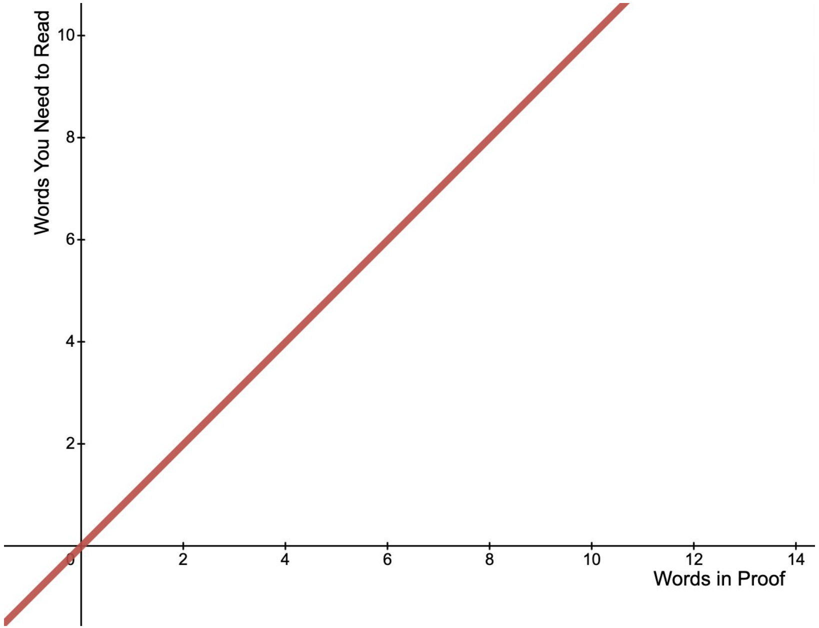
# Classical and Quantum Probabilistically Checkable Proofs

Jon Rosario and Laker Newhouse
Mentored by David Cui

# *Breaking News:* Researchers Discover Proofs That Are Faster To Read



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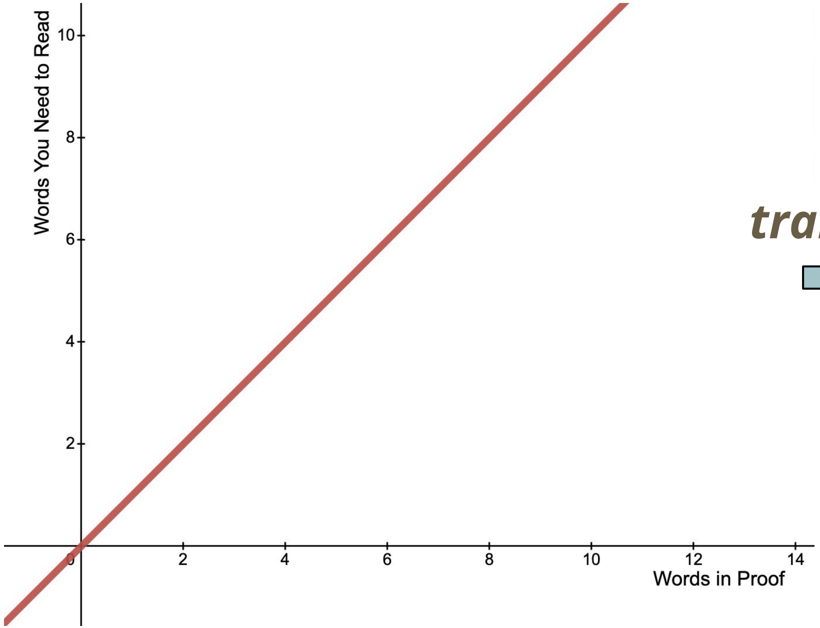


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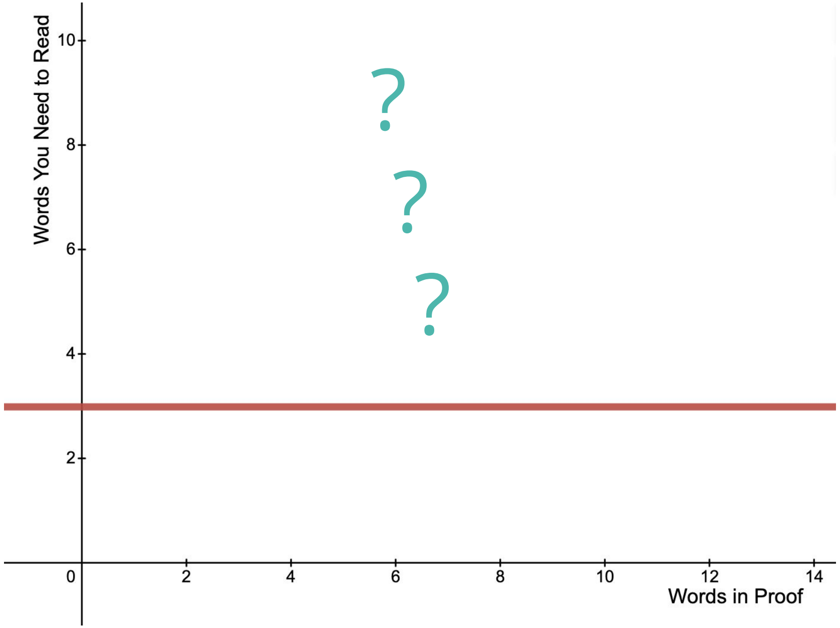

# Quick Preliminaries: The Class P

Verifier

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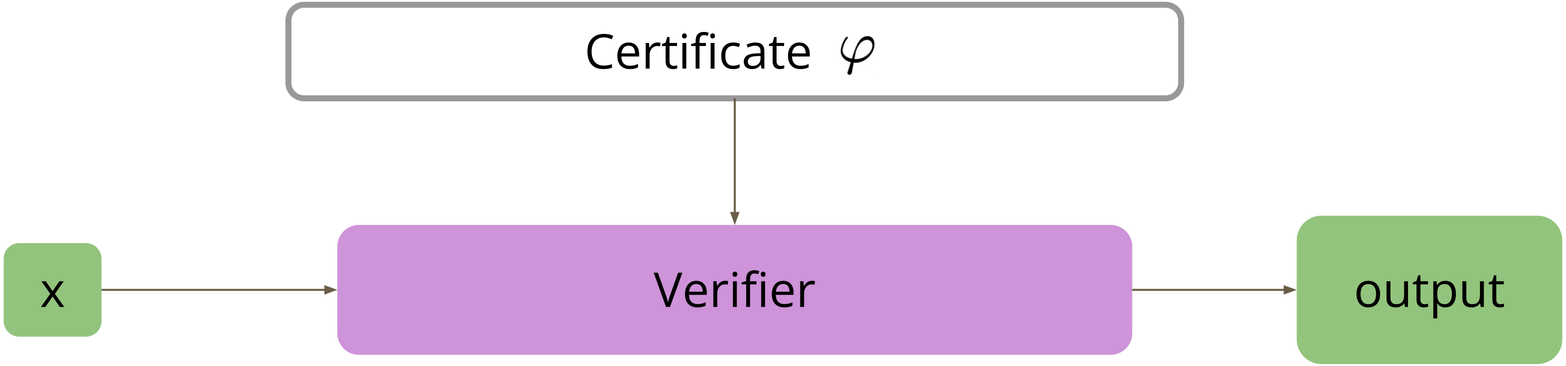


