CMB Fisher

Xianzhe Tang

December 11, 2024

1 Fisher CMB

1. Redshift at Recombination (z_*)

$$g_1 = \frac{0.0783 \times (\Omega_b h^2)^{-0.238}}{1.0 + 39.5 \times (\Omega_b h^2)^{0.763}}$$

$$g_2 = \frac{0.560}{1.0 + 21.1 \times (\Omega_b h^2)^{1.81}}$$

$$z_* = 1048.0 \times \left(1 + 0.00124 \times (\Omega_b h^2)^{-0.738}\right) \times \left(1 + g_1 \times (\Omega_m h^2)^{g_2}\right)$$

2. Hubble Parameter in Flat $w_0 w_a$ Cosmology

$$H(a) = H_0 \times \sqrt{\Omega_{m0}\, a^{-3} + \Omega_{r0}\, a^{-4} + \Omega_{\mathrm{de},0}\, e^{-3[w_0(1-a) + w_a a - w_a \ln a]}}$$

$$H(z) = H_0 \times \sqrt{\Omega_{m0}\,(1+z)^3 + \Omega_{r0}\,(1+z)^4 + \Omega_{\mathrm{de},0}\,(1+z)^{3(1+w_0+w_a)}\, e^{-3w_a \frac{z}{1+z}}}$$

3. Comoving Sound Horizon ($r_s(z_*)$)

$$r_s(z_*) = c \int_0^{a_{\mathrm{end}}} \frac{1}{a^2 H(a)} \times \frac{1}{\sqrt{3}} \times \sqrt{1 + \frac{3\Omega_b h^2}{4\Omega_\gamma h^2}\, a}\, da$$

where $a_{\mathrm{end}} = \dfrac{1}{1+z_*}$

4. Parameters la and r

$$la = \frac{(1+z_*)\,\pi\, D_A(z_*)}{r_s(z_*)}$$

$$r = (1+z_*) \times D_A(z_*) \times \frac{\sqrt{\Omega_{m0}}\, H_0}{c}$$

where $D_A(z_*) = \dfrac{1}{1+z_*} \displaystyle\int_0^{z_*} \frac{c}{H(z)}\, dz$

$$
\begin{aligned}
\frac{\partial D_A}{\partial \theta} &= \frac{c}{(1+z_*)H(z_*)}\cdot\frac{\partial z_*}{\partial \theta} + \frac{c}{1+z_*}\int_0^{z_*}\left(-\frac{1}{H(z)^2}\cdot\frac{\partial H(z)}{\partial \theta}\right)dz - \frac{D_A(z_*)}{1+z_*}\cdot\frac{\partial z_*}{\partial \theta}\\
\frac{\partial r_s}{\partial \theta} &= \frac{c}{\sqrt{3}}\frac{\sqrt{1+\frac{3\Omega_b h^2}{4\Omega_\gamma h^2}\,a_{\rm end}}}{a_{\rm end}^2 H(a_{\rm end})}\cdot\left(-\frac{1}{(1+z_*)^2}\cdot\frac{\partial z_*}{\partial \theta}\right)\\
&\quad + \int_0^{a_{\rm end}}\frac{c}{\sqrt{3}}\left(\frac{1}{a^2H(a)}\cdot\frac{\partial}{\partial \theta}\sqrt{1+\frac{3\Omega_b h^2}{4\Omega_\gamma h^2}\,a} - \frac{\sqrt{1+\frac{3\Omega_b h^2}{4\Omega_\gamma h^2}\,a}}{a^2H(a)^2}\cdot\frac{\partial H(a)}{\partial \theta}\right)da\\
\frac{\partial D_A}{\partial z_*} &= \frac{c}{1+z_*}\left(\frac{1}{H(z_*)} - \frac{1}{1+z_*}\int_0^{z_*}\frac{c}{H(z)}\,dz\right)\\
\frac{\partial r_s}{\partial z_*} &= -\frac{c}{\sqrt{3}H(a_{\rm end})}\cdot\sqrt{1+\frac{3\Omega_b h^2}{4\Omega_\gamma h^2}\cdot a_{\rm end}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z_*}{\partial \Omega_m} &= 1048.0\left(1+0.00124\,(\Omega_b h^2)^{-0.738}\right)g_1 g_2(\Omega_m h^2)^{g_2-1}h^2\\
\frac{\partial z_*}{\partial H_0} &= 1048.0\left(1+0.00124\,(\Omega_b h^2)^{-0.738}\right)g_1(2g_2)(\Omega_m h^2)^{g_2}\,\frac{H_0^{2g_2-1}}{100^{2g_2}}\\
\text{where } h &= \frac{H_0}{100}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\frac{\partial l_a}{\partial \Omega_m} &= \left[\pi\frac{1}{r_s(z_*)}\left(D_A(z_*)+(1+z_*)\frac{\partial D_A}{\partial z_*}\right) - \pi(1+z_*)D_A(z_*)\frac{1}{r_s(z_*)^2}\frac{\partial r_s}{\partial z_*}\right]\frac{\partial z_*}{\partial \Omega_m}\\
&\quad + \frac{\pi(1+z_*)}{r_s(z_*)}\frac{\partial D_A}{\partial \Omega_m} - \frac{\pi(1+z_*)D_A(z_*)}{r_s(z_*)^2}\frac{\partial r_s}{\partial \Omega_m},\\
\frac{\partial l_a}{\partial H_0} &= \left[\pi\frac{1}{r_s(z_*)}\left(D_A(z_*)+(1+z_*)\frac{\partial D_A}{\partial z_*}\right) - \pi(1+z_*)D_A(z_*)\frac{1}{r_s(z_*)^2}\frac{\partial r_s}{\partial z_*}\right]\frac{\partial z_*}{\partial H_0}\\
&\quad + \frac{\pi(1+z_*)}{r_s(z_*)}\frac{\partial D_A}{\partial H_0} - \frac{\pi(1+z_*)D_A(z_*)}{r_s(z_*)^2}\frac{\partial r_s}{\partial H_0},\\
\frac{\partial r}{\partial \Omega_m} &= \left(D_A(z_*)\frac{\sqrt{\Omega_m}H_0}{c}\right)\frac{\partial z_*}{\partial \Omega_m} + \left((1+z_*)\frac{\sqrt{\Omega_m}H_0}{c}\right)\frac{\partial D_A}{\partial \Omega_m} + \left((1+z_*)\frac{H_0}{c}\frac{D_A(z_*)}{2\sqrt{\Omega_m}}\right),\\
\frac{\partial r}{\partial H_0} &= \left(D_A(z_*)\frac{\sqrt{\Omega_m}H_0}{c}\right)\frac{\partial z_*}{\partial H_0} + \left((1+z_*)\frac{\sqrt{\Omega_m}H_0}{c}\right)\frac{\partial D_A}{\partial H_0} + \left((1+z_*)\frac{\sqrt{\Omega_m}}{c}D_A(z_*)\right).
\end{aligned}
}
$$