UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 12
Fall 2017

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1. **Orthogonal LLSE**

   (a) Consider zero-mean random variables X, Y, Z such that Y, Z are orthogonal. Show that $L[X \mid Y, Z] = L[X \mid Y] + L[X \mid Z]$.

   (b) Show that for any zero-mean random variables X, Y, Z it holds that:

$$L[X \mid Y, Z] = L[X \mid Y] + L\big[X \bigm| Z - L[Z \mid Y]\big]$$

   Solution:

   (a) Let $U(Y) = L[X \mid Y]$, $V(Z) