UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Discussion 1
Fall 2017

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1. **Deriving Facts from the Axioms**

   (a) Let $n \in \mathbb{Z}_{>0}$ and $A_1, \dots, A_n$ be any events. Prove the **union bound**: $\mathbb{P}(\bigcup_{i=1}^n A_i) \le \sum_{i=1}^n \mathbb{P}(A_i)$.

   (b) Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{i=1}^\infty A_i)$. [This can be viewed as a **continuity** property for probability measures.]

   (c) Let $A_1, A_2, \dots$ be a sequence of events. Prove that the union bound holds for countably many events: $\mathbb{P}(\bigcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \mathbb{P}(A_i)$.

Solution:

(a) From inclusion-exclusion, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \le \mathbb{P}(A) + \mathbb{P}(B)$. The result now follows from induction. Formally, the case of $n = 1$ is trivial and the case of $n = 2$ was proven above; let $n \ge 3$.

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \mathbb{P}\left(A_1 \cup \bigcup_{i=2}^n A_i\right) \le \mathbb{P}(A_1) + \mathbb{P}\left(\bigcup_{i=2}^n A_i\right)$$
$$\le \mathbb{P}(A_1) + \sum_{i=2}^n \mathbb{P}(A_i) = \sum_{i=1}^n \mathbb{P}(A_i).$$

(b) Write $A_1' = A_1$ and $A_i' = A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for $i \in \mathbb{N}$, $i \ge 2$. Now, the A_i' for $i \in \mathbb{Z}_{>0}$ are disjoint, and $\bigcup_{i=1}^n A_i' = \bigcup_{i=1}^n A_i = A_n$, so $\mathbb{P}(A_n) = \mathbb{P}(\bigcup_{i=1}^n A_i') = \sum_{i=1}^n \mathbb{P}(A_i')$. Hence,

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}(A_i') = \sum_{i=1}^\infty \mathbb{P}(A_i') = \mathbb{P}\left(\bigcup_{i=1}^\infty A_i'\right) = \mathbb{P}\left(\bigcup_{i=1}^\infty A_i\right),$$

by countable additivity.

Note: A function $f : \mathbb{R} \to \mathbb{R}$ is called **continuous** if for every $x \in \mathbb{R}$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, we have $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$. If we view $\bigcup_{i=1}^\infty A_i$ as "$\lim_{i\to\infty} A_i$", then the continuity property of probability says that $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n)$, which explains the name.

Note 2: Some students noticed that by inclusion-exclusion, $\mathbb{P}(\bigcup_{i=1}^\infty A_i) = \mathbb{P}(A_1) + \mathbb{P}(\bigcup_{i=2}^\infty A_i) - \mathbb{P}(A_1 \cap (\bigcup_{i=2}^\infty A_i))$, but $A_1 \cap (\bigcup_{i=2}^\infty A_i) = A_1$, so $\mathbb{P}(\bigcup_{i=1}^\infty A_i) = \mathbb{P}(\bigcup_{i=2}^\infty A_i)$; indeed, one can prove by induction that for any $n \in \mathbb{Z}_{>0}$, $\mathbb{P}(\bigcup_{i=1}^\infty A_i) = \mathbb{P}(\bigcup_{i=n}^\infty A_i)$. This is *not* sufficient to prove the problem, but it can be explained by noticing that the LHS, $\lim_{n\to\infty} \mathbb{P}(A_n)$, does not depend on finitely many events (since we are taking a limit), so "throwing away" the events $A_1, \dots, A_n$ does not change the probability.

(c) As in the previous part, define $A_1' = A_1$ and $A_i' = A_i \setminus \bigcup_{j=1}^{i-1} A_i$ for $i \in \mathbb{N}$, $i \geq 2$. Now, $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \mathbb{P}(\bigcup_{i=1}^{\infty} A_i') = \sum_{i=1}^{\infty} \mathbb{P}(A_i')$, and for all $i \in \mathbb{Z}_{>0}$ we have $\mathbb{P}(A_i') \leq \mathbb{P}(A_i)$ since $A_i' \subseteq A_i$, so $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Note: The fact we used above is that if $B \subseteq A$, then $\mathbb{P}(B) \leq \mathbb{P}(A)$; this follows because $A = B \cup (A \setminus B)$ is a disjoint union, so $\mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) \geq \mathbb{P}(B)$.

