UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 6

Fall 2017

Self-Graded Scores Due: 5 PM, Monday, October 16, 2017
Submit your self-graded scores via the Google form:
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1. **Spatial Poisson Process**

   A two-dimensional Poisson process of rate $\lambda > 0$ is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA, and (ii) the number of special points in non-overlapping regions is independent. For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance between two points (x_1, y_1) and (x_2, y_2) is defined as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$). Show that:

   (a) $\mathbb{P}(X > t) = \exp(-\lambda \pi t^2)$ for $t > 0$.

   (b) $\mathbb{E}[X] = \dfrac{1}{2\sqrt{\lambda}}$.

   Solution:

   (a) Given an arbitrary location, $X > t$ if and only if there are no special points in the circle of radius t around the given point. The expected number in that circle is $\lambda \pi t^2$, and since the number in that circle is Poisson with expected value $\lambda \pi t^2$, the probability that number is 0 is $\exp(-\lambda \pi t^2)$. Thus $\mathbb{P}(X > t) = \exp(-\lambda \pi t^2)$.

   (b) Since $X \geq 0$, we have

   $$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda \pi t^2}\,\mathrm{d}t.$$

   We can look this up in a table of integrals, or recognize its resemblance to the Gaussian PDF. If we define $\sigma^2 = 1/(2\pi\lambda)$, the above integral is

   $$\mathbb{E}[X] = \sigma\sqrt{2\pi} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{t^2}{2\sigma^2}\Big)\,\mathrm{d}t = \frac{\sigma\sqrt{2\pi}}{2} = \frac{1}{2\sqrt{\lambda}}.$$

2. **Running Sum of a Markov Chain**

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with two states, -1 and 1, and transition probabilities $P(-1,1) = P(1,-1) = a$ for $a \in (0,1)$. Define

$$Y_n = X_0 + X_1 + \cdots + X_n.$$

Is $(Y_n)_{n\in\mathbb{N}}$ a Markov chain? Prove or disprove.

Solution:

If $a \neq 1/2$, $(Y_n)_{n\in\mathbb{N}}$ is not a Markov chain. Consider the following probability:

$$\mathbb{P}(Y_4 = 3 \mid Y_2 = 1, Y_3 = 2) = \mathbb{P}(X_4 = 1 \mid X_3 = 1) = 1 - a.$$

On the other hand,

$$\mathbb{P}(Y_4 = 3 \mid Y_2 = 3, Y_3 = 2) = \mathbb{P}(X_4 = 1 \mid X_3 = -1) = a.$$

So the distribution of Y_4 given the past is not only dependent on Y_3, which shows that Markov property does not hold.

However, when $a = 1/2$, $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain. This is because when $a = 1/2$, then $(X_n)_{n\in\mathbb{N}}$ is i.i.d. with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Then, for any positive integer n,

$$Y_n = \begin{cases} Y_{n-1} + 1, & \text{with probability } 1/2, \\ Y_{n-1} - 1, & \text{with probability } 1/2. \end{cases}$$

Then we know that given Y_{n-1}, Y_n is conditionally independent of all the previous states $Y_0, \ldots, Y_{n-2}$. Then $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain.

3. **Umbrellas**

A professor has n umbrellas, for some positive integer n. Every morning, she commutes from her home to her office, and every night she commutes from her office back home. On every commute, if it is raining outside, she takes an umbrella (if there is at least one umbrella at her starting location); otherwise she does not take any umbrellas. Assume that on each commute, it rains with probability $p \in (0,1)$ independently of all other times. Give the state space and transition probabilities for the Markov chain which corresponds to the number of umbrellas she has at her current location.

Solution:

The state space is $\mathcal{X} := \{0, \ldots, n\}$. Suppose she is in state $i \in \mathcal{X}$. If it does not rain, she will end up at state $n - i$; otherwise, if it rains she will end up at state $\max(n - i + 1, n)$. Thus, for $i \in \mathcal{X} \setminus \{0\}$,

$$P(i, n-i) = 1 - p,$$
$$P(i, n-i+1) = p,$$

and $P(0, n) = 1$.

4. **Markov Chain Practice**

Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are $P(0,1) = P(0,2) = 1/2$, $P(1,0) = P(1,1) = 1/2$, and $P(2,0) = 2/3$, $P(2,2) = 1/3$.

(a) Classify the states in the chain. Is this chain periodic or aperiodic?

