UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 2
Fall 2017

Self-Graded Scores Due: 5 PM, Monday, September 18, 2017
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1. Packet Routing

Consider a system with n inputs and n outputs. At each input, a packet appears independently with probability p. If a packet appears, it is destined for one of the n outputs uniformly randomly, independently of the other packets.

(a) Let X denote the number of packets destined for the first output. What is the distribution of X?

(b) What is the probability of a collision, that is, more than one packet heading to the same output?

Solution:

(a) The probability that there exists a packet at an input and the packet is destined for the first output is p/n. By the independence over inputs, X has the binomial distribution $(n, p/n)$.

(b) Let C be the event of a collision and let N be the total number of packets in all inputs.

$$\begin{aligned}\mathbb{P}(C) &= 1 - \mathbb{P}(\bar{C}) = 1 - \sum_{k=0}^{\infty} \mathbb{P}(\bar{C} \mid N = k)\mathbb{P}(N = k) \\ &= 1 - \sum_{k=0}^{n} \frac{n!}{(n-k)!n^k} \binom{n}{k} p^k (1-p)^{n-k}\end{aligned}$$

2. Numbered Balls

A bin contains balls numbered $1, 2, \ldots, n$. You reach in and select k balls at random (where $k \le n$ is a positive integer). Note that you are not putting the balls back into the bin after each draw, i.e., you are sampling the balls *without* replacement. Let T be the sum of the numbers on the balls you picked.

(a) Say $k = 1$, what is $\mathbb{E}[T]$?

(b) Find $\mathbb{E}[T]$ for general values of k.

(c) What is $\text{var}(T)$ for general values of k?

Solution:

(a) If $k = 1$, then we can think of this as picking one ball randomly from the bin. Each of the n balls is equally likely to be selected, so we have:

$$\mathbb{E}[T] = \sum_{i=1}^{n} \frac{i}{n} = \frac{n+1}{2}.$$

(b) Now, let T_i be the value of the ith ball picked. We see that:

$$\mathbb{E}[T] = \sum_{i=1}^{k} \mathbb{E}[T_i] = k\,\mathbb{E}[T_i] = \frac{k(n+1)}{2}.$$

(c) The variance is slightly harder to calculate since the T_i are not independent. We need to find $\mathbb{E}[T^2]$. Thus, we have:

$$\begin{aligned}
\mathbb{E}[T^2] = \mathbb{E}\left[\left(\sum_{i=1}^{k} T_i\right)^2\right] &= k\,\mathbb{E}[T_1^2] + k(k-1)\,\mathbb{E}[T_1T_2] \\
&= \frac{k}{n}\sum_{i=1}^{n} j^2 + \frac{k(k-1)}{n(n-1)}\sum_{i\neq j} ij \\
&= \frac{k(n+1)(2n+1)}{6} + \frac{k(k-1)}{n(n-1)}\sum_{i\neq j} ij.
\end{aligned}$$

We note that:

$$\sum_{i\neq j} ij = \sum_{i,j} ij - \sum_{i=1}^{n} i^2 = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}$$

so we have that:

$$k(k-1)\,\mathbb{E}[T_1T_2] = \frac{k(k-1)}{n(n-1)}\left(\frac{n^2(n+1)^2}{4} - \frac{n(2n+1)(n+1)}{6}\right).$$

Additionally, we have:

$$\begin{aligned}
&\text{var}(T) \\
&\quad = \mathbb{E}[T^2] - \mathbb{E}[T]^2 \\
&\quad = \frac{k(n+1)(2n+1)}{6} + \frac{k(k-1)}{n(n-1)}\left(\frac{n^2(n+1)^2}{4} - \frac{n(2n+1)(n+1)}{6}\right) \\
&\qquad - \frac{k^2(n+1)^2}{4}.
\end{aligned}$$

Simplification is not necessary.

3. Poisson Properties

(a) Suppose X and Y are independent Poisson random variables with mean λ and μ respectively. Prove that $X+Y$ has the Poisson distribution with mean $\lambda+\mu$. (This is known as **Poisson merging**.)

(b) Suppose X is an exponential random variable with mean $1/\lambda$, that is, X is a continuous random variable with density $f_X(x) = \lambda \exp(-\lambda x)$ for $x > 0$. Show that

$$\mathbb{E}(X^k) = \frac{k!}{\lambda^k}.$$

Solution:

(a) For $z \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}(X+Y=z) &= \sum_{j=0}^{z} \mathbb{P}(X=j, Y=z-j) = \sum_{j=0}^{z} \frac{\mathrm{e}^{-\lambda}\lambda^j}{j!} \frac{\mathrm{e}^{-\mu}\mu^{z-j}}{(z-j)!} \\
&= \frac{\mathrm{e}^{-(\lambda+\mu)}}{z!} \sum_{j=0}^{z} \frac{z!}{j!(z-j)!} \lambda^j \mu^{z-j} \\
&= \frac{\mathrm{e}^{-(\lambda+\mu)}}{z!} \sum_{j=0}^{z} \binom{z}{j} \lambda^j \mu^{z-j} = \frac{\mathrm{e}^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}.
\end{aligned}$$

(b) $\mathbb{E}(X^k) = \int_0^\infty x^k \lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x$. Integrating by parts, with proper limits,

$$\mathbb{E}(X^k) = \frac{k}{\lambda} \mathbb{E}(X^{k-1}).$$

Continuing, and with the base case

$$\mathbb{E}(X) = \frac{1}{\lambda},$$

we get

$$\mathbb{E}(X^k) = \frac{k!}{\lambda^k}.$$

Remark: These properties will be used extensively when we discuss the Poisson process model.

