Definition of *linear algebra* : a branch of mathematics that is concerned with mathematical structures closed under the operations of addition and scalar multiplication and that includes the theory of systems of linear equations, matrices, determinants, vector spaces, and linear transformations.

Matrix

A matrix is a rectangular array of numbers (real or complex) enclosed by a pair of brackets (or double vertical rolls) and the numbers in the array are called the entries or the elements of the matrix, that is , a rectangular array of (real or complex) numbers of the form

$$\begin{bmatrix} 1 & 4 & 1 \\ 3 & 0 & 3 \\ -1 & 5 & 2 \end{bmatrix}$$

Is called a matrix.

Or Definition: A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

Example:-1 $\begin{bmatrix} 1 & 4 & 1 \\ 3 & 0 & 3 \\ -1 & 5 & 2 \end{bmatrix}$ **this is square Matrix.**

Example:-2 $\begin{bmatrix} 2 & 1 & 0 & 4 \end{bmatrix}$ this is Row Matrix.

Example:-3 $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ this is Column Matrix.

Notation of Matrices

One of the most important tools used throughout linear algebra, and thus one of the key points to learn on this course, is matrix mathematics. Due to linear algebra being all about finding the solutions to systems of linear equations, matrix math and the study of vector spaces become a tool to represent and orderly solve such systems in an orderly and intuitive fashion.

The lesson of today focuses on the introduction of this new kind of mathematical information arrangement: the matrix (plural: matrices). On our lesson we will introduce the concept of a matrix, how can it be operated on mathematically, and then there will be a little introduction preparing you for the next lessons on this course, where you will learn how useful matrices are and the methods to solve mathematical equations with them.

Element of a Matrix

One of the entries in a matrix. The address of an element is given by listing the row number then the column number.

General 3x3 matrix: $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example of a 3x3 matrix: $$\mathbf{A} = \begin{bmatrix} 3 & 0 & -5 \\ 2 & -6 & 1 \\ 4 & -1 & 7 \end{bmatrix}$$

The elements of this matrix are the numbers 3, 0, -5, 2, etc.
Element a_{23} is 1.

Notation

A matrix is usually shown by a **capital letter** (such as A, or B)

Each entry (or "element") is shown by a **lower case letter** with a "subscript" of **row, column**:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Operations on Matrices.

1. Addition of Matrix: If A = $\begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$, Then **A+B=?**

Solution: A+B = $\begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$

= $\begin{bmatrix} 7 & 1 \\ 7 & 1 \end{bmatrix}$ **Ans.**

2. Subtraction of Matrix: If A = $\begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$, Then **A-B=?**

Solution: A-B = $\begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$

= $\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ **Ans.**

3. Multiplication of Matrix: if A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, Then **AB=?**

Solution: AB = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1x0 + 1x0 & 1x1 + 1x(-1) \\ 0x0 + 0x0 & 0x1 + 0x(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ **Ans.**

Types of Matrices

There are various types of matrices, depending on their structure. Let's explore the most common types:

Null Matrix

A matrix that has all 0 elements is called a **null matrix**. It can be of any order. For example, we could have a null matrix of the order 2 X 3. It's also a **singular matrix**, since it does not have an inverse and its determinant is 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any matrix that does have an inverse can be called a **regular matrix**.

Row Matrix

A **row matrix** is a matrix with only one row. Its order would be 1 X *C*, where *C* is the number of columns. For example, here's a row matrix of the order 1 X 5:

$$\begin{bmatrix} 3 & 5 & 7 & 9 & 11 \end{bmatrix}$$

Column Matrix

A **column matrix** is a matrix with only one column. It is represented by an order of *R* X 1, where R is the number of rows. Here's a column matrix of the order 3 X 1:

$$\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Square Matrix

A matrix where the number of rows is equal to the number of columns is called a **square matrix**. Here's a square matrix of the order 2 X 2:

$$\begin{bmatrix} 4 & 9 \\ 15 & 2 \end{bmatrix}$$

Diagonal Matrix

A **diagonal matrix** is a square matrix where all the elements are 0 except for those in the diagonal from the top left corner to the bottom right corner. Let's take a look at a diagonal matrix of order 4 X 4:

$$\begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Scalar matrix

A special type of diagonal matrix, where all the diagonal elements are equal is called a **scalar matrix**. We can see a 3 X 3 scalar matrix here:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Unit matrix or identity matrix

A scalar matrix whose diagonal elements are all 1 is called a **unit matrix**, or **identity matrix**.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Triangular Matrix

