UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 11
Fall 2017

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1. **MMSE from Joint Density**

   Let the joint density of two random variables X and Y be

   $$f_{X,Y}(x,y) = \frac{1}{4}(2x+y)\mathbb{1}\{0 \le x \le 1\}\mathbb{1}\{0 \le y \le 2\}.$$

   First show that this is a valid joint distribution. Suppose you observe Y drawn from this joint density. Find $\text{MMSE}[X \mid Y]$.

   Solution:

   First, we make sure that the PDF integrates to 1.

   $$\int_0^2 \int_0^1 \frac{1}{4}(2x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \frac{1}{4}(1+y)\,\mathrm{d}y = 1.$$

   We find

   $$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

   for $x \in [0,1]$, $y \in [0,2]$, via

   $$f_Y(y) = \int_0^1 f(x,y)\,\mathrm{d}x = \frac{1}{4}(1+y)\mathbb{1}\{0 \le y \le 2\}.$$

   For $0 \le y \le 1$,

   $$f_{X|Y}(x \mid y) = \frac{2x+y}{1+y}\mathbb{1}\{0 \le x \le 1\}.$$

   Thus,

   $$\mathbb{E}[X \mid Y = y] = \int_0^1 x\frac{2x+y}{1+y}\,\mathrm{d}x = \frac{1}{1+y}\Big(\frac{2}{3} + \frac{1}{2}y\Big).$$

2. **MMSE for Jointly Gaussian**

   Let $\begin{bmatrix} X & Y & Z \end{bmatrix}^\mathsf{T} \sim \mathcal{N}(\mu, \Sigma)$, and

   $$\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   and

   $$\Sigma = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Find $\mathbb{E}[X \mid Y, Z]$.

Solution:

Since $\mu = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\mathsf{T}$, we have

$$\begin{aligned}\mathbb{E}[X \mid Y, Z] &= \begin{bmatrix}\mathbb{E}[XY] & \mathbb{E}[XZ]\end{bmatrix}\begin{bmatrix}\mathbb{E}[Y^2] & \mathbb{E}[YZ] \\ \mathbb{E}[YZ] & \mathbb{E}[Z^2]\end{bmatrix}^{-1}\begin{bmatrix}Y \\ Z\end{bmatrix} \\ &= \begin{bmatrix}3 & 1\end{bmatrix}\begin{bmatrix}9 & 3 \\ 3 & 2\end{bmatrix}^{-1}\begin{bmatrix}Y \\ Z\end{bmatrix} = \frac{1}{3}Y.\end{aligned}$$

3. Recursive JG MMSE

Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Define

$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

(a) What is the distribution of X_n, where n is a positive integer?

(b) Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.

(c) Find u so that the distribution of X_n is the same for all $n \in \mathbb{N}$.

Solution:

(a) First, we find X_n as a function of X_0 and $(V_n)_{n\in\mathbb{N}}$.

$$\begin{aligned}X_1 &= aX_0 + V_0 \\ X_2 &= aX_1 + V_1 = a^2X_0 + aV_0 + V_1 \\ X_3 &= aX_2 + V_2 = a^3X_0 + a^2V_0 + aV_1 + V_2.\end{aligned}$$

Thus, if we proceed doing this recursively, we find that

$$X_n = a^nX_0 + \sum_{i=0}^{n-1} a^iV_{n-1-i}.$$

Since X_0 and $(V_n)_{n\in\mathbb{N}}$ are independent Gaussian random variables, X_n is also Gaussian, so we need to find the mean and variance. X_0 and $(V_n)_{n\in\mathbb{N}}$ are zero-mean so

$$\mathbb{E}(X_n) = 0.$$

We know that

$$\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}.$$

Thus,

$$\operatorname{var} X_n = a^{2n}\operatorname{var} X_0 + \sum_{i=0}^{n-1} a^{2i}\operatorname{var} V_{n-1-i} = a^{2n}u^2 + \frac{1-a^{2n}}{1-a^2}\sigma^2.$$

Hence,

$$X_n \sim \mathcal{N}\Big(0, a^{2n}u^2 + \frac{1-a^{2n}}{1-a^2}\sigma^2\Big).$$

(b) Similarly, by a shift of index

$$X_{n+m} = a^m X_n + \sum_{i=0}^{m-1} a^i V_{n+m-1-i}.$$

Now suppose that we have zero-mean random variables X, Y, and Z where $X = aY + Z$ and Y and Z are independent, then

$$\mathrm{LLSE}[X \mid Y] = aY.$$

(Why?) Now since the random variables are jointly Gaussian, the MMSE is actually linear. Furthermore, X_n is independent of $\sum_{i=0}^{m-1} a^i V_{n+m-1-i}$. Thus,

$$\mathbb{E}(X_{n+m} \mid X_n) = a^m X_n.$$

(c) This is equivalent to X_1 having the same variance as X_0. Thus,

$$a^2 u^2 + \sigma^2 = u^2.$$

Thus,

$$u^2 = \frac{\sigma^2}{1 - a^2}.$$