4. **Indicators & Markov's Inequality**

An **indicator random variable** is a discrete random variable defined in the following way (informally): $\mathbb{1}_A = 1$ if event A occurs, 0 otherwise. Show that:

(a) $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$.

(b) If X is a non-negative random variable, then for $c > 0$, $\mathbb{P}(X \geq c) \leq \mathbb{E}(X)/c$. (Remark: This is known as Markov's Inequality.)
[*Hint*: Consider the random variable $\mathbb{1}_{\{X \geq c\}}$.]

(c) Now suppose Y is a random variable (not necessarily non-negative). Provide an upper bound for $\mathbb{P}(Y \geq c)$ for $c > 0$.

Solution:

(a) $\mathbb{E}(\mathbb{1}_A) = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^{\mathsf{c}}) = \mathbb{P}(A)$.

(b) Consider the random variable $\mathbb{1}_{\{X \geq c\}}$. For $c > 0$, observe

$$\mathbb{1}_{\{X \geq c\}} \leq \frac{X}{c}.$$

Taking expectation will yield the answer.

(c)

$$\mathbb{P}(Y \geq c) = \mathbb{P}\big(\exp(\lambda Y) \geq \exp(\lambda c)\big) \leq \frac{\mathbb{E}(\exp(\lambda Y))}{\exp(\lambda c)}$$

(for some λ). Note: The answer to this part is not unique.

5. Generating Random Variables

Consider a continuous random variable $U \sim \text{Uniform}[0, 1]$. Let $F : \mathbb{R} \to [0, 1]$ be a strictly increasing distribution function. Show that $F^{-1}(U)$ has the cumulative distribution function (CDF) F.

Solution:

Let $Y = F^{-1}(U)$. The CDF of Y is $G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(U \leq F(y)) = F(y)$. The last equality follows from the CDF of a uniform random variable. Hence, $F^{-1}(U)$ has CDF F.

6. Auction Theory

This problem explores auction theory and is meant to be done at the same time as the lab.

In auction theory, n bidders have **valuations** which represent how much they value an item; we will make the simplifying assumption that the valuations are i.i.d. with density $f(x)$. In the first-price auction, the bidder who makes the highest bid wins the item and pays his/her bid. In the second-price auction, the bidder who makes the highest bid wins the auction, and pays an amount equal to the *second-highest* bid. A strategy for the auction is a **bidding function** $\beta(x)$, where x is the bidder's valuation. The bidding function determines how much to bid as a function of the bidder's valuation, and the goal is to find a bidding function $\beta(\cdot)$ which maximizes your expected utility (0 if you do not win, and your valuation minus the amount of money you bid if you do win).

(a) For the first-price auction, consider the following scenario: each person draws his/her valuation uniformly from the interval $(0, 1)$ (so $f(x) = 1$ for $x \in (0, 1)$). Suppose that the other bidders bid their own valuations (they use $\beta(x) = x$, the identity bidding function). Consider the case where there is only one other bidder. Your Stanford friend insists that you should always bid $\beta(x) = 1$. Your Berkeley friend tells your Stanford friend that it would be better to bid

$$\beta(x) = \frac{x}{2}.$$

Who is correct? [Do not simply compute the expected profit and state that one of the friends has a better bidding function—your job is to prove that your friend's bidding function is optimal.]

(b) Consider the same situation as the previous part, but now assume that there are n other bidders. Your Stanford friend again suggests that $\beta(x) = 1$ is the best bid. Your Berkeley friend suggests

$$\beta(x) = \frac{n}{n+1}x.$$

Who is correct this time? [Again, prove that your friend's bidding function is optimal.]

(c) Consider a second-price auction with n bidders where the bidders' valuations are i.i.d. with the exponential density (with parameter λ). Again, they use the identity bidding function, $\beta(x) = x$. What is the distribution of the price P at which the item sells?

Solution:

(a) Suppose that your valuation is x, and you choose to bid b. The probability that you win the auction is the probability that the other bidder has a valuation which is less than b, which occurs with probability b. Therefore, the expected utility is the probability that you win the auction, multiplied by $x - b$, which gives $b(x - b)$. The optimal bid b is therefore

$$\beta(x) = \frac{1}{2}x.$$

(b) Now, the probability that you win is the probability that all n other bidders have a valuation less than b, which is b^n. The expected utility is $b^n(x - b)$, and optimizing over b gives $nb^{n-1}(x - b) - b^n = 0$, or

$$\beta(x) = \frac{n}{n+1}x.$$

(c) We let $X^{(2)}$ be the second-largest bid. We can specify the distribution simply by specifying the CDF, so we aim to find $\mathbb{P}(X^{(2)} < x)$. There are two disjoint events in which $X^{(2)} < x$. The first case is when each of the $X_i < x$ and the second is when exactly $n - 1$ of the $X_i < x$ and one is greater. Concretely, we may let A be the event that each of the $X_i < x$ and B be the event that $n - 1$ of the $X_i < x$. We thus have:

$$\begin{aligned}\mathbb{P}(X^{(2)} < x) &= \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)\\ &= (1 - \mathrm{e}^{-\lambda x})^n + (1 - \mathrm{e}^{-\lambda x})^{n-1} \cdot n \cdot \mathrm{e}^{-\lambda x}\end{aligned}$$

We may take the derivative to compute the density:

$$f_{X^{(2)}} = \lambda \cdot n \cdot (n - 1) \cdot (1 - \mathrm{e}^{-\lambda x})^{n-2} \cdot \mathrm{e}^{-2\lambda x}$$