UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 5
Fall 2017

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1. **Convergence of Exponentials**

   Let $X_1, X_2, \dots$ be i.i.d. Exponential(λ) random variables. Show that
   $$\frac{X_n}{\ln n} \to 0 \qquad \text{in probability as } n \to \infty.$$

   Solution:

   Fix $\varepsilon > 0$.
   $$\mathbb{P}\Big(\frac{X_n}{\ln n} \ge \varepsilon\Big) = \mathbb{P}(X_n \ge \varepsilon \ln n) = \exp(-\lambda\varepsilon \ln n) = n^{-\lambda\varepsilon} \to 0$$
   as $n \to \infty$.

2. **Exponential Bounds**

   Let $X \sim$ Exponential(λ). For $x > \lambda^{-1}$, calculate bounds on $\mathbb{P}(X \ge x)$ using Markov's Inequality, Chebyshev's Inequality, and the Chernoff Bound.

   Solution:

   Since $\mathbb{E}[X] = \lambda^{-1}$, Markov's Inequality gives
   $$\mathbb{P}(X \ge x) \le \frac{\mathbb{E}[X]}{x} = \frac{1}{\lambda x},$$
   and from $\operatorname{var} X = \lambda^{-2}$, Chebyshev's Inequality gives
   $$\begin{aligned}\mathbb{P}(X \ge x) &= \mathbb{P}(X - \lambda^{-1} \ge x - \lambda^{-1}) \le \mathbb{P}(|X - \lambda^{-1}| \ge x - \lambda^{-1}) \\ &\le \frac{\operatorname{var} X}{(x - \lambda^{-1})^2} = \frac{1}{(\lambda x - 1)^2}.\end{aligned}$$
   For the Chernoff Bound, for any $s > 0$,
   $$\mathbb{P}(X \ge x) = \mathbb{P}\big(\exp(sX) \ge \exp(sx)\big) \le \frac{M_X(s)}{\exp(sx)} = \frac{\lambda}{(\lambda - s)\exp(sx)}.$$
   We wish to optimize this bound over $s > 0$. It suffices to maximize the denominator $(\lambda - s)\exp(sx)$. Taking derivatives,
   $$-\exp(sx) + x(\lambda - s)\exp(sx) = 0,$$
   so $1 = x(\lambda - s)$, that is, $s = \lambda - x^{-1}$. Thus,
   $$\begin{aligned}\mathbb{P}(X \ge x) &\le \frac{\lambda}{(\lambda - (\lambda - x^{-1}))\exp((\lambda - x^{-1})x)} = \frac{\lambda}{x^{-1}\exp(\lambda x - 1)} \\ &= \lambda x \exp\big(-(\lambda x - 1)\big).\end{aligned}$$
   Observe that the Chernoff Bound is the only one which decreases exponentially with x, which is the true behavior: $\mathbb{P}(X \ge x) = \exp(-\lambda x)$.