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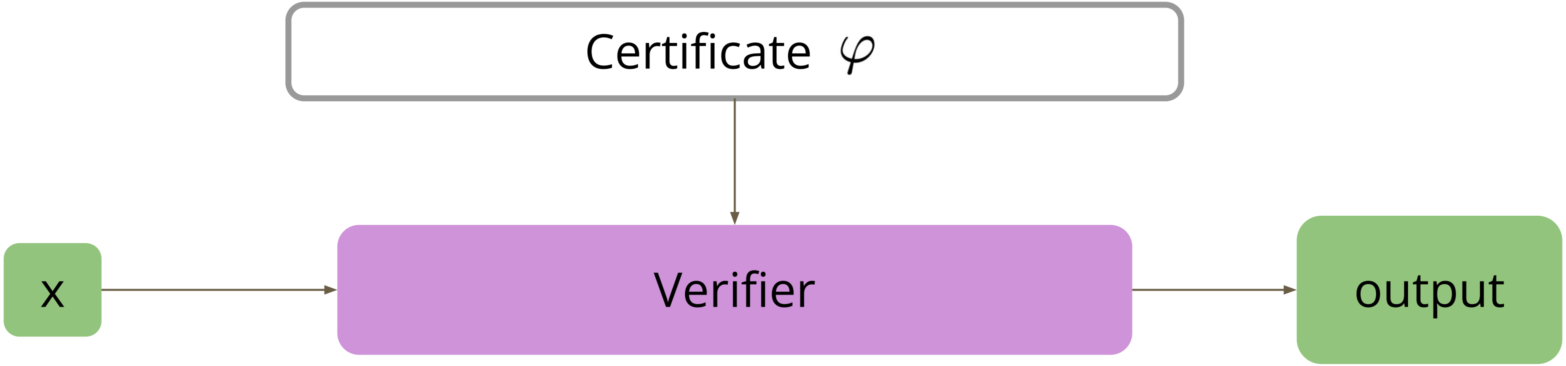


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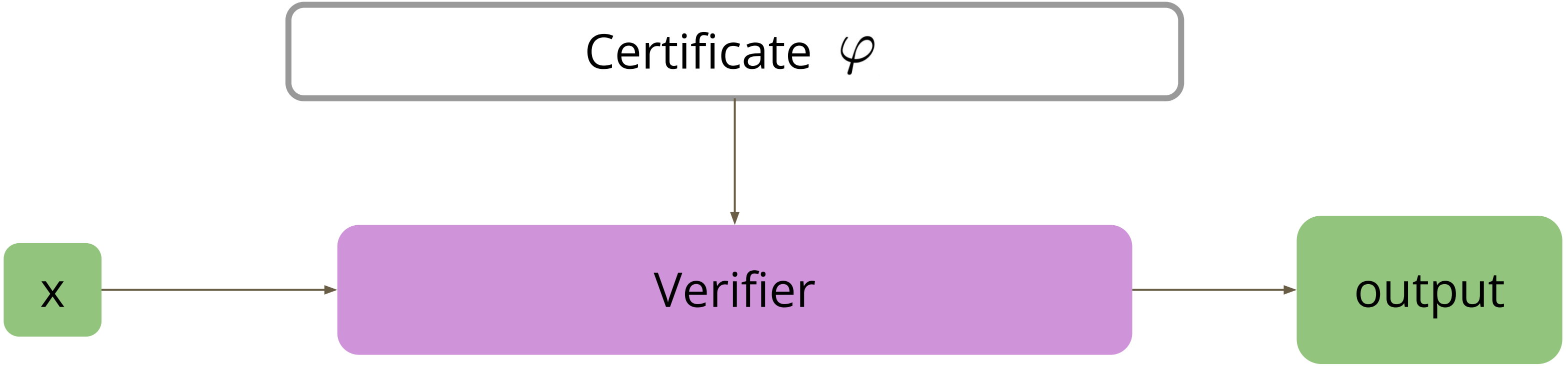


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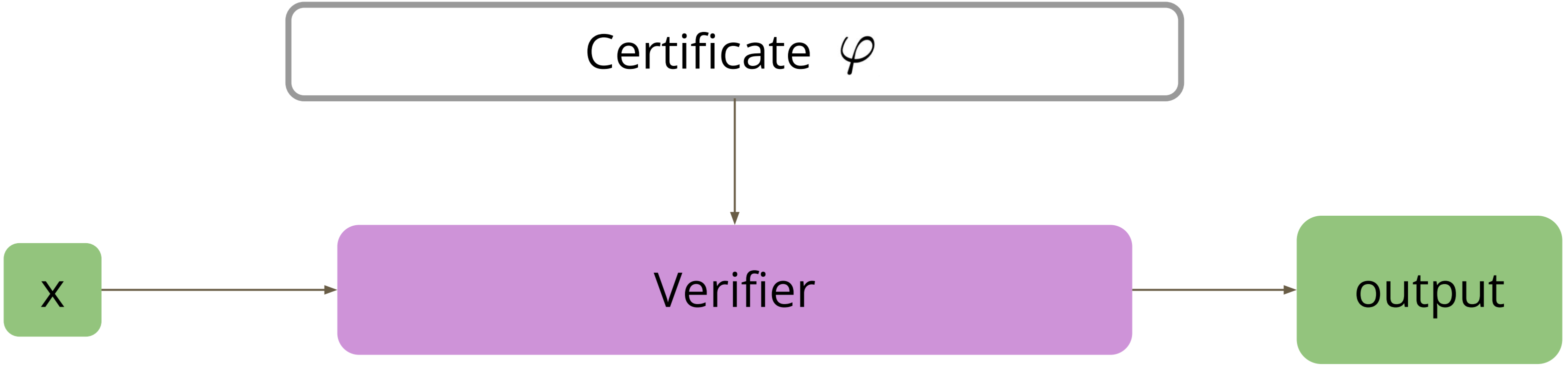


