FEDERAL STATE AUTONOMOUS EDUCATIONAL INSTITUTION
OF HIGHER EDUCATION
ITMO UNIVERSITY

Report
on the practical task No. 2
Algorithms for unconstrained nonlinear optimization. Direct methods.

Performed by:
Roman Bezaev
J4133c

Accepted by:
Dr Petr Chunaev

St. Petersburg
2021

1 Goal

The use of direct methods (one-dimensional methods of exhaustive search, dichotomy, golden section search; multidimensional methods of exhaustive search, Gauss (coordinate descent), Nelder-Mead) in the tasks of unconstrained nonlinear optimization

2 Formulation of the problem

Task I.
Use the one-dimensional methods of exhaustive search, dichotomy and golden section search to find an approximate (with precision $\varepsilon = 0.001$) solution $x : f(x) \to min$ for the following functions and domains:

1. $f(v) = x^3, x \in [0, 1]$
2. $f(v) = |x - 0.2|, x \in [0, 1]$
3. $f(v) = xsin\frac{1}{x}, x \in [0, 01, 1]$

Calculate the number of f-calculations and the number of iterations performed in each method and analyze the results. Explain differences (if any) in the results obtained.

Task II. Generate random numbers $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. Furthermore, generate the noisy data $\{x_K, y_k\}$ where $k = 0, \ldots, 100$, according to the following rule:

$$y_k = \alpha x_k + \beta + \delta_k, where\ x_k = \frac{k}{100} \tag{1}$$

$\delta_k \sim N(0, 1)$ are values of a random variable with standard normal distribution. Approximate the data by the following linear and rational functions:

1. $F(x, a, b) = ax + b$ (linear)
2. $F(x, a, b) = \frac{a}{1+bx}$ (rational)

by means of least squares through the numerical minimization (with precision ε= 0.001) of the following function:

$$D(a, b) = \sum_{k=0}^{100} (F(x_k, a, b) - y_k)^2 \tag{2}$$

To solve the minimization problem, use the methods of exhaustive search, Gauss and Nelder-Mead. If necessary, set the initial approximations and other parameters of the methods. Visualize the data and the approximants obtained in a plot separately for each type of approximant so that one can compare the results for the numerical methods used. Analyze the results obtained (in terms of number of iterations, precision, number of function evaluations, etc.).

3 Brief theoretical part

Exhaustive search is a brute-force method of finding optimal solution of the problem. When finding optimum of one-dimensional function, it calculates the value of the function at each point in the given range with defined precision. For example, if we want to find minimum of the function in the range from 0 to 1 with precision 0.001, we have to make 1000 calls of the function.

For the multiple-dimensional problems the number of function calls needed to find optimum using exhaustive search increases as exponential function of number of dimensions. If we wanted to find minimum of the 2D function in the range from 0 to 1 with precision 0.001, we should make 10^6 calls. In practice, this method is used to find initial approximations in broad search space.

Dichotomic search in a broad sense is a search algorithm that operates by selecting between two distinct alternatives (dichotomies) at each step. It is a specific type of divide and conquer algorithm. Therefore, it has to make substantially less steps than exhaustive search to find an optimum of the function.

The golden-section search is a technique for finding an extremum of a function inside a specified interval. For a strictly unimodal function with an extremum inside the interval, it will find that extremum, while for an interval containing multiple extrema (possibly including the interval boundaries), it will converge to one of them. If the only extremum on the interval is on a boundary of the interval, it will converge to that boundary point. The method operates by successively narrowing the range of values on the specified interval, which makes it relatively slow, but very robust. It uses less f-calculations than dichotomy because of its choosing of β. It is chosen by rules of golden ratio.

The idea of **Gauss method** of multidimensional function optimization is that in each iteration, the minimisation is carried out only with respect to one vector component of the multidimensional variable x. The method is simple but hardly efficient.

