UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 7
Fall 2017

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1. **Backwards Markov Property**

   Let $(X_n, n \in \mathbb{N})$ be a discrete-time Markov chain with state space $\mathcal{S}$. Show that for every $m, k \in \mathbb{N}$, $m \geq 1$, we have

   $$\mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1, \dots, X_{k+m} = i_m) = \mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1)$$

   for all states $i_0, i_1, \dots, i_m$.

   Solution:

   We can write

   $$\begin{aligned}&\mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1, \dots, X_{k+m} = i_m)\\ &\quad = \frac{\mathbb{P}(X_k = i_0, X_{k+1} = i_1, \dots, X_{k+m} = i_m)}{\mathbb{P}(X_{k+1} = i_1, \dots, X_{k+m} = i_m)}.\end{aligned}$$

   Now use Markov property to compute the joint probability. After several term cancellations, the result follows.

2. **Seven-State Chain**

   A discrete-time Markov chain with seven states has the following transition probabilities:

   $$p_{i,j} = \begin{cases} 0.5, & (i,j) = (3,2), (3,4), (5,6), \text{ and } (5,7) \\ 1, & (i,j) = (1,3), (2,1), (4,5), (6,7), \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}$$

   In the questions below, let X_k be the state of the Markov chain at time k, for each $k \in \mathbb{N}$.

   (a) Give a pictorial representation of the discrete-time Markov chain.

   (b) For what values of n is $\mathbb{P}(X_n = 5 \mid X_0 = 1) > 0$?

   (c) What is the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \dots, 7$?

   (d) If $X_0 = 1$, what is the expected time for the Markov chain to reach state 7 for the first time?

   Solution:

   (a) See the pictorial representation in Figure 2.

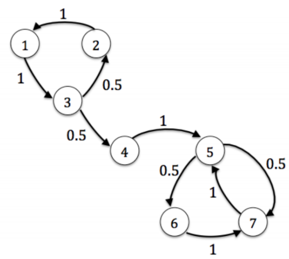

Figure 1: Pictorial representation of the discrete-time Markov chain.

(b) State 5 is reachable from state 1 with three transitions. Paths from state 1 to state 5 also include paths with a loop from 1 back to 1 (of length 3) and/or a loop from 5 back to 5 by way of state 7 (either length 2 or length 3). Therefore potential path lengths are $3+2m+3n$, for $m, n \in \mathbb{N}$. Therefore, $\mathbb{P}(X_n = 5 \mid X_0 = 1) > 0$ for $n = 3$ or $n \geq 5$.

(c) From states 1, 2, and 3, all states are accessible because there is a non-zero probability path from these states by way of state 3 to any other state. From states, 4, 5, 6, and 7, paths only exist to states 5, 6, and 7.

(d) Let $F(i, j)$ be the expected time that the system needs to take to get to state j for the first time, starting from state i. We can see that $F(1, 7) = 1 + F(3, 4) + 1 + F(5, 7)$. Then we have

$$F(3,4) = 0.5 + 0.5\big(F(2,4) + 1\big),$$
$$F(2,4) = 2 + F(3,4).$$

Then we get $F(3, 4) = 4$. To find $F(5, 7)$, we have

$$F(5,7) = 0.5 + 0.5\big(F(6,7) + 1\big),$$
$$F(6,7) = 1.$$

Then we get $F(5, 7) = 1.5$. Therefore, $F(1, 7) = 7.5$.

3. Two-State Chain with Linear Algebra

Consider the Markov chain $(X_n,\ n \in \mathbb{N})$, shown in Figure 2, where $\alpha, \beta \in (0, 1)$.

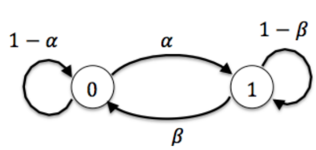

Figure 2: Markov chain for Problem 3.

(a) Find the probability transition matrix P.

(b) Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U\Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

Hint: This is called the eigendecomposition of a matrix.

(c) Find P^n in terms of U and Λ for each $n \in \mathbb{N}$.

(d) Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \in \mathbb{N}$.

Solution:

(a) The probability transition matrix is

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

(b) Since $(P - \lambda_i I)x = 0$ has non-zero solution u_i, we have $\det(P - \lambda_i I) = 0$, i.e., λ_1 and λ_2 are solutions to

$$\det \begin{bmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{bmatrix} = \lambda^2 - (2-\alpha-\beta)\lambda + 1 - \alpha - \beta.$$

Then we get $\lambda_1 = 1$, and $\lambda_2 = 1 - \alpha - \beta$. Then we can get u_1 and u_2: $u_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\intercal$ and $u_2 = \begin{bmatrix} \alpha & -\beta \end{bmatrix}^\intercal$. Further, we can see that if we let

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 1 & -\beta \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix},$$

we have $PU = U\Lambda$, which is equivalent to $P = U\Lambda U^{-1}$.

(c) We have

$$P^n = U\Lambda U^{-1} \cdots U\Lambda U^{-1} = U\Lambda^n U^{-1}.$$

(d) Let $\pi(n) = \begin{bmatrix} \mathbb{P}(X_n = 0) & \mathbb{P}(X_n = 1) \end{bmatrix}$ be the PMF of X_n. Then we have

$$\pi(n) = \pi(0)P^n = \pi(0)U\Lambda^n U^{-1}.$$

Since we have $\pi(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$, by some computation, we get

$$\pi(n) = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta + \alpha(1-\alpha-\beta)^n & \alpha - \alpha(1-\alpha-\beta)^n \end{bmatrix}.$$