UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 3
Fall 2017

Self-Graded Scores Due: N/A
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Make sure you use your **SORTABLE NAME** on CalCentral.

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1. **Conditional Distribution of a Poisson Random Variable with Exponentially Distributed Parameter**

   Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, having an exponential density with parameter $\theta > 0$.

   (a) What is the distribution of X?

   (b) Determine the conditional density of λ given $X = k$, where $k \in \mathbb{N}$.

   Solution:

   (a) The PDF of λ is: $f(\lambda) = \theta \exp(-\theta\lambda)\mathbb{1}\{\lambda > 0\}$.
   The PMF of X conditioned on λ is

   $$\mathbb{P}(X = k \mid \lambda) = \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}, \qquad k \in \mathbb{N}.$$

   Applying the total law of probability yields, for $k \in \mathbb{N}$,

   $$\mathbb{P}(X = k) = \int_0^\infty \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\theta \exp(-\theta\lambda)\,\mathrm{d}\lambda = \frac{\theta}{(1+\theta)^{k+1}}.$$

   (b)

   $$f(\lambda \mid X = k) = \frac{\mathbb{P}(X = k \mid \lambda) f(\lambda)}{\mathbb{P}(X = k)} = \frac{\mathrm{e}^{-(1+\theta)\lambda}\lambda^k(1+\theta)^{k+1}}{k!}, \qquad \lambda > 0.$$

   Remember here that θ is fixed and λ is the argument. You should check that the integral of this over $[0, \infty)$ is 1.

2. **Records**

   Let n be a positive integer and $X_1, X_2, \dots, X_n$ be a sequence of i.i.d. continuous random variable with common probability density f_X. For any integer $2 \leq k \leq n$, define X_k as a record-to-date of the sequence if $X_k > X_i$ for all $i = 1, \dots, k-1$. (X_1 is automatically a record-to-date.)

(a) Find the probability that X_2 is a record-to-date.
Hint: You should be able to do it without rigorous computation.

(b) Find the probability that X_n is a record-to-date.

(c) Find the expected number of records-to-date that occur over the first n trials (Hint: Use indicator functions.) Compute this when $n \to \infty$.

Solution:

(a) X_2 is record-to-date with probability $1/2$. The reason is that X_1 and X_2 are i.i.d., so either one is larger than other with probability $1/2$. This uses the fact that they are equal with probability 0, since they have a density.

(b) Now, by the same symmetry argument, each X_i for $i = 1, \dots, n$ is equally likely to the the largest, so that each is largest with probability $1/n$. Since X_n is the record-to-date if it is the largest among $X_1, \dots, X_n$, it is a record with probability $1/n$.

(c) For $i = 1, \dots, n$, let $\mathbb{1}_i$ be 1 if X_i is a record-to-date, 0 otherwise. Thus $\mathbb{E}(\mathbb{1}_i)$ is the expected value of the number of records-to-date on trial i. Thus,

$$\mathbb{E}(\mathbb{1}_i) = \mathbb{P}(\mathbb{1}_i = 1) = \frac{1}{i}.$$

Thus,

$$\mathbb{E}(\text{records to date in } n \text{ trials}) = \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_i) = \sum_{i=1}^{n} \frac{1}{i}.$$

This is a harmonic series, and if $n \to \infty$, it diverges to ∞.

3. **First Exponential**

(a) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, X and Y independent, compute $\mathbb{P}(X < Y)$.

(b) For any integer $n \geq 2$, if $X_1, \dots, X_n$ are exponentially distributed with parameters $\lambda_1, \dots, \lambda_n$, show that (for $i = 1, \dots, n$)

$$\mathbb{P}\Big(X_i = \min_{k=1,\dots,n} X_k\Big) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}.$$

Solution:

(a)

$$\mathbb{P}(X < Y) = \int_{y=0}^{\infty} \mathbb{P}(X < y \mid Y = y) f_Y(y) \, \mathrm{d}y$$