The Nelder–Mead method is a commonly applied numerical method used to find the minimum or maximum of an objective function in a multidimensional space. The Nelder–Mead technique is a heuristic search method that can converge to nonstationary points on problems that can be solved by alternative methods. The idea of the method is to find function values at the vertices of simplex in the search space, define a vertex with maximum function value, and then reflect this point with respect to the gravity center of the other points.

4 Results

4.1 One-dimensional optimization problem

f-calculations table

.	x^3	$\|x-0.2\|$	$xsin\frac{1}{x}$
brute	10001	10001	9901
dichotomy	30	30	30
golden section	21	21	21

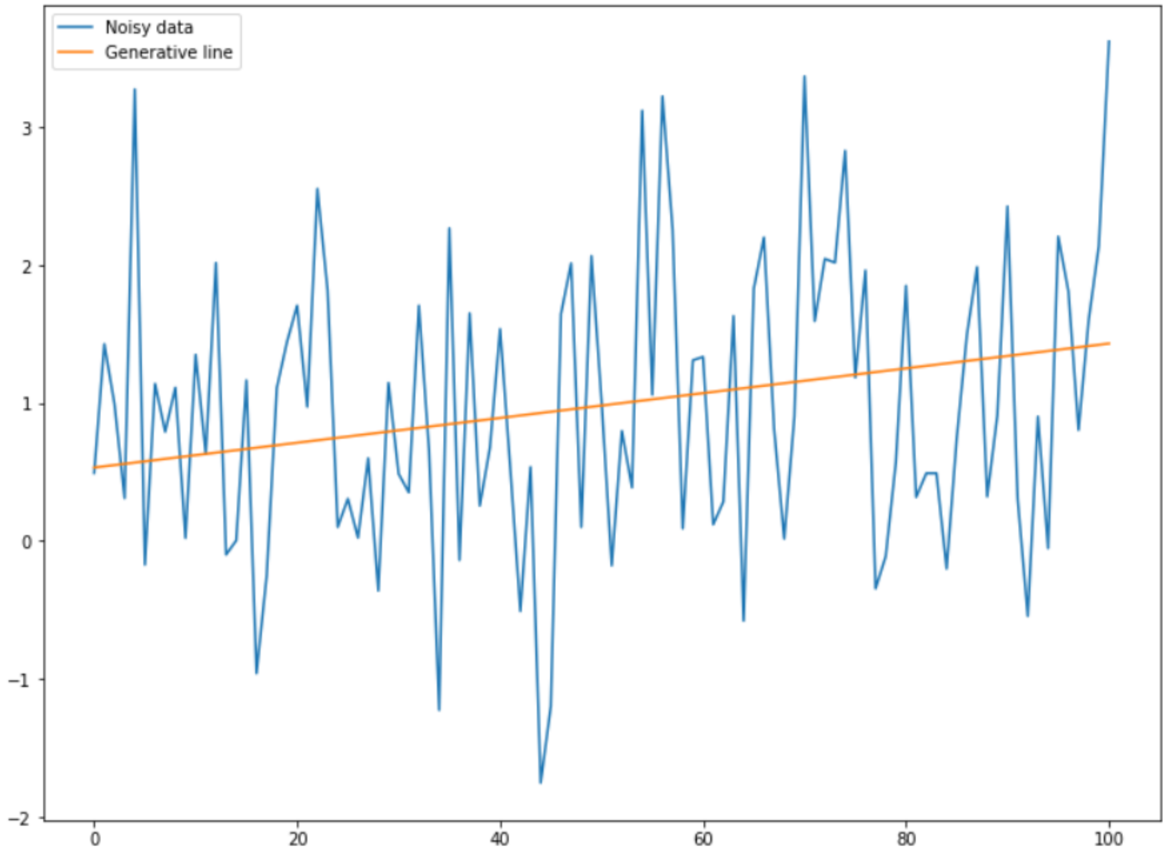
Number of iterations table

.	x^3	$\|x-0.2\|$	$xsin\frac{1}{x}$
brute	10000	10000	9900
dichotomy	15	15	15
golden section	20	20	20

As we see, golden section method requires less f-calculations than any other method, but you have to do more iterations, than in dichotomy method. It is because in dichotomy method the interval become almost two times smaller on every iteration, but in golden section search method interval shrinks in the golden ratio proportion, which smaller than half. Obviously, brute-force method is worst by all measures except its simplicity.