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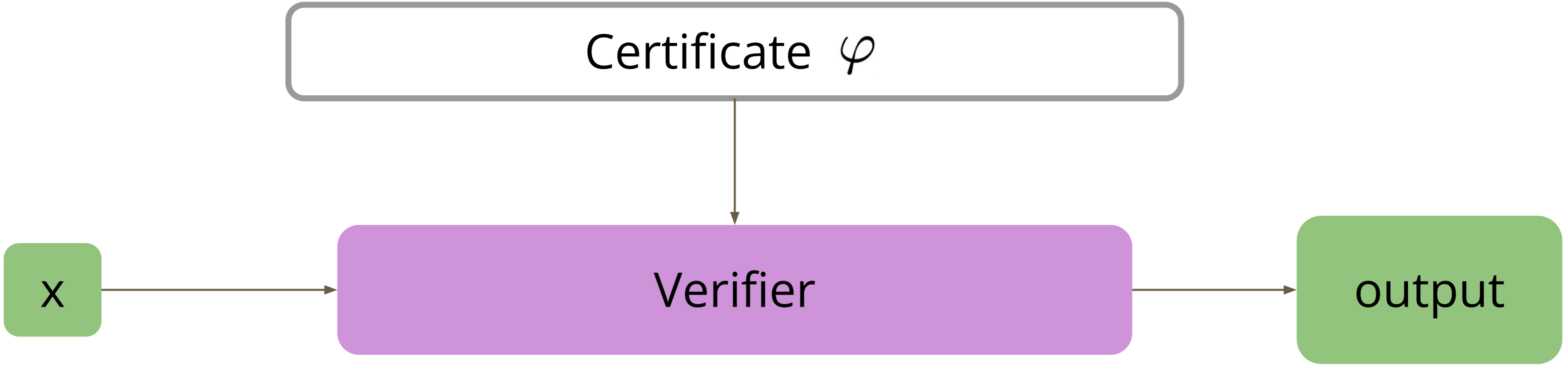
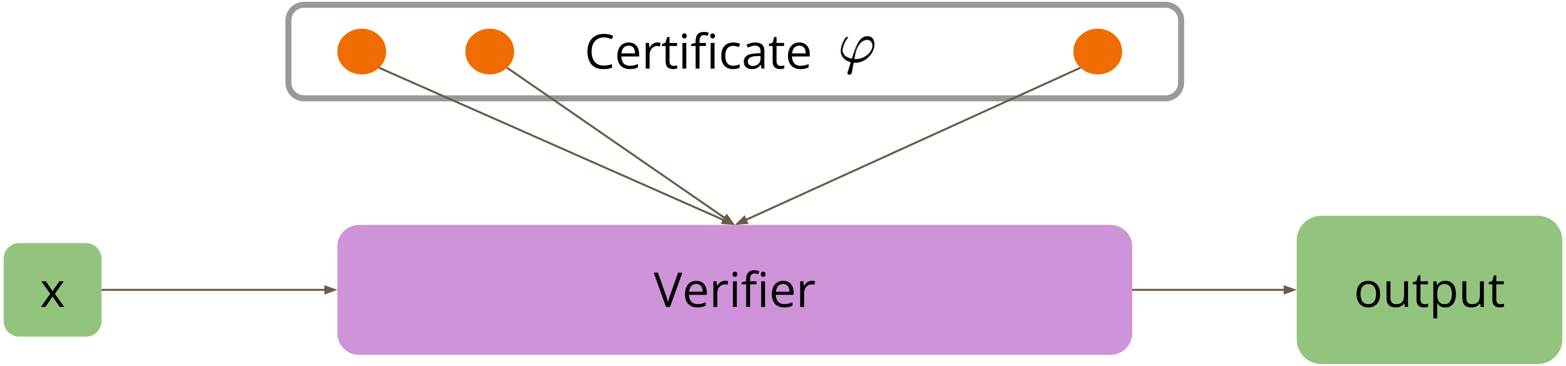
# Natural Extension #1: Probabilistic Verifier




- $x \in L \implies \exists \varphi$ such that $P[V(x,\varphi) = 1] = 1$ (At least one good certificate works)
- $x \notin L \implies \forall \varphi$, we have $P[V(x,\varphi) = 1] \leq 1/2$ (All false certificates probably fail)

# Natural Extension #2: Bounded Queries

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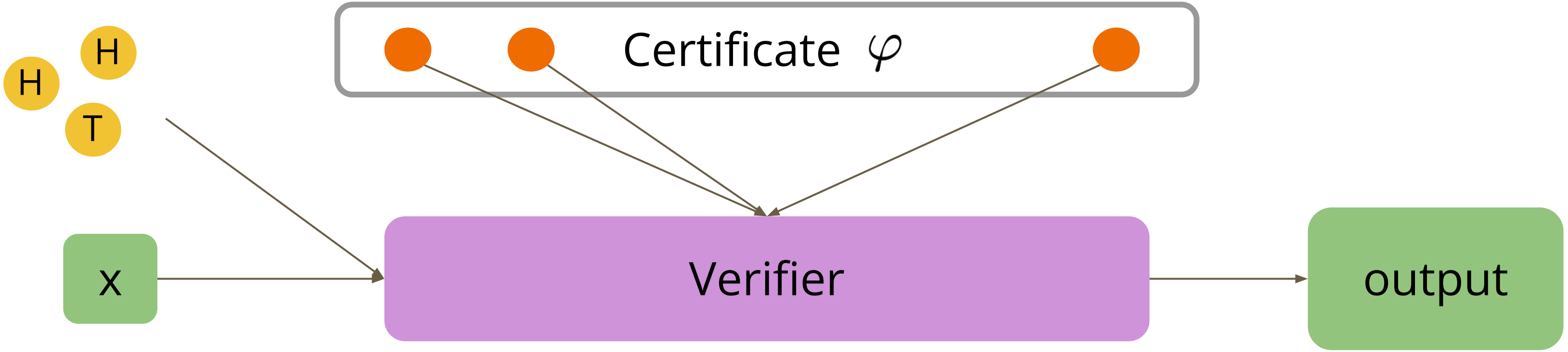
# Natural Extension #3: Bounded Randomness

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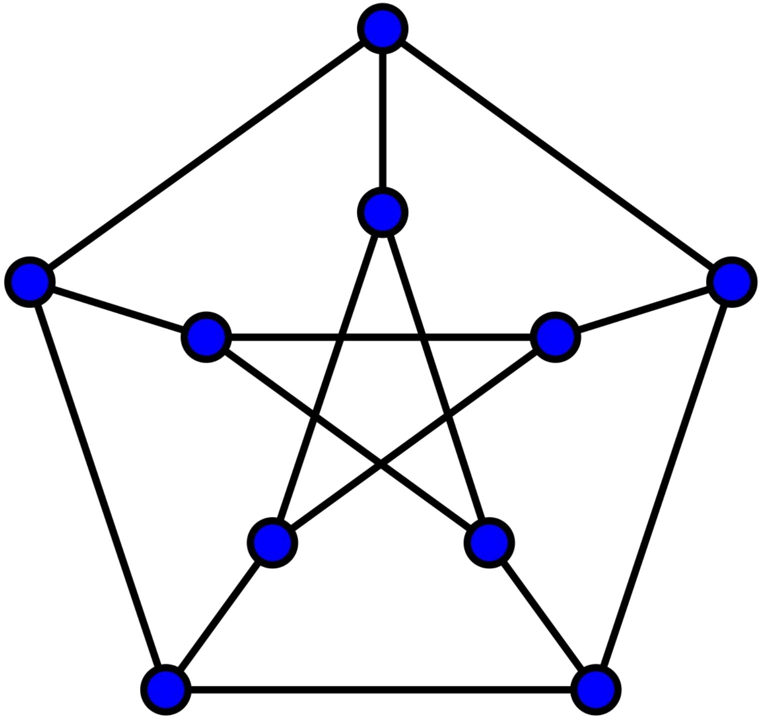
# The Theorem That Rocked The ‘90s

$$\mathsf{NP} = \mathsf{PCP}(O(\log n), O(1))$$

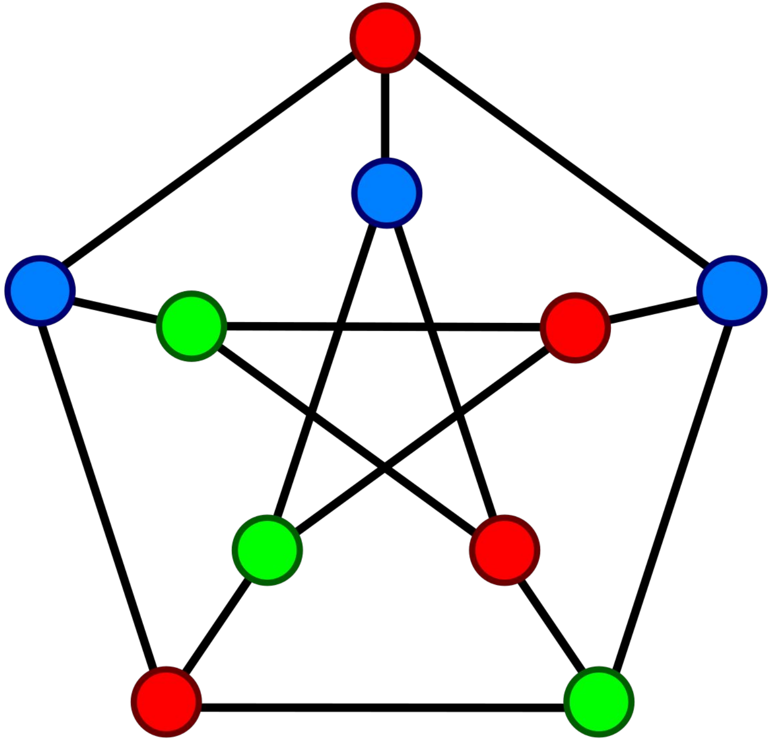
“randomness” “queries”

# Example: Graph Coloring



# Example: Graph Coloring

In NP: certificate is a color for each vertex



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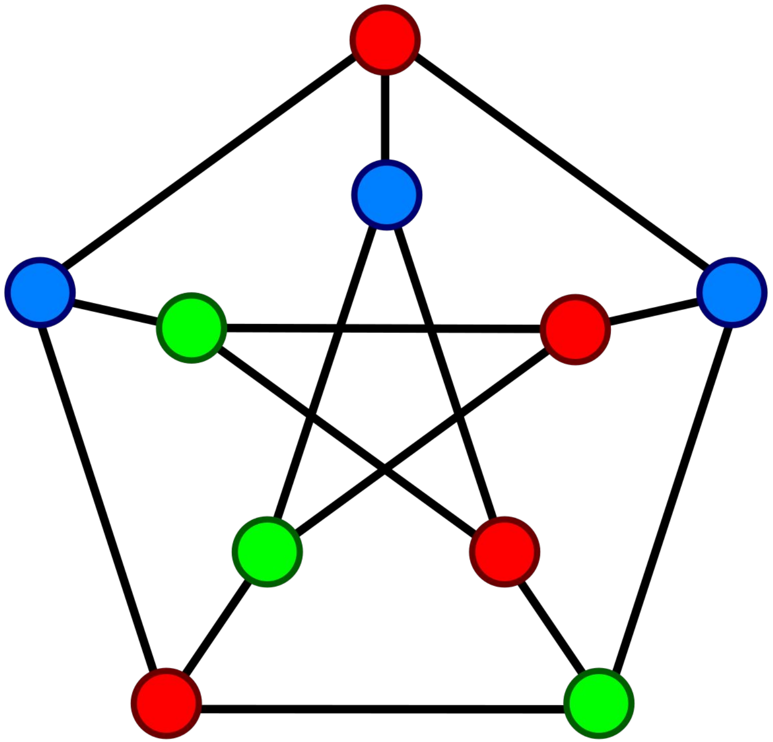

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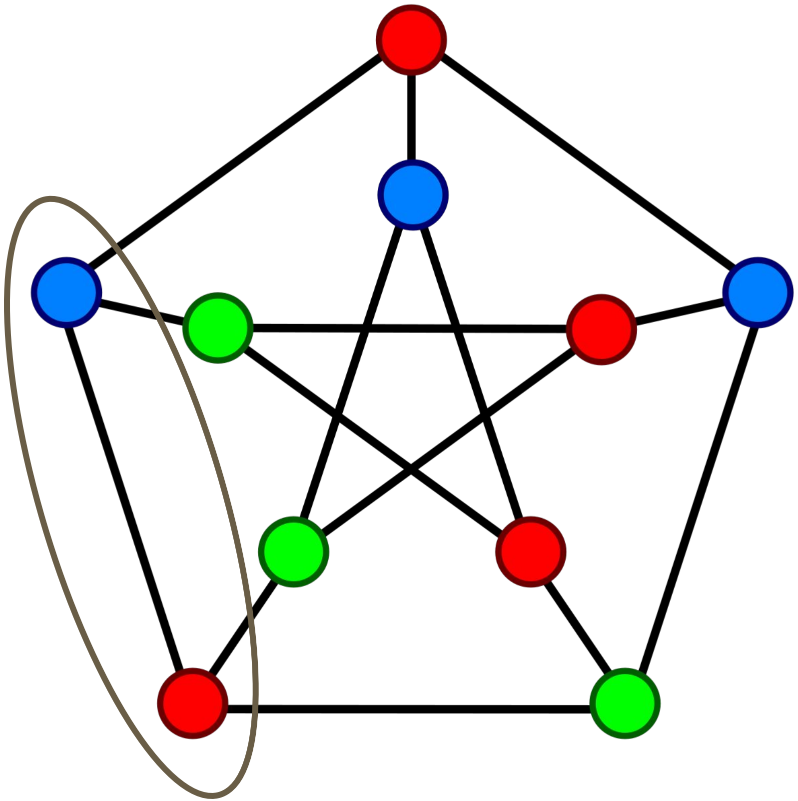

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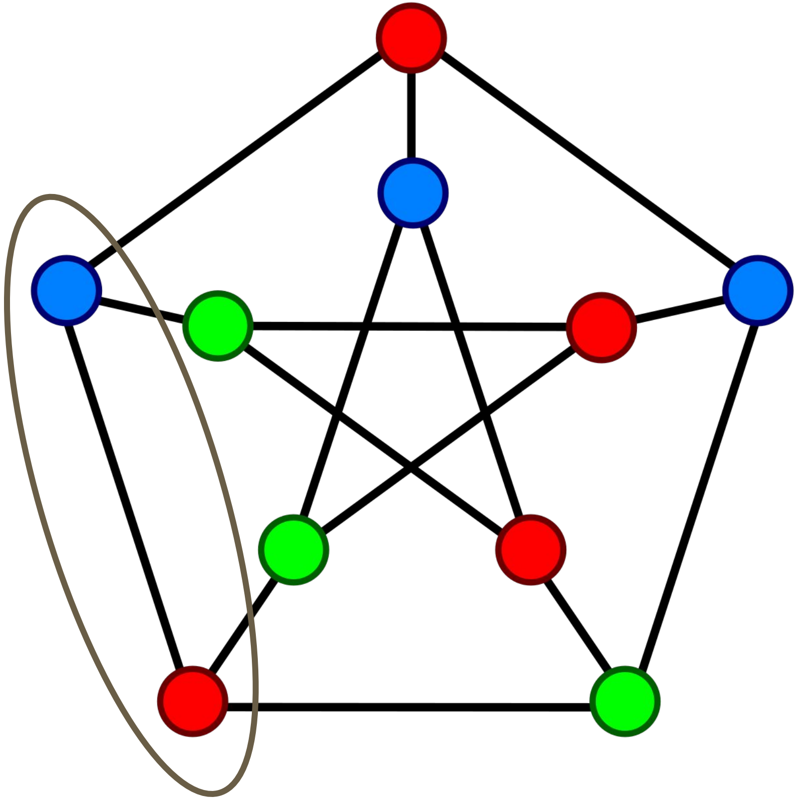

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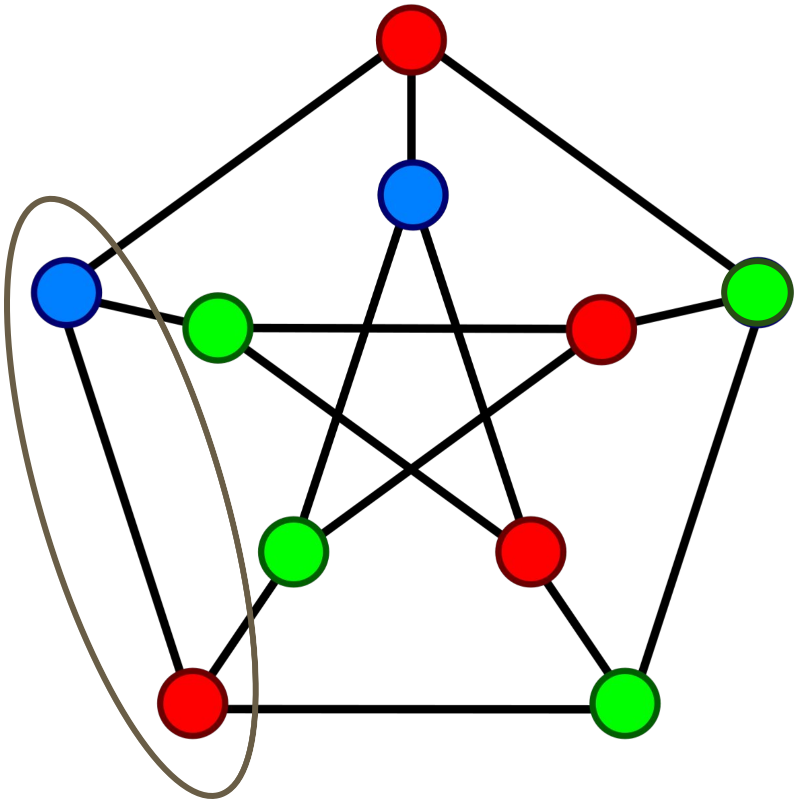

What if just one rogue vertex wrong?

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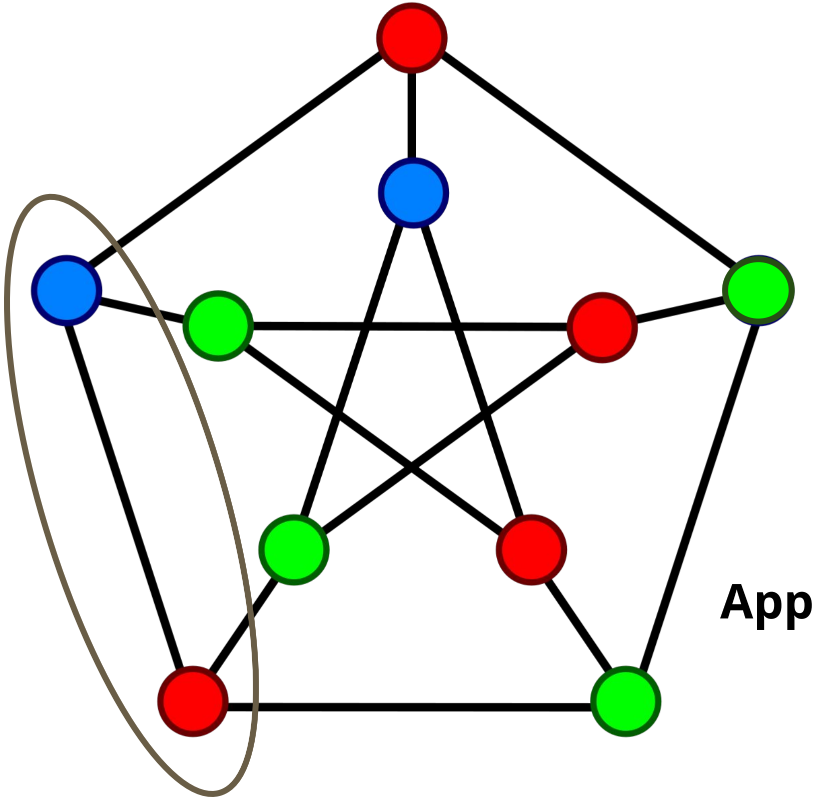

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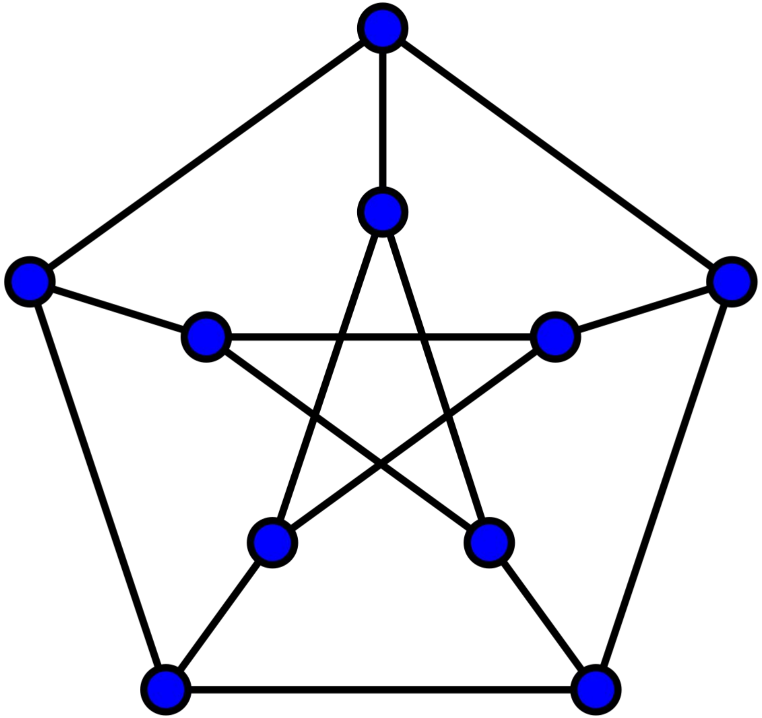
**Apply PCP Transformation**

# Example: Graph Coloring - CSP Formulation

CSP:

- One constraint per edge
- Constraints "query" constant number of vertices
- Goal: distinguish between cases in the promise

"constraint satisfaction problem"



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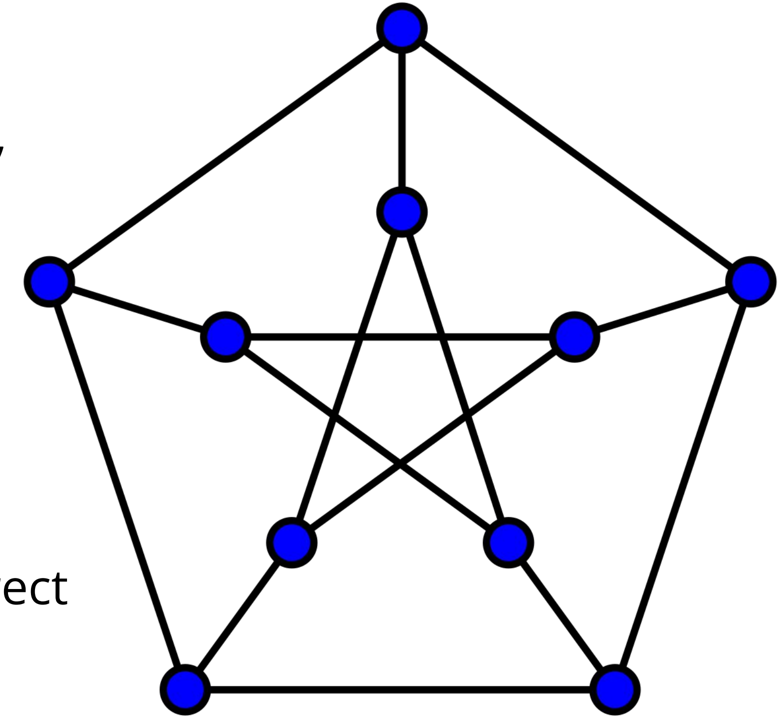

