EECS 126 Probability and Random Processes
Spring 2018 Kannan Ramchandran **HW 10**

1 Midterm

1. (a) $\mathbb{P}(|Y_n - 1| \geq \epsilon) = \mathbb{P}(Y_n \leq 1-\epsilon) = \mathbb{P}(X_1 \leq 1-\epsilon, \ldots, X_n \leq 1-\epsilon) = (1-\epsilon)^n$. $\lim_{n\to\infty} \mathbb{P}(Y_n \leq 1-\epsilon) = \lim_{n\to\infty}(1-\epsilon)^n = 0$. $\therefore Y_n \xrightarrow{\mathbb{P}} 1$.

   (b) Yes. Since $N(t) \sim \text{Poisson}(\lambda t), t \geq 0)$, this means that $N(n) = \sum_{i=1}^n N(1)$, so that by the SLLN, $\lim_{n\to\infty} \frac{N(n)}{n} = \mathbb{E}[N(1)] = \lambda$.

   (c) Let $X_1, \ldots, X_i, \ldots X_N$ represent the interarrival times of a Poisson process process with rate λ. Let each arrival be accepted with probability p and rejected with probability $1-p$. We can view this as a splitting process, where the original process is split into Poisson processes for arrival acceptances and arrival rejections. In particular, S_N represents the time of the first arrival acceptance, and $S_N \sim \text{Exponential}(\lambda p)$.

   (d) See Figure 1.

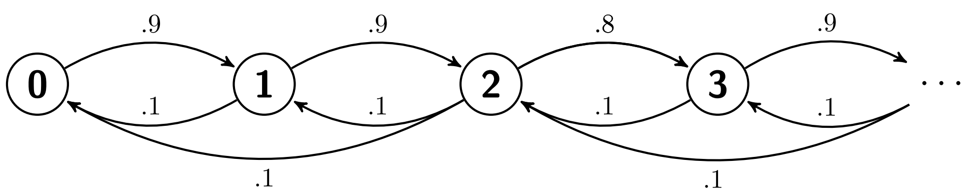

Figure 1: Irreducible, transient, aperiodic MC with no self-loops

   (e) Let state i be the Hamming distance from 000, and $\alpha(i) = \mathbb{P}(T_0 < T_3 \mid X_0 = 2)$. Then the first step equations are:

$$\begin{aligned}
\alpha(0) &= 1 \\
\alpha(1) &= \frac{1}{3}\alpha(0) + \frac{2}{3}\alpha(2) \\
\alpha(2) &= \frac{1}{3}\alpha(3) + \frac{2}{3}\alpha(1) \\
\alpha(3) &= 0
\end{aligned}$$

   We are looking for $\alpha(2) = \frac{2}{3}\alpha(1) = \frac{2}{3}\left(\frac{1}{3} + \frac{2}{3}\alpha(2)\right)$, so $\alpha(2) = \frac{2}{5}$.

   (f)

   (g) Note that that $\mathbb{P}(1 - Y_i = 2) = \mathbb{P}(Y_i = -1) = p$ and $\mathbb{P}(1 - Y_i = 0) = \mathbb{P}(Y_i = 1) = 1 - p$. As such, $\hat{p} = \frac{\sum_{i=1}^n B_i}{2n}$, where each $B_i = \frac{1-Y_i}{2} \sim \text{Bernoulli}(p)$. We would like to determine an interval $(\hat{p} - \epsilon, \hat{p} + \epsilon)$ such that $\mathbb{P}(|\hat{p} - p| \leq \epsilon) = .95$. Using the CLT, we get that:

$$\begin{aligned}
\mathbb{P}(|\hat{p} - p| \leq \epsilon) &= .95 \\
\mathbb{P}\left(\frac{\sqrt{n}|\hat{p} - p|}{\sqrt{p(1-p)}} \leq \frac{\epsilon\sqrt{n}}{\sqrt{p(1-p)}}\right) &= .95 \\
\mathbb{P}\left(|Z| \leq \frac{\epsilon\sqrt{n}}{\sqrt{p(1-p)}}\right) &= .95
\end{aligned}$$

