using a parenthesis $($ means the start value is not included in the set

Using a parentheses $)$ means the end value is not included in the set

Try it Now

Given the following interval, write its meaning in words, set builder notation, and interval notation.



a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3

b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$

c. $(-\infty, -2] \cup [-1, 3)$

Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Usually a dot is used to indicate a graph stops at a certain point and an arrow is used to indicate a graph continues beyond the window that is shown. If no dot or arrow is present, you may need more contextual information to decide if the graph stops or continues, but usually it means the graph still continues beyond the window shown.

Domains and Ranges of the Toolkit functions

We will now return to our set of toolkit functions to note the domain and range of each.

Constant Function: $f(x) = c$

The domain here is not restricted; x can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.

Domain: $(-\infty, \infty)$

Range: $[c]$

Since there is only one output value, we list it by itself in square brackets.

Identity Function: $f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Quadratic Function: $f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Multiplying a negative or positive number by itself can only yield a positive output.

Cubic Function: $f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Reciprocal: $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

We cannot divide by 0 so we must exclude 0 from the domain.

One divide by any value can never be 0, so the range will not include 0.

Reciprocal squared: $f(x) = \frac{1}{x^2}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(0, \infty)$

We cannot divide by 0 so we must exclude 0 from the domain.

Cube Root: $f(x) = \sqrt[3]{x}$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Square Root: $f(x) = \sqrt[2]{x}$, commonly just written as, $f(x) = \sqrt{x}$

Domain: $[0, \infty)$

Range: $[0, \infty)$

When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.

Absolute Value Function: $f(x) = |x|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Since absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

Example 4.5

Find the domain of each function: a) $f(x) = 2\sqrt{x+4}$ b) $g(x) = \frac{3}{6-3x}$

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$x + 4 \geq 0$ when $x \geq -4$.

The domain of $f(x)$ is $[-4, \infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.

$6 - 3x = 0$ when $x = 2$, so we must exclude 2 from the domain.

The domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.

Important Topics of this Section

Definition of a function
Input (independent variable)
Output (dependent variable)
Toolkit Functions
Definition of domain
Definition of range
Interval notation
Set builder notation
Domain and Range of toolkit functions

S.1 Exercises

1. Match each function name with its equation.

a. $y = x$

i. Cube root

b. $y = x^3$

ii. Reciprocal

c. $y = \sqrt[3]{x}$

iii. Linear

d. $y = \frac{1}{x}$

iv. Square Root

v. Absolute Value

e. $y = x^2$

vi. Quadratic

f. $y = \sqrt{x}$

vii. Reciprocal Squared

g. $y = |x|$

viii. Cubic

h. $y = \frac{1}{x^2}$

2. Match each graph with its equation.

a. $y = x$

b. $y = x^3$

c. $y = \sqrt[3]{x}$

d. $y = \frac{1}{x}$

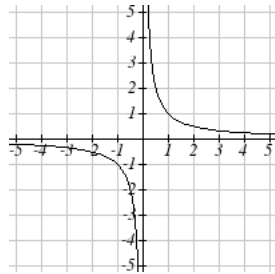
e. $y = x^2$

f. $y = \sqrt{x}$

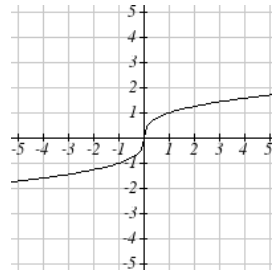
g. $y = |x|$

h. $y = \frac{1}{x^2}$

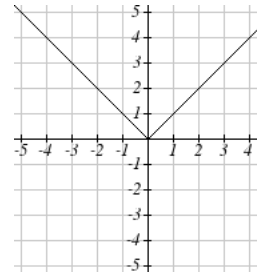
i.



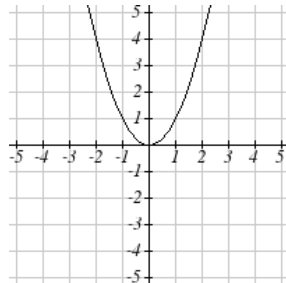
ii.



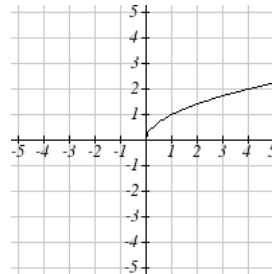
iii.