4.2 Two-dimensional optimization problem

Using the formula (1) generating line with a noise.



4.2.1 Linear approximation

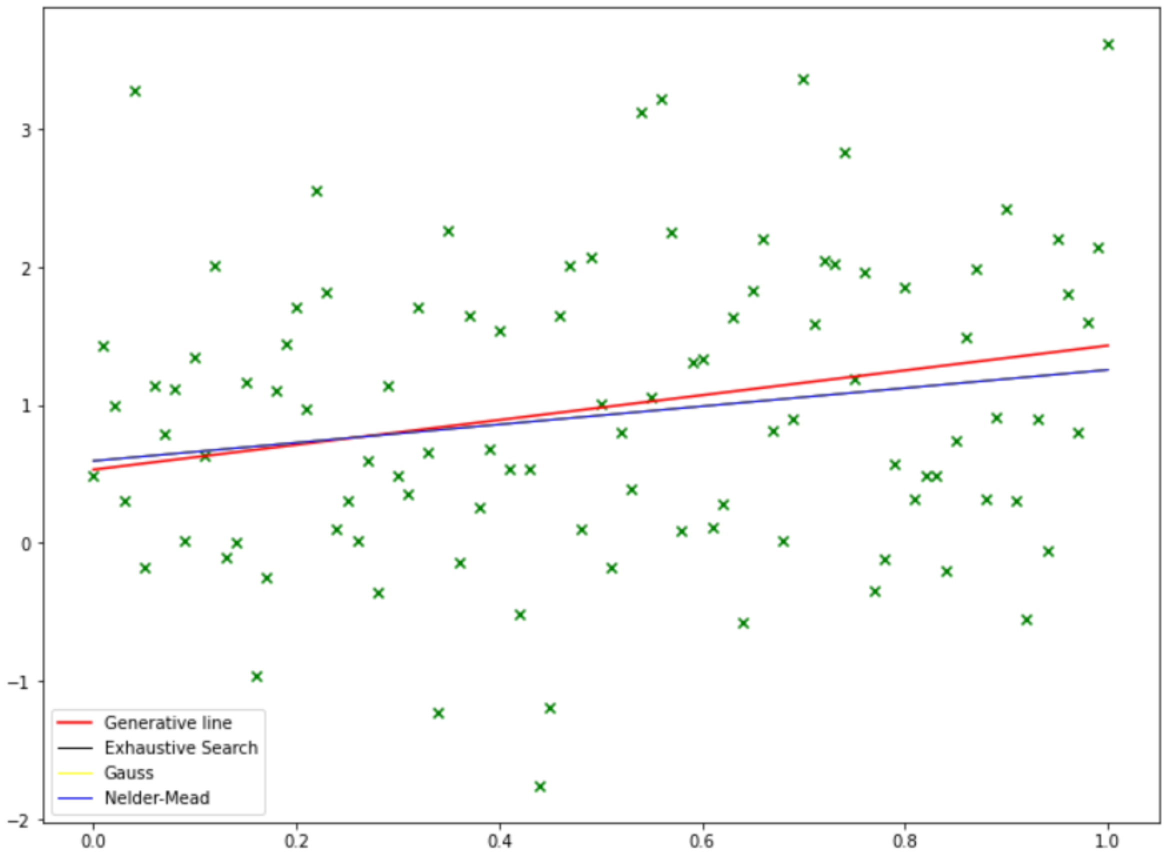


Figure 1: Linear approximation

All algorithms of search for optimal values of parameters “a” and “b” converged to the same values. Nelder-Mead algorithm had found optimal solution using the least number of function calls.

4.2.2 Rational approximation

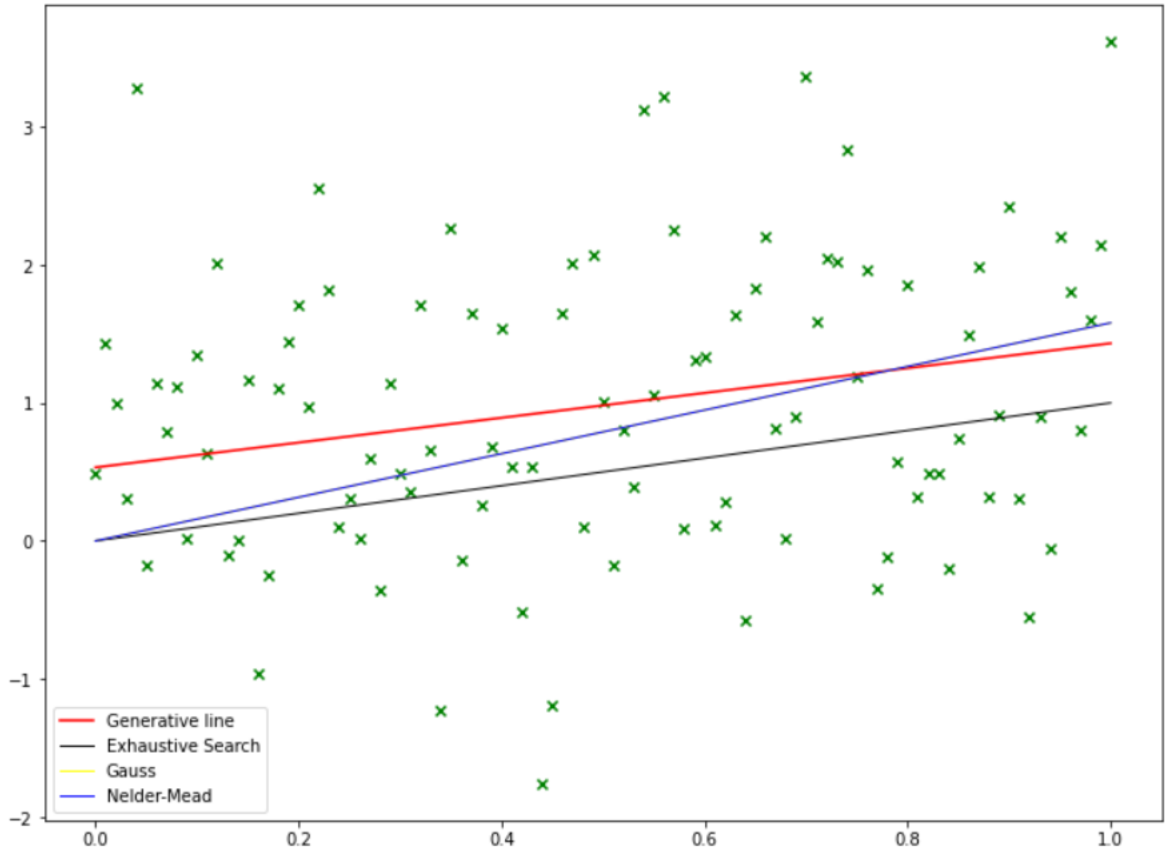


Figure 2: Rational approximation

As we see on the Figure 2, Gauss and Nelder-Mead methods found almost the same values for parameters. Exhaustive search found the parameters, that lies pretty far from the generative line, but it doesn't mean that it has low accuracy according to the least square method. Nelder-Mead algorithm had found optimal solution using the least number of function calls.

5 Conclusions

Direct methods of optimization were used to solve one-dimensional and two-dimensional optimization problems. Brute-force method proved to be the most ineffective for both types. Dichotomy method happened to be less effective than golden-section search for one-dimensional optimization. Nelder-Mead algorithm was more effective than Gauss method for two-dimensional optimization.

6 Appendix

https://github.com/trixdade/ITMO_algorithms/blob/master/Algorithms_Lab2.ipynb