   Since $Z \sim \mathcal{N}(0,1)$ by the CLT, we have that $\frac{\epsilon\sqrt{n}}{\sqrt{p(1-p)}} \approx 2$, so $\epsilon \approx \frac{2\sqrt{p(1-p)}}{\sqrt{n}}$. As such, our interval is thus $\left(\hat{p} - \frac{2\sqrt{p(1-p)}}{\sqrt{n}}, \hat{p} + \frac{2\sqrt{p(1-p)}}{\sqrt{n}}\right)$.

(h) Since $\sum_{i=1}^{10} |x_i - 6| \leq 1$, for a sequence to be in the typical set with $n = 10, p = .6$ and $\epsilon = .1$, this means there must only be either 5, 6, or 7 $x_i \mid x_i = 1\ \forall i \in \{1, \ldots, 10\}$. As such, we get that $|A_{.1}^{(10)}| = \binom{10}{5} + \binom{10}{6} + \binom{10}{7}$.

2. (a)
   (b)
   (c)
   (d)

3. (a)
   (b)
   (c)

4. (a) Let N_t be a random variable indicating the total number of transitions at time t. Then $N_t \sim \text{Poisson}(\lambda t)$, as the transitions between the states can be modeled as a Poisson arrival process with rate λ. As such, we are looking for $\mathbb{P}(N_1 = 2 \mid N_3 = 6)$. We thus get that:

$$\begin{aligned}
\mathbb{P}(N_1 = 2 \mid N_3 = 6) &= \frac{\mathbb{P}(N_3 = 6, N_1 = 2)}{\mathbb{P}(N_3 = 6)} \\
&= \frac{\frac{\lambda^3 e^{-\lambda}}{2!} \cdot \frac{(2\lambda)^4 e^{-2\lambda}}{4!}}{\frac{(3\lambda)^6 e^{-3\lambda}}{6!}} \\
&= \binom{6}{2} \frac{2^4 \lambda^6}{3^6 \lambda^6} \\
&= \binom{6}{2} \frac{2^4}{3^6}
\end{aligned}$$

(b) Let $\beta(i) = \mathbb{E}[T_n \mid X = i]$, where T_n is the expected time to hit state n. We would then like to find $\beta(0)$. Writing out the first step equations, we get that:

$$\begin{aligned}
\beta(-1) &= \frac{1}{\lambda} + \beta(0) \\
\beta(0) &= \frac{1}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda}\beta(1) + \frac{\mu}{\mu + \lambda}\beta(-1) \\
&\vdots \\
\beta(i) &= \frac{1}{\lambda} + \beta(i+1) \\
&\vdots \\
\beta(n) &= 0
\end{aligned}$$

Solving these for $\beta(0)$, bet get that:

$$\begin{aligned}
\beta(0) &= \frac{1}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} \cdot \frac{n-1}{\lambda} + \frac{\mu}{\mu + \lambda}\left(\frac{1}{\lambda} + \beta(0)\right) \\
\frac{\lambda \beta(0)}{\mu + \lambda} &= \frac{1}{\mu + \lambda} + \frac{n-1}{\mu + \lambda} + \frac{\mu}{\lambda(\mu + \lambda)} \\
\beta(0) &= \frac{n-1}{\lambda} + \frac{\mu}{\lambda^2}
\end{aligned}$$

(c) Too hard.

2 Exponential: MLE & MAP

(a)

(b)

3 BSC: MLE & MAP

(a)

(b)

(c)

4 Fun with Linear Regression

(a)

(b)

(c)

(d)

5 Community Detection Using MAP

6 [Bonus] Bayesian Estimation of Poisson Distribution

(a)

(b)

(c)