Università degli Studi della Campania Luigi Vanvitelli

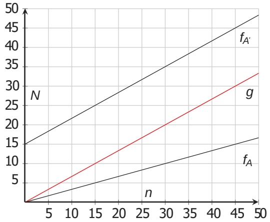
Future Computing Architecture and Programming Paradigms

Quantum Complexity

Estimating the run time of an algorithm

Big-O Notation

Let $f_A : \mathbb{N} \to \mathbb{N}$ be a function that returns the number of elementary calculation steps for an algorithm A, given an input with size n. We write $f_A \in O(g(n))$ if f grows asymptotically as fast or slower than g.



Common asymptotic functions

Function	designation	example algorithm
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2^n	exponential	naive calculation of the n-th Fibonacci number

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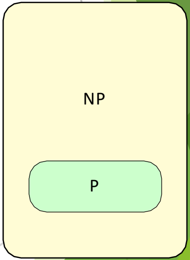

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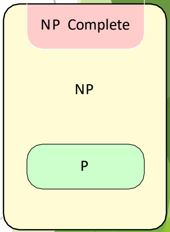

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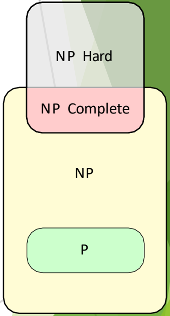

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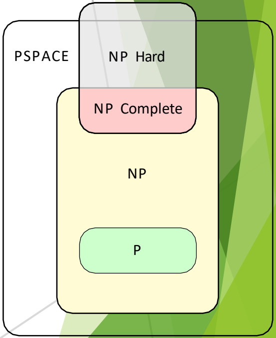
- **PSPACE**: The problem is solvable in polynomial **space**.



Bounded-error probabilistic polynomial time (BPP)

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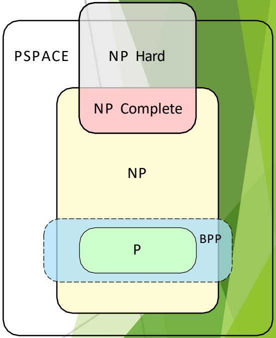

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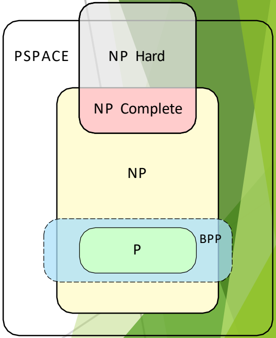

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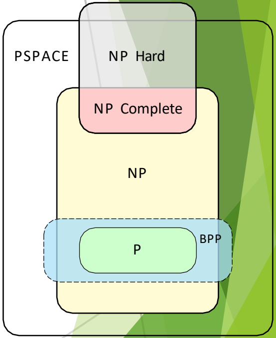

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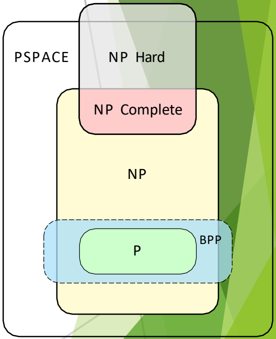
- BPP = *frontier of feasibility* for classical computers.



Bounded-error quantum polynomial time (BQP)

BQP

A decision problem E is in the class BQP if there is an algorithm A:

- If the correct solution $E(x) = 1$ for an input x, the algorithm generates the result $A(x) = 1$ with probability greater than $\frac{1}{2}$.
- If the correct solution $E(x) = 0$ for an input x, the algorithm generates the result $A(x) = 0$ with probability greater than $\frac{1}{2}$.
- The solution can be calculated with uniform quantum circuits of polynomial size.

Back to BQP

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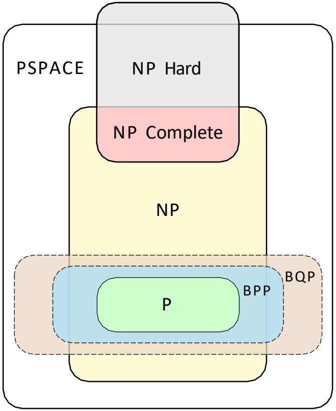
⇒ $P \subseteq BQP$

Back to BQP



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- Functionality is hidden from outside (Black box model)
- Quantum oracle U_f : $|\psi\rangle - \boxed{U_f} - |\psi'\rangle$
- Quantum computing allows to reduce calls to an oracle!