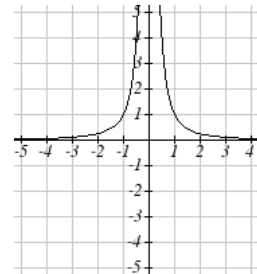
iv.



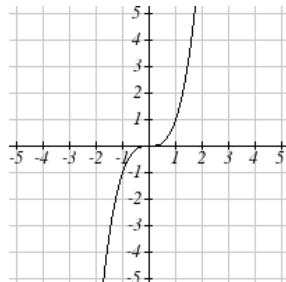
v.



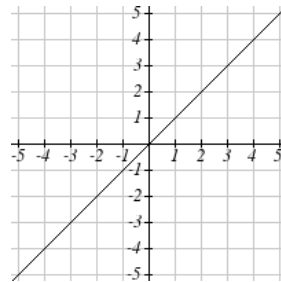
vi.



vii.



viii.



3. Match each table with its equation.

a. $y = x^2$

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

b. $y = x$

c. $y = \sqrt{x}$

d. $y = 1/x$

e. $y = |x|$

f. $y = x^3$

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

4. Match each equation with its table

a. Quadratic

i.

In	Out
-2	-0.5
-1	-1
0	-
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

b. Absolute Value

c. Square Root

d. Linear

e. Cubic

f. Reciprocal

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

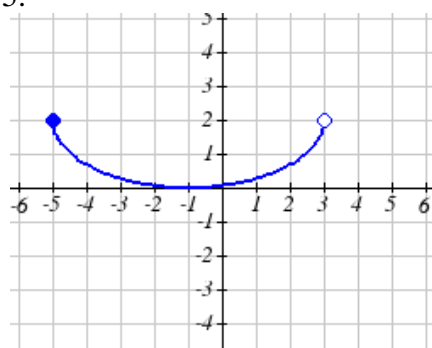
In	Out
-2	-
-1	-
0	0
1	1
4	2
9	3

vi.

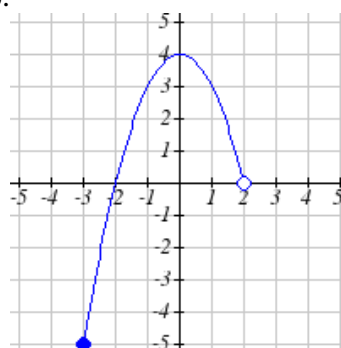
In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

Write the domain and range of the function using interval notation.

5.

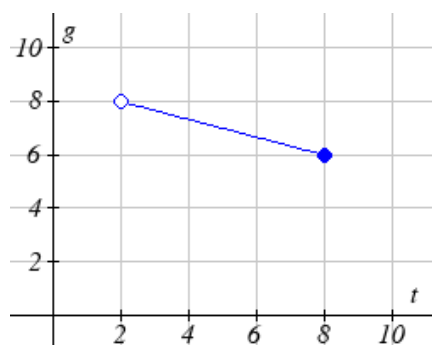


6.



Write the domain and range of each graph as an inequality.

7.

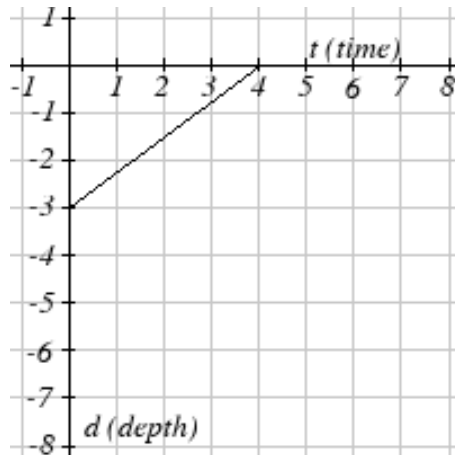


8.

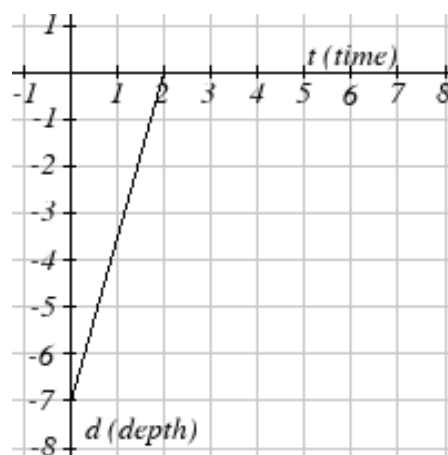


Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time, stopping once the sub surfaces. What is the domain and range of the function in the graph?