Since X and Y are independent, $\mathbb{P}(X < y \mid Y = y) = \mathbb{P}(X < y)$, and since $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, $\mathbb{P}(X < y) = 1 - \mathrm{e}^{-\lambda y}$ and $f_Y(y) = \mu \mathrm{e}^{-\mu y}$. Plugging in, we get

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}.$$

(b) We need to verify a nice fact about a collection of independent exponentially distributed random variable. Given a collection of random variables, $Y_i \sim \text{Exponential}(\mu_i)$, for $i = 1, \ldots, n$, $Y := \min\{Y_1, \ldots, Y_n\}$ is exponentially distributed with parameter $\sum_{i=1}^{n} \mu_i$. This can be easily checked by considering the CDF of Y. (Try it out!)

Now, $\mathbb{P}(X_i = \min_{i=k=1,\ldots,n} X_k) = \mathbb{P}(X_i \leq \min_{k=1,\ldots,n,k\neq i} X_k)$. From the previous argument, $\min_{k=1,\ldots,n,k\neq i} X_k \sim \text{Exponential}(\sum_{j=1}^{n} \lambda_j - \lambda_i)$. Using the result of Part (a), the claim follows.

4. **Gaussian Densities**

(a) Let $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

(b) Show that all linear combinations of i.i.d. finitely many Gaussians are Gaussian.

Solution:

(a) Let $Z = X_1 + X_2$. To make the principles stand out, without loss of generality, assume $\sigma_1 = \sigma_2 = 1$. Then,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(z - x)\,\mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\Big\{-\Big(x^2 - xz + \frac{z^2}{2}\Big)\Big\}\,\mathrm{d}x \\
&= \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-z^2/4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-(x-z/2)^2}\,\mathrm{d}x = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-z^2/4}
\end{aligned}$$

The last integral is 1, and it follows from transforming the integration from Cartesian coordinates to polar coordinates.

We have proved the result for $\sigma_1 = \sigma_2 = 1$. A simple extension will conclude the claim.

(b) Given $X \sim \mathcal{N}(\mu, \sigma^2)$, one can easily verify that $aX \sim \mathcal{N}(a\mu, a^2\sigma^2)$. Also the results in Part (a) can be generalized as follows: if $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, $aX_1 + bX_2 \sim \mathcal{N}(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$. For a finite collection, using simple induction, we get the result.

5. **Triangle Density**

Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0), (1,0), (0,1)$.

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$.

(e) Find $\mathbb{E}[X]$.

Solution:

(a) Note that the joint PDF is uniform on the triangle, which has area 1/2, so for all valid x, y, $f_{X,Y}(x, y) = 2$.

(a) In order to find the marginal PDF, we integrate out:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x = \int_0^{1-y} 2\,\mathrm{d}x = 2(1 - y)$$

where $0 \le y \le 1$.

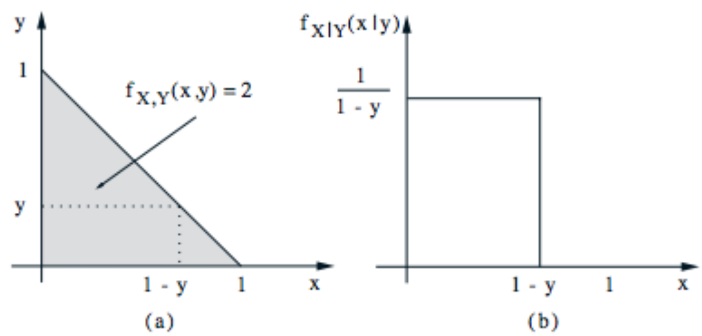

Figure 1: Joint density of (X, Y) (a) and the conditional density $X \mid Y$ (b). Image taken from Bertsekas and Tsitsiklis.

(c) The conditional density is given by, for $0 \le y \le 1$,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, \qquad 0 \le x \le 1 - y.$$

This should agree with your intuition that given $Y = y$, X should be uniform.