Interference changes the behavior of oracles

Hadamard "Sandwich": $$|\psi\rangle \;-\!\!-\boxed{H}\!-\!\!-\boxed{U_f}\!-\!\!-\boxed{H}\!-\!\!-\; |\psi'\rangle$$

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Quantum oracles

Behavior of gates in Hadamard sandwich

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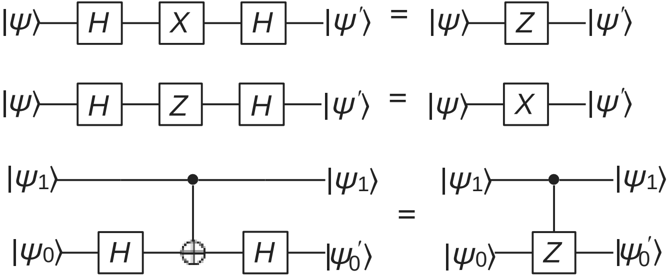

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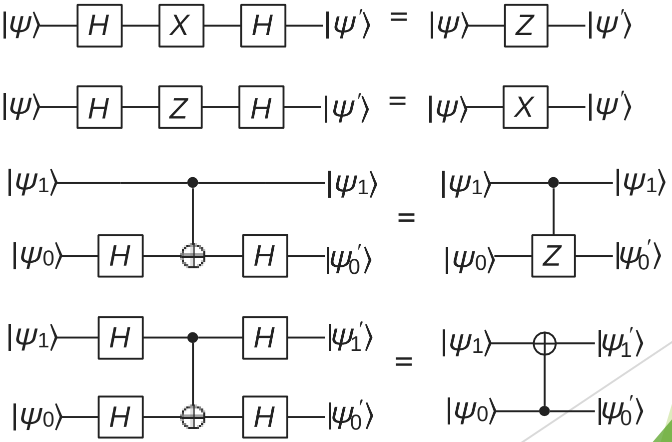

Summary

$\Rightarrow$ With interference we can find out information of an oracle which we would not get in the classical case.

Algorithm of Deutsch

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- $\Rightarrow$ Classically we need two calls to the oracle for solving the problem.
- $\Rightarrow$ On a quantum computer we need only one!

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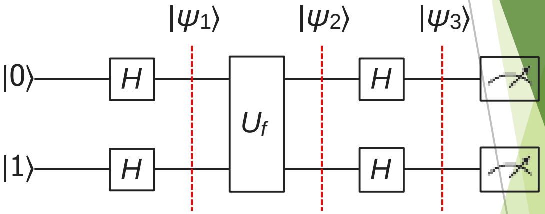
$$U_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$$



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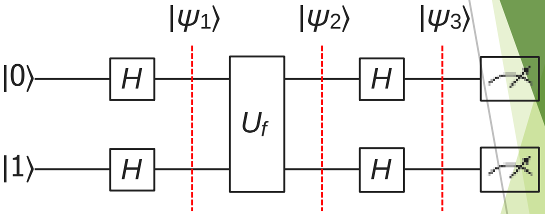

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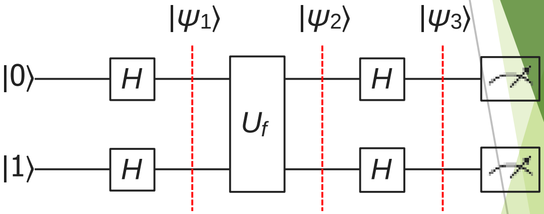

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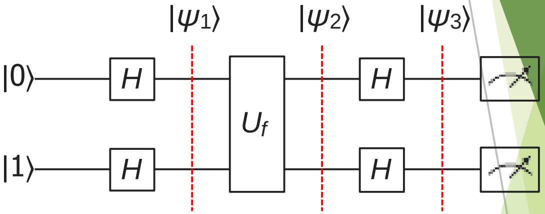