9.



10.



Find the domain of each function.

11. $f(x) = 3\sqrt{x-2}$

12. $f(x) = 5\sqrt{x+3}$

13. $f(x) = 3 - \sqrt{6-2x}$

14. $f(x) = 5 - \sqrt{10-2x}$

15. $f(x) = \frac{9}{x-6}$

16. $f(x) = \frac{6}{x-8}$

17. $f(x) = \frac{3x+1}{4x+2}$

18. $f(x) = \frac{5x+3}{4x-1}$

19. $f(x) = \frac{\sqrt{x+4}}{x-4}$

20. $f(x) = \frac{\sqrt{x+5}}{x-6}$

21. $f(x) = \frac{x-3}{x^2+9x-22}$

22. $f(x) = \frac{x-8}{x^2+8x-9}$

S.1 Solutions to Exercises

1. (a) $y = x$ (iii. Linear) (b) $y = x^3$ (viii. Cubic)
(c) $y = \sqrt[3]{x}$ (i. Cube Root) (d) $y = \frac{1}{x}$ (ii. Reciprocal)
(e) $y = x^2$ (vi. Quadratic) (f) $y = \sqrt{x}$ (iv. Square Root)
(g) $y = |x|$ (v. Absolute Value) (h) $y = \frac{1}{x^2}$ (vii. Reciprocal Squared)
3. (a) $y = x^2$ (iv.) (b) $y = x$ (ii.)
(c) $y = \sqrt{x}$ (v.) (d) $y = \frac{1}{x}$ (i.)
(e) $y = |x|$ (vi.) (f) $y = x^3$ (iii.)

5. The domain is $[-5, 3]$; the range is $[0, 2]$

7. The domain is $2 < t \leq 8$; the range is $6 \leq g < 8$

9. The domain is $0 \leq t \leq 4$; the range is $-3 \leq d \leq 0$

11. Since the function is not defined when there is a negative number under the square root, x cannot be less than 2 (it can be equal to 2, because $\sqrt{0}$ is defined). So the domain is $x \geq 2$. Because the inputs are limited to all numbers greater than 2, the number under the square root will always be positive, so the outputs will be limited to positive numbers. So the range is $f(x) \geq 0$. In interval notation, the domain is $[2, \infty)$, and the range is $[0, \infty)$.

13. Since the function is not defined when there is a negative number under the square root, x cannot be greater than 3 (it can be equal to 3, because $\sqrt{0}$ is defined). So the domain is $x \leq 3$. Because the inputs are limited to all numbers less than 3, the number under the square root will always be positive, and there is no way for 3 minus a positive number to equal more than three, so the outputs can be any number less than 3. So the range is $f(x) \leq 3$. In interval notation, the domain is $(-\infty, 3]$, and the range is $(-\infty, 3)$.

15. Since the function is not defined when there is division by zero, x cannot equal 6. So the domain is all real numbers except 6, or $\{x | x \in \mathbb{R}, x \neq 6\}$. In interval notation, the domain is $(-\infty, 6) \cup (6, \infty)$.

17. Since the function is not defined when there is division by zero, x cannot equal $-1/2$. So the domain is all real numbers except $-1/2$, or $\{x | x \in \mathbb{R}, x \neq -1/2\}$. In interval notation, the domain is $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$.

19. Since the function is not defined when there is a negative number under the square root, x cannot be less than -4 (it can be equal to -4 , because $\sqrt{0}$ is defined). Since the function is also not defined when there is division by zero, x also cannot equal 4. So the domain is all real

numbers less than -4 excluding 4 , or $\{x|x \geq -4, x \neq 4\}$. In interval notation, the domain is $(-4, \infty)$.

21. It is easier to see where this function is undefined after factoring the denominator. This gives $f(x) = \frac{x-3}{(x+11)(x-2)}$. It then becomes clear that the denominator is undefined when $x = -11$ and when $x = 2$ because they cause division by zero. Therefore, the domain is $\{x|x \in \mathbb{R}, x \neq -11, x \neq 2\}$. In interval notation, the domain is $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$.