2. Balls & Bins

Let $n \in \mathbb{Z}_{>1}$. You throw n balls, one after the other, into n bins, so that each ball lands in one of the bins uniformly at random. What is an appropriate sample space to model this scenario? What is the probability that exactly one bin is empty?

Solution:

An appropriate sample space is to take $\Omega = \{1, \dots, n\}^n$, the set of n-tuples where each coordinate is a number in $\{1, \dots, n\}$. An outcome $\omega \in \Omega$ represents a scenario as follows: the first coordinate gives the label of the bin into which the first ball fell; the second coordinate gives the label of the bin into which the second ball fell; and so on.

Notice that this choice of sample space treats all of the balls as distinguishable and all of the bins as distinguishable. The reason for making this choice is that the sample space is *uniform*, that is, all outcomes have the same probability.

In contrast, if we chose a sample space corresponding to *indistinguishable balls* (and distinguishable bins), then the sample space would *not* be uniform, which makes the problem harder to analyze. The reason why the sample space is no longer uniform is that some outcomes can happen in more ways, so they have higher probabilities. Concretely, the outcome that all balls land in the first bin will have a smaller probability than the outcome that half the balls land in the first bin and the other half land in the second bin, because in the latter outcome you have the freedom to change *which* balls land in first bin (because the balls are indistinguishable).

Now, we return to our uniform sample space with distinguishable balls. The probability of each outcome is n^{-n}, so we must count how many outcomes correspond to exactly one empty bin. There are n ways to choose which bin is empty; then $n-1$ ways to choose which of the remaining bins will have two balls; then, there are $\binom{n}{2}$ ways to choose *which* two of the n balls will land in the bin with two balls; finally, there are $(n-2)!$ ways to throw the remaining $n-2$ balls into the $n-2$ other bins. Therefore, the total number of outcomes is $n(n-1)(n-2)!\binom{n}{2} = n!\binom{n}{2}$, so the desired probability is $n!\binom{n}{2}/n^n$.

3. Monty Hall Mixed Strategies

We showed that in the Monty Hall problem, it is better to switch than to stay. Now we will consider mixed strategies: that is, you choose to switch with probability $p \in [0, 1]$. What is the optimal value of p?

Solution:

Switching leads to success exactly when your initial guess of the door was *incorrect*, which has probability $2/3$; not switching leads to success exactly when your initial guess of the door was *correct*, which has probability $1/3$. So the probability of winning under the mixed strategy is $(2/3)p + (1/3)(1-p) = (1/3)p + 1/3$, which is maximized when $p = 1$.

4. **Borel-Cantelli Lemma**

   Prove the **Borel-Cantelli Lemma**: If $A_1, A_2, \ldots$ is a sequence of events with $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, then

$$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) = 0.$$

Solution:

If infinitely many of $A_1, A_2, \ldots$ occur, then at least one of $A_n, A_{n+1}, \ldots$ occurs for any $n \in \mathbb{Z}_{>0}$. So,

$$\mathbb{P}(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) \leq \mathbb{P}\left(\bigcup_{m=n}^{\infty} A_m\right) \leq \sum_{m=n}^{\infty} \mathbb{P}(A_m) \xrightarrow[n\to\infty]{} 0$$

because $\sum_{i=1}^{\infty} \mathbb{P}(A_i)$ converges. In more detail,

$$\sum_{m=n}^{\infty} \mathbb{P}(A_m) = \sum_{m=1}^{\infty} \mathbb{P}(A_m) - \sum_{m=1}^{n-1} \mathbb{P}(A_m),$$

and as $n \to \infty$, the second term converges to $\sum_{m=1}^{\infty} \mathbb{P}(A_m)$, so $\sum_{m=n}^{\infty} \mathbb{P}(A_m)$ converges to 0 as $n \to \infty$.

Note: This result is incredibly useful for proving convergence results.