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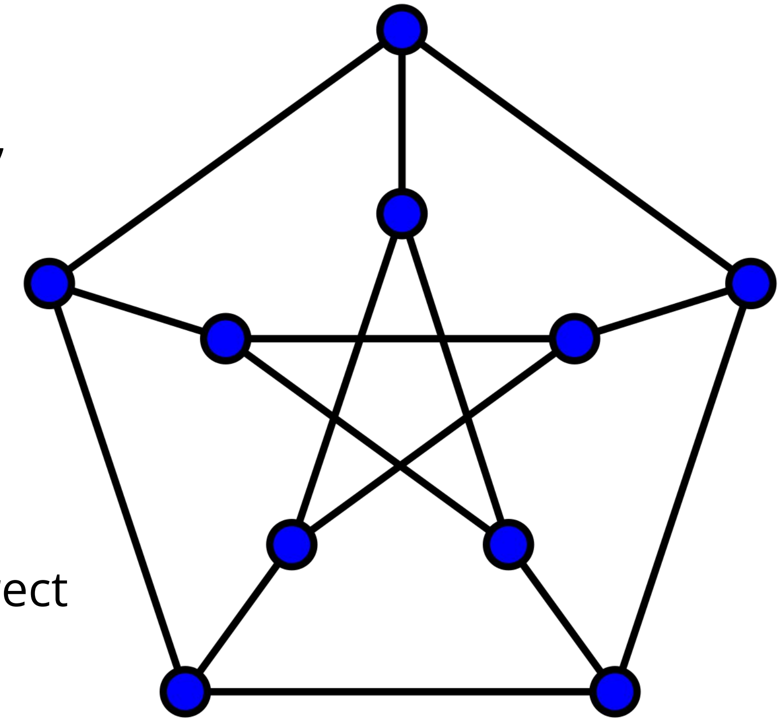

CSP Formulation

$\Longleftrightarrow$

PCP Formulation

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Two-player game:

- Edge player “E”
- Vertex player “V”
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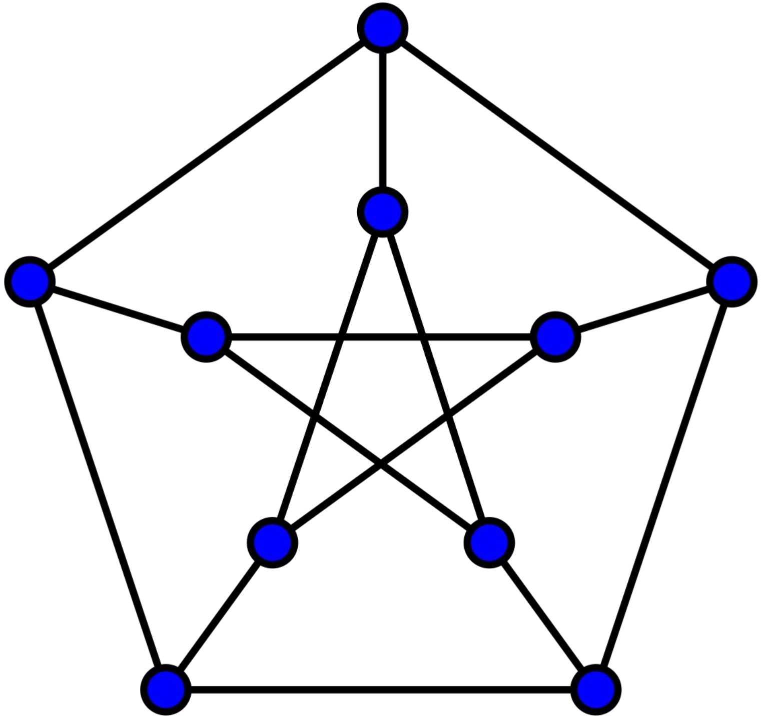

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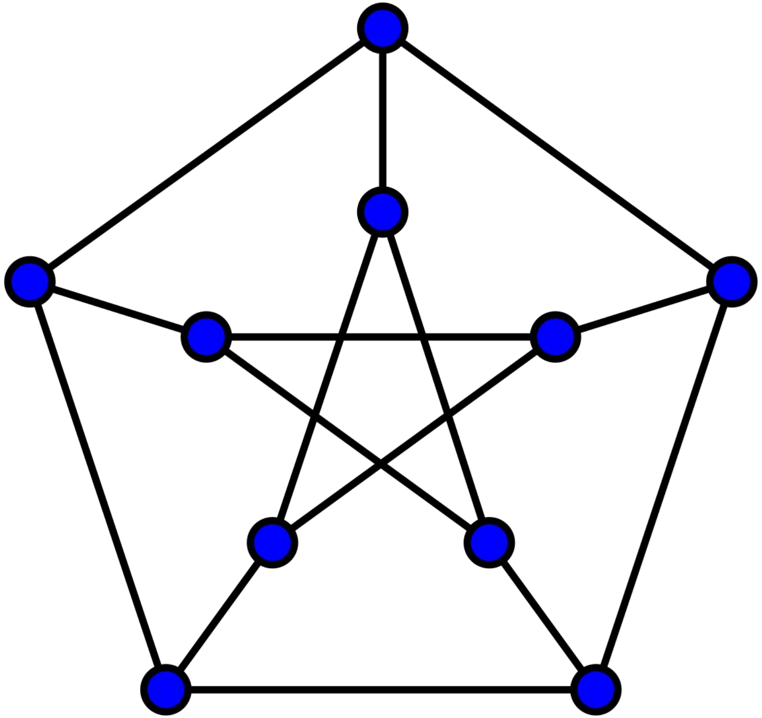

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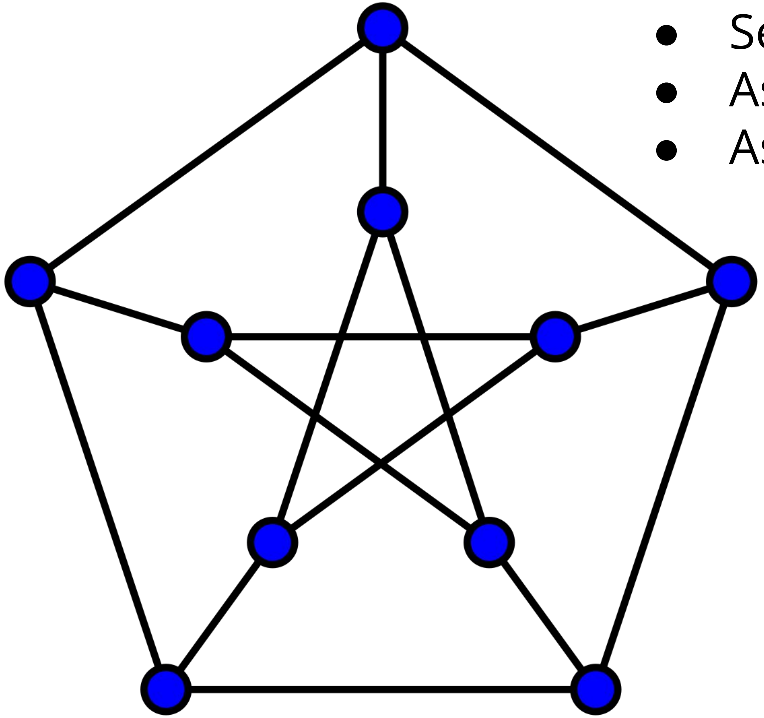