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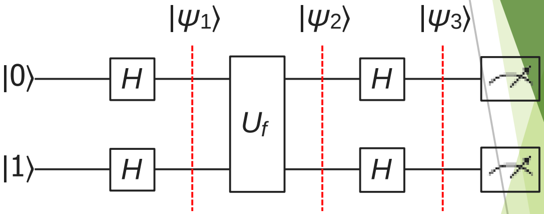

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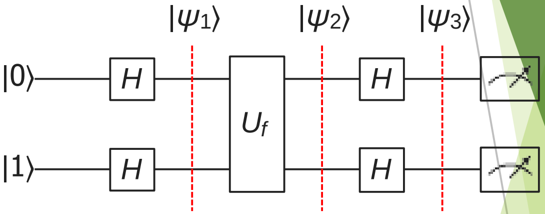

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Algorithm of Deutsch

Case U_f constant

State of upper qubit:

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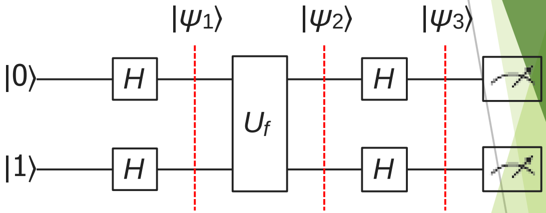

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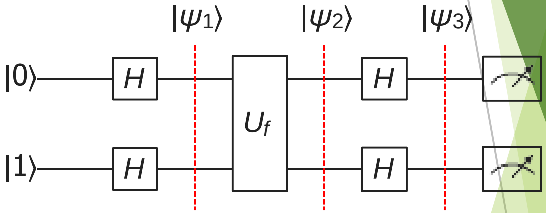

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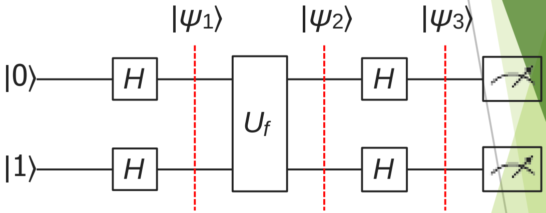

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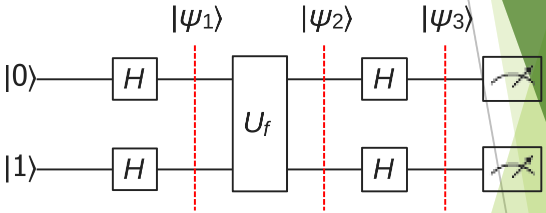
$\Rightarrow$ We always measure $|0\rangle$ on the upper qubit.

Algorithm of Deutsch

Case U_f balanced



State of upper qubit:

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- It applies: $(-1)^{f(0)} \neq (-1)^{f(1)}$, i.e. the state is $\pm\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

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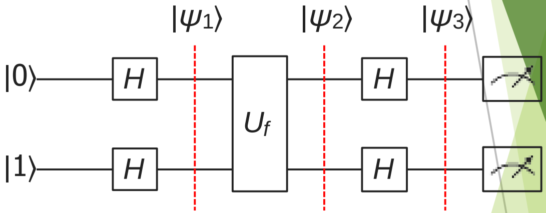

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$\Rightarrow$ Hadamard Gate transforms state to $\pm\, |1\rangle$

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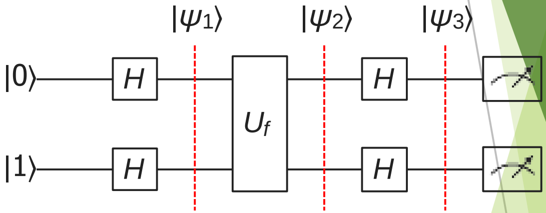

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⇒ Hadamard Gate transforms state to $\pm\,|1\rangle$

⇒ We always measure $|1\rangle$ on the upper qubit.

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- Quantum oracle U_f is only applied **once** to determine whether f is constant or balanced.
- Can be **extended** to boolean functions with **arbitrary input size**.
- **Theoretical** speedup over classical counterpart, but small **practical** value.
- However, most quantum algorithms with provable speedup rely on **quantum oracles** in combination with **interference**.