UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 6
Fall 2017

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1. **Poisson Practice**

   Let $(N(t), t \geq 0)$ be a Poisson process with rate λ. Let T_k denote the time of k-th arrival, for $k \in \mathbb{N}$, and given $0 \leq s < t$, we write $N(s,t) = N(t) - N(s)$. Compute:

   (a) $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

   (b) $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

   (c) $\mathbb{E}(T_2 \mid N(2) = 1)$.

   Solution:

   (a) The event $\{N(1) + N(2,4) + N(3,5) = 0\}$ is the same as the intersection of 2 events, $\{N(1) = 0\}$ and $\{N(2,5) = 0\}$. These are independent with probabilities $\exp(-\lambda)$ and $\exp(-3\lambda)$. Hence $\mathbb{P}(N(1)+N(2,4)+N(3,5) = 0) = \exp(-4\lambda)$.

   (b) $N(1,3) = N(1,2) + N(2,3)$. We know $N(2,3)$ is independent of $N(1,2)$. Hence, $\mathbb{E}(N(1,3) \mid N(1,2) = 3) = 3 + \lambda$.

   (c) Since $N(2) = 1$, the second interarrival time T_2 hasn't lapsed yet at $t = 2$. From the memoryless property of the exponential distribution:

   $$\mathbb{E}\big(T_2 - 2 \bigm| N(2) = 1\big) = \frac{1}{\lambda}.$$

   Hence the answer is $2 + \lambda^{-1}$.

2. **Customers in a Store**

   Consider two independent Poisson processes with rates λ_1 and λ_2. Those processes measure the number of customers arriving in store 1 and 2.

   (a) What is the probability that a customer arrives in store 1 before any arrives in store 2?

   (b) What is the probability that in the first hour exactly 6 customers arrive at the two stores? (The total for both is 6.)

   (c) Given exactly 6 have arrived at the two stores, what is the probability all 6 went to store 1?

   Solution:

(a) Consider the sum of two processes which is a Poisson process with rate $\lambda_1 + \lambda_2$. You mark each customer in this process as 1 with probability $\lambda_1/(\lambda_1 + \lambda_2)$ and mark as 2 otherwise. The resulting two processes are Poisson processes of rates λ_1 and λ_2. Thus, the probability of having the first customer going to store 1 is equal to the probability of marking the first customer as 1 which is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Other solutions: The arrival times of the first customer of the two stores are $X \sim \text{Exponential}(\lambda_1)$ and $Y \sim \text{Exponential}(\lambda_2)$, respectively. Then we can compute the probability

$$\begin{aligned}\mathbb{P}(X < Y) &= \int_{y=0}^{+\infty} \lambda_2 \mathrm{e}^{-\lambda_2 y} \mathbb{P}(X < Y \mid Y = y) \, \mathrm{d}y \\ &= \int_{y=0}^{+\infty} \lambda_2 \mathrm{e}^{-\lambda_2 y} (1 - \mathrm{e}^{-\lambda_1 y}) \, \mathrm{d}y = \frac{\lambda_1}{\lambda_1 + \lambda_2}.\end{aligned}$$

(b) $\dfrac{\mathrm{e}^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^6}{6!}$.

(c) Similar to the argument of Part (a), the answer is $\left(\dfrac{\lambda_1}{\lambda_1 + \lambda_2}\right)^6$.

3. Sum-Quota Sampling

Consider the problem of estimating the mean interarrival time of a Poisson process. In what follows, recall that N_t denotes the number of arrivals by time t, where $t \geq 0$.

Sum-quota sampling is a form of sampling in which the number of samples is not fixed in advance; instead, we wait until a fixed *time* t, and take the average of the interarrival times seen so far. If we let X_i denote the ith interarrival time, then

$$\bar{X} = \frac{X_1 + \cdots + X_{N_t}}{N_t}.$$

Of course, the above quantity is not defined when $N_t = 0$, so instead we must condition on the event $\{N_t > 0\}$. Compute $\mathbb{E}[\bar{X} \mid N_t > 0]$, assuming that $(N_t,\ t \geq 0)$ is a Poisson process of rate λ.

Solution:

We proceed by conditioning on the values of N_t. Note that

$$\mathbb{P}(N_t = n \mid N_t > 0) = \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!(1 - \mathrm{e}^{-\lambda t})}, \qquad n \in \mathbb{Z}_+.$$

Now, we use the law of total probability:

$$\mathbb{E}[\bar{X} \mid N_t > 0] = \sum_{n=1}^{\infty} \mathbb{E}[N_t^{-1}(X_1 + \cdots + X_{N_t}) \mid N_t = n] \mathbb{P}(N_t = n \mid N_t > 0)$$

Conditioned on $\{N_t = n\}$, the sum $X_1 + \cdots + X_{N_t}$ is the maximum of n uniform $[0, t)$ random variables:

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\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n} \frac{tn}{n+1} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!(1-\mathrm{e}^{-\lambda t})} = \frac{\mathrm{e}^{-\lambda t}}{\lambda(1-\mathrm{e}^{-\lambda t})} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} \\
&= \frac{\mathrm{e}^{-\lambda t}}{\lambda(1-\mathrm{e}^{-\lambda t})}(\mathrm{e}^{\lambda t} - 1 - \lambda t) = \frac{1}{\lambda}\Big(1 - \frac{\lambda t \mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\Big).
\end{aligned}
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The expectation $\mathbb{E}[\bar{X} \mid N_t > 0]$ does not quite equal $1/\lambda$, which is what we want to estimate.