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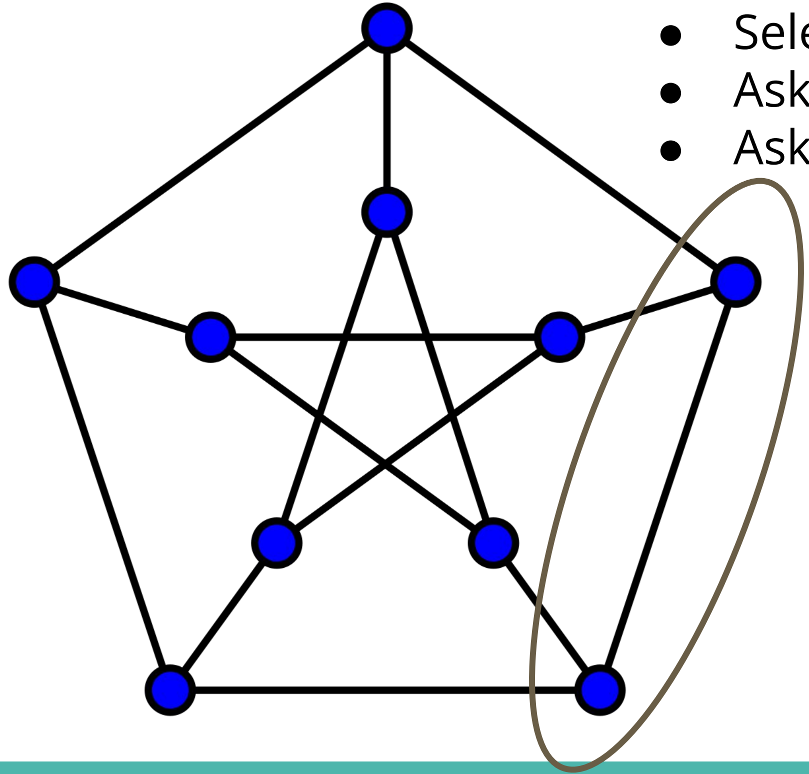

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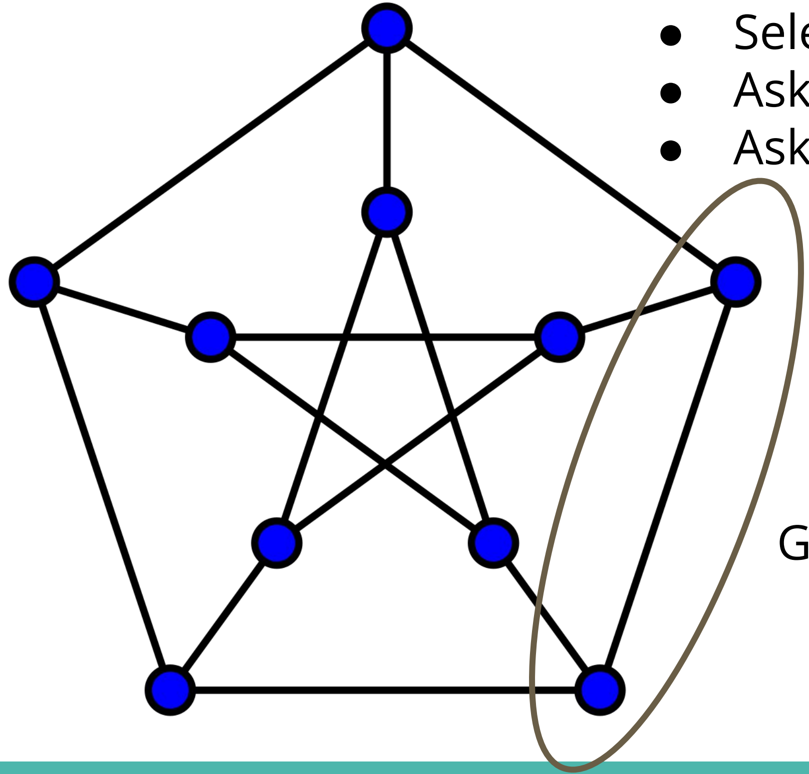

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$\Longleftrightarrow$

PCP Formulation

# Key Takeaways

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CSP Formulation

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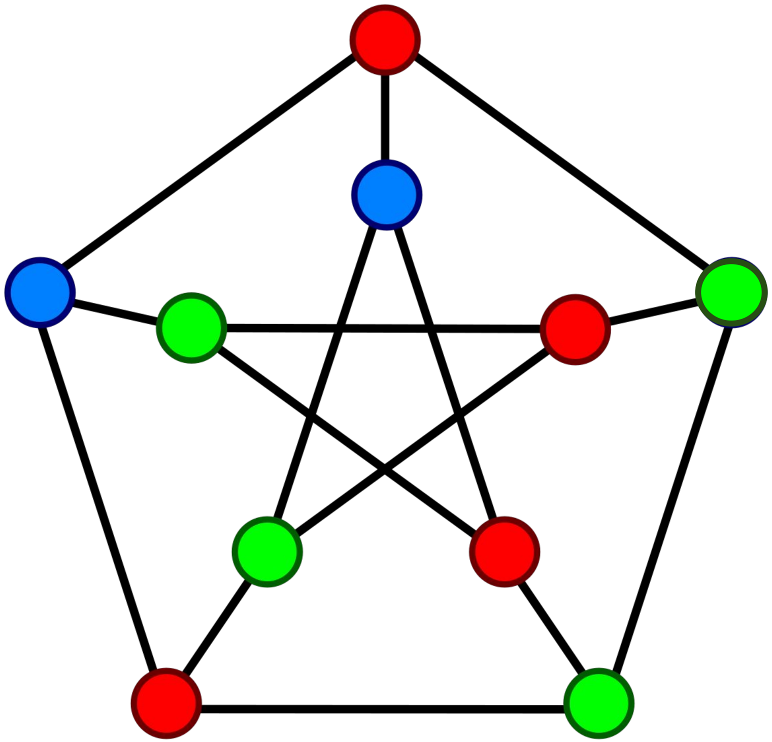
Proofs Formulation

(the easy part)



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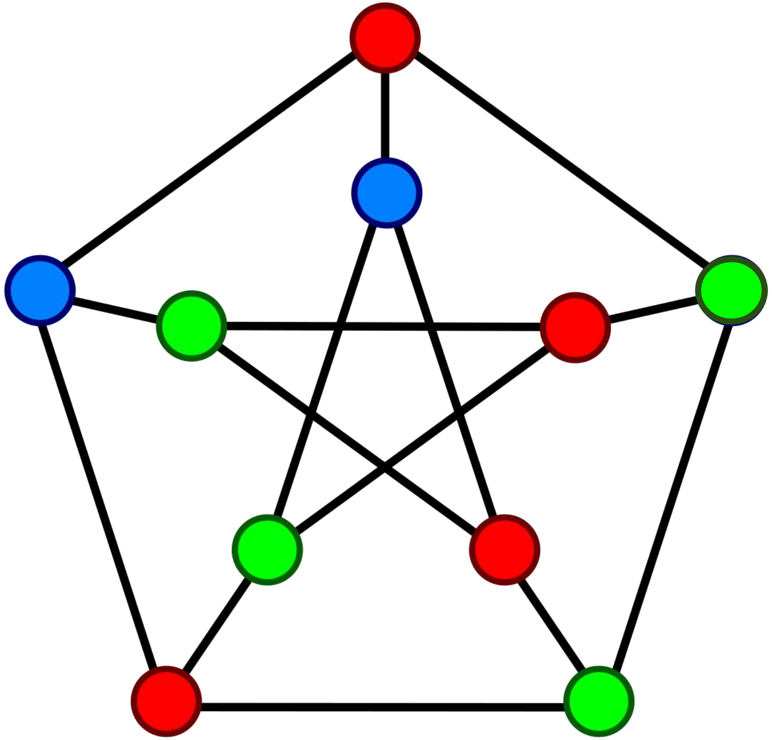