(d) We use the tower property: $\mathbb{E}[\mathbb{E}(X \mid Y)] = \mathbb{E}[X]$. Note that for $0 \le y \le 1$,

$$\begin{aligned}\mathbb{E}[X \mid Y = y] &= \int_0^{1-y} x f_{X|Y}(x \mid y)\,\mathrm{d}x = \int_0^{1-y} x \frac{1}{1-y}\,\mathrm{d}x \\ &= \frac{1}{1-y}\left[\frac{(1-y)^2}{2}\right] = \frac{1-y}{2}.\end{aligned}$$

Thus, we have:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}(X \mid Y)] = \int_0^1 \mathbb{E}[X \mid Y = y] f_Y(y)\,\mathrm{d}y.$$

Note that we are simply trying to find $\mathbb{E}[X]$ in terms of $\mathbb{E}[Y]$, so there is no need to expand out $f_Y(y)$, so we have:

$$\begin{aligned}\mathbb{E}[X] &= \int_0^1 \mathbb{E}[X \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_0^1 \frac{1-y}{2} f_Y(y)\,\mathrm{d}y \\ &= \frac{1}{2} - \frac{1}{2}\int_0^1 y f_Y(y)\,\mathrm{d}y = \frac{1 - \mathbb{E}[Y]}{2}.\end{aligned}$$

(e) Finally, we note that by symmetry, $\mathbb{E}[X]$ should be equal to $\mathbb{E}[Y]$, so we have

$$\mathbb{E}[X] = \frac{1-\mathbb{E}[X]}{2},$$

and

$$\mathbb{E}[X] = \frac{1}{3}.$$

6. **Transform Practice**

Consider a random variable Z with transform:

$$M_Z(s) = \frac{a-3s}{s^2-6s+8}$$

(a) Find the numerical value for the parameter a.

(b) Find $\mathbb{P}(Z \geq 0.5)$.

(c) Find $\mathbb{E}[Z]$ by using the probability distribution of Z.

(d) Find $\mathbb{E}[Z]$ by using the transform of Z and without explicitly using the probability distribution of Z.

(e) Find $\text{var}(Z)$ by using the probability distribution of Z.

(f) Find $\text{var}(Z)$ by using the transform of Z and without explicity using the probability distribution of Z.

Solution:

(a) By definition, we know that $M_Z(s) = \mathbb{E}[e^{sZ}]$. Thus, we know the following must be true:

$$M_Z(0) = \mathbb{E}[e^{0Z}] = 1 = \frac{a}{8}$$

It follows that $a = 8$.

(b) We should find the PDF of Z. Expanding the transform, we write

$$M_Z(s) = \frac{A}{s-4} + \frac{B}{s-2}.$$

We may solve for A, B to see that $A = -2$ and $B = -1$. Thus:

$$M_Z(s) = \frac{1}{2}\Big(\frac{4}{4-s} + \frac{2}{2-s}\Big)$$

It follows that $f_Z(z) = (2\mathrm{e}^{-4z} + \mathrm{e}^{-2z})\mathbb{1}\{z \geq 0\}$. We may thus integrate to see that:

$$\int_{0.5}^{\infty} f_Z(z)\,\mathrm{d}z = \frac{1}{2}(\mathrm{e}^{-2} + \mathrm{e}^{-1})$$

(c) From the expectation of exponential random variables, we see that:

$$\mathbb{E}[Z] = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8}$$

(d)
$$\mathbb{E}[Z] = \frac{\mathrm{d}}{\mathrm{d}s} M_Z(s)\Big|_{s=0} = \frac{2}{(4-s)^2} + \frac{1}{(2-s)^2}\Big|_{s=0} = \frac{3}{8}.$$

(e) We see that:
$$\mathbb{E}[Z^2] = \int_0^\infty z^2 f_Z(z)\,\mathrm{d}z = \frac{5}{16}$$
and thus
$$\mathrm{var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \frac{11}{64}.$$

(f) Note that
$$\mathbb{E}[Z^2] = \frac{\mathrm{d}^2}{\mathrm{d}s^2} M_Z(2)\Big|_{s=0} = \frac{4}{(4-s)^3} + \frac{2}{(2-s)^3}\Big|_{s=0} = \frac{5}{16}.$$

Thus,
$$\mathrm{var}(Z) = \frac{11}{64}.$$