(b) In the long run, what fraction of time does the chain spend in state 1?

(c) Suppose that X_0 is chosen according to the steady state distribution. What is $\mathbb{P}(X_0 = 0 \mid X_2 = 2)$?

(d) Suppose that $X_0 = 0$, and let T denote the first time by which the process has visited all the states. Find $\mathbb{E}[T]$.

Solution:

(a) The Markov chain has one recurrent, aperiodic class.

(b) By solving $\pi P = \pi$, we have

$$\pi = \frac{1}{11}\begin{bmatrix}4 & 4 & 3\end{bmatrix}.$$

Thus, $\pi(1) = 4/11$.

(c) By the definition of conditional probability,

$$\begin{aligned}\mathbb{P}(X_0 = 2 \mid X_2 = 2) &= \frac{\mathbb{P}(X_0 = 2, X_2 = 2)}{\mathbb{P}(X_2 = 2)} \\ &= \frac{\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 2)}{\mathbb{P}(X_2 = 2)},\end{aligned}$$

where we exploited the structure of the Markov chain in the last equality. Note that $\mathbb{P}(X_2 = 2) = \mathbb{P}(X_0 = 2)$ as X_0 is chosen according to π. Thus,

$$\frac{\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 2)}{\mathbb{P}(X_2 = 2)} = \frac{\pi(0)\cdot(1/2)\cdot(1/3)}{\pi(2)} = \frac{2}{9}.$$

(d) Conditioning on the first move, $\mathbb{E}[T] = .5\,\mathbb{E}[T \mid X_1 = 1] + .5\,\mathbb{E}[T \mid X_1 = 2]$. Consider the first conditional expectation term. As the chain already visited state 0 and 1, this quantity is equal to the hitting time to state 2 starting from state 1 plus one. The hitting time from 0 to 2 can be found by solving:

$$\begin{aligned}a_2 &= 0 \\ a_0 &= 1 + .5a_2 + .5a_1 = 1 + .5a_1 \\ a_1 &= 1 + .5a_1 + .5a_0.\end{aligned}$$

Thus, $a_0 = 4$ and $a_1 = 6$, and hence $\mathbb{E}[T \mid X_1 = 1] = a_1 + 1 = 7$. Similarly, you can find $\mathbb{E}[T \mid X_1 = 2] = 6$, giving the final answer $13/2$.

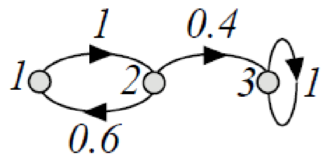

Figure 1: A Markov chain.

5. **Before Absorption**

   Consider the Markov chain in Figure 1. Suppose that $X(0) = 1$. Calculate the expected number of times that the chain is in state 1 before being absorbed in state 3. ($X(0) = 1$ is included in this number.)

   Solution:

   Let
   $$\gamma(i) = \mathbb{E}\Big(\sum_{n=0}^{T_3} g\big(X(n)\big) \Bigm| X(0) = i\Big),$$
   where $g(X(n)) = \mathbb{1}\{X(n) = 1\}$ and T_3 is the hitting time of state 3. We are interested in computing $\gamma(1)$. The first-step equations are:
   $$\gamma(1) = 1 + \gamma(2)$$
   $$\gamma(2) = 0.6\gamma(1)$$
   Thus, $\gamma(1) = 1/0.4 = 2.5$.

6. **Random Walk on an Undirected Graph**

   Consider a random walk on an undirected connected finite graph (that is, define a Markov chain where the state space is the set of vertices of the graph, and at each time step, transition to a vertex chosen uniformly at random out of the neighborhood of the current vertex). What is the stationary distribution?

   Solution:

   Let $\mathcal{X}$ be the state space. The stationary distribution is
   $$\pi(v) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')}, \qquad v \in \mathcal{X}.$$
   Clearly, π is a valid probability distribution. We check that the chain is reversible. Note that if u and v are neighbors, then
   $$\pi(u)P(u,v) = \frac{\deg(u)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(u)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$
   Also,
   $$\pi(v)P(v,u) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(v)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$
   So, $\pi(u)P(u,v) = \pi(v)P(v,u)$ if u and v are neighbors. If u and v are not neighbors, then $P(u,v) = P(v,u) = 0$, so the equation holds in this case as well. The chain is reversible and so π is stationary.