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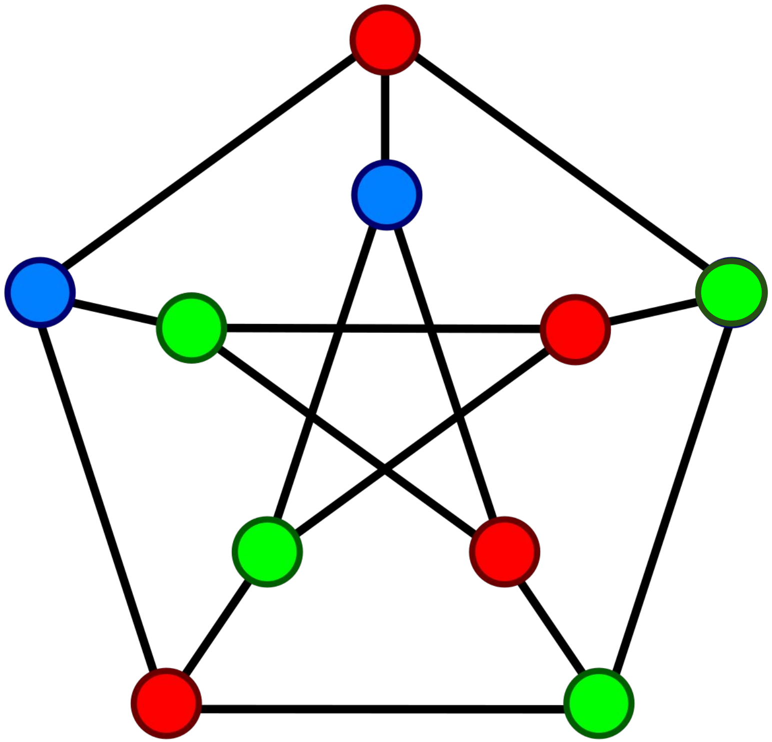

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(the hard part)

CSP equivalence: it is even hard to approximate NP problems

# Into the Quantum Realm...

# q-Local Hamiltonian Problem

- Given *m* Hermitian matrices acting on q < n qubits $\mathbf{H_i} \in \mathbf{C}^{(2^n)^2}$
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- Quantum **Cook-Levin Theorem** says the above is **QMA-hard** when
  - $b - a = 1/poly(n)$

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Constraints <---> Hamiltonians

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  - We can now interpret the $H_i$'s as constraints on assignments to n bits
- CSP problem of finding a satisfying assignment
  - ...same to LH problem to find if it's possible to get a cost of 0
  - But this is just a special case of LH!

# Quantum PCP Conjecture (qLH Formulation)

- QPCP conjecture says ∃γ>0 and q such that it is QMA-hard to distinguish YES and NO instances of q-LH on m Hermitian matrices with **b-a = γm**

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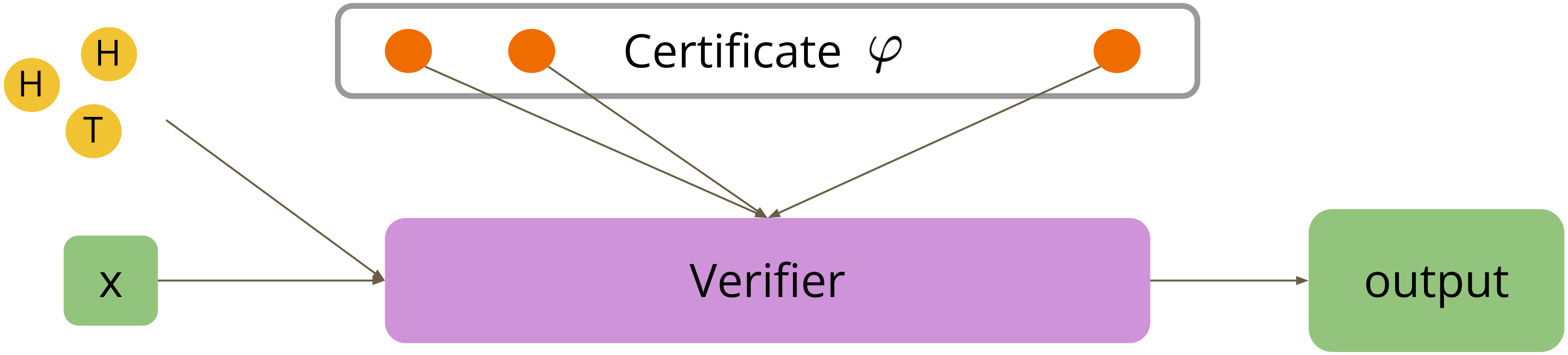
- How is this different from Quantum Cook-Levin?
  - QCL says **b-a = 1/poly(n)**
  - This is a statement the hardness of approximating the eigenvalue, similar to the classical promise problem of CSP

# Quantum PCP Conjecture (Proofs & Games)

- Proofs are easily adaptable to a quantum setting
  - Just allow for quantum certificates on a quantum computer
  - Allow our proofs to measure exactly **q qubits** before making a decision
  - Few more details...
  - Conjectured that there exists a **constant q** s.t. all languages in QMA can be solved in this setting



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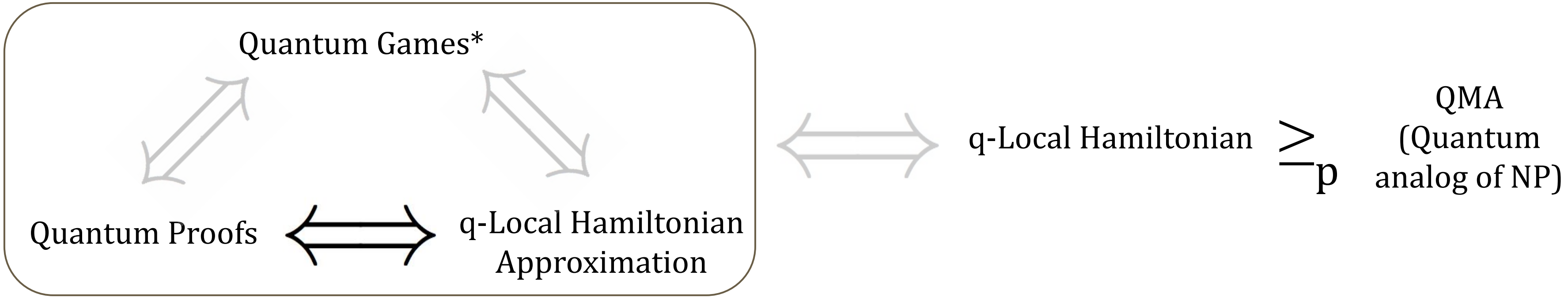
- Games also extend nicely to a quantum setting
  - Allow our players to compute their answers to any questions on a quantum computer
  - For anything interesting to happen, our players have to share entanglement
  - Few more details...

# Summary of Quantum Knowledge

- The quantum Local Hamiltonian and Proofs Formulations are proven to be equivalent
- Quantum games are complicated but conjectured to also be equivalent
- What is the full picture so far?



# Concluding Thoughts

- Beautiful theory, powerful applications
- Hot area, not well understood
- **MIP* = RE**

**Classical PCP**

Games Formulation

$\Longleftrightarrow$

CSP Formulation

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Proofs Formulation

Quantum Games*

Quantum Proofs $\Longleftrightarrow$ q-Local Hamiltonian Approximation

$\Longleftrightarrow$ q-Local Hamiltonian $\geq_p$ QMA (Quantum analog of NP)

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And now **YOU** are equipped to go forth and learn more!

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