A square matrix where all the elements below the left-right diagonal are 0 is called an **upper triangular matrix**. Here's an upper triangular matrix of order 3 X 3:

$$\begin{bmatrix} 2 & 5 & 8 \\ 0 & 6 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

Lower Triangular Matrix

A square matrix where all the elements above the left-right diagonal are 0 is called a **lower triangular matrix**. Here's what a lower triangular matrix of order 3 X 3 could look like:

$$\begin{bmatrix} 10 & 0 & 0 \\ 8 & 7 & 0 \\ 3 & 2 & 9 \end{bmatrix}$$

Symmetric Matrix

A matrix whose transpose is the same as the original matrix is called a **symmetric matrix**. Only a square matrix can be a symmetric matrix. The **transpose** of a matrix is another matrix that is formed by switching the rows and columns of a given matrix. The given matrix A is a 3 X 3 symmetric matrix, since it's the same as its transpose A^T.

$$A = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 8 & 6 \\ 9 & 6 & 5 \end{bmatrix} ; A^T = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 8 & 6 \\ 9 & 6 & 5 \end{bmatrix}$$

Antisymmetric Matrix

A square matrix whose transpose is its negation is an **antisymmetric matrix**, or **skew-symmetric matrix**. The negation of a matrix is a matrix formed by negating the signs of all the entries:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix} , A^T = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$$

Rectangular Matrix

A rectangular matrix is formed by a different number of rows and columns, and its dimension is noted as: **mxn**.

$$\begin{pmatrix} 1 & 2 & 5 \\ 9 & 1 & 3 \end{pmatrix}$$

Regular Matrix

A regular matrix is a square matrix that has an inverse.

Singular Matrix

A singular matrix is a square matrix that has no inverse.

Idempotent Matrix

The matrix A is idempotent if:

$\mathbf{A^2 = A}$.

Involutive Matrix

The matrix A is involutive if:

$\mathbf{A^2 = I}$.

Symmetric Matrix

A symmetric matrix is a square matrix that verifies:

$\mathbf{A = A^t}$.

Antisymmetric Matrix

An antisymmetric matrix is a square matrix that verifies:

$\mathbf{A = -A^t}$.

Orthogonal Matrix

A matrix is orthogonal if it verifies that:

$\mathbf{A \cdot A^t = I}$.

Square Matrix

A matrix where the number of rows is equal to the number of columns is called a **square matrix**. Here's a square matrix of the order 2 X 2:

$$\begin{bmatrix} 4 & 9 \\ 15 & 2 \end{bmatrix}$$

The transpose of a matrix

The transpose of a matrix is a new matrix whose rows are the columns of the original. This makes the columns of the new matrix the rows of the original). Here is a matrix and its transpose:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix}^{T} = \begin{pmatrix} 5 & 4 & 7 \\ 4 & 0 & 10 \\ 3 & 4 & 3 \end{pmatrix}$$

The superscript "T" means "transpose"

What is:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{T}_{2\times 3} = \quad ?$$

Invertible Matrices

Definition: An $n \times n$ matrix A is called **nonsingular** or **invertible** iff there exists an $n \times n$ matrix B such that

$$A\,B = B\,A = I_n$$

where I_n is the identity matrix. The matrix B is called the **inverse** matrix of A.

Example. Let

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 3/2 \\ 1 & -1 \end{pmatrix}.$$

One may easily check that

$$A\,B = B\,A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 .$$

Hence A is invertible and B is its inverse.

Notation. A common notation for the inverse of a matrix A is A^{-1}. So

$$A\,A^{-1} = A^{-1}\,A = I_n .$$

Example. Find the inverse of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

Write

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since

$$A\,A^{-1} = \begin{pmatrix} a+c & b+d \\ -a+2c & -b+2d \end{pmatrix} = I_2$$

we get

$$\begin{cases} a + c = 1 \\ -a + 2c = 0 \\ b + d = 0 \\ -b + 2d = 1 \end{cases}$$

Easy algebraic manipulations give

$$a = \frac{2}{3},\quad b = -\frac{1}{3},\quad c = \frac{1}{3},\quad d = \frac{1}{3}$$

or

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

The inverse matrix is unique when it exists. So if A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A\,.$$

The following basic property is very important:

If A and B are invertible matrices, then $A\,B$ is also invertible and

$$(A\,B)^{-1} = B^{-1}\,A^{-1}\,.$$

NB. In the definition of an invertible matrix A, we used both $A\,B$ and $B\,A$ to be equal to the identity matrix. In fact, we need only one of the two. In other words, for a matrix A, if there exists a matrix B such that $AB = I_n$, then A is invertible and $B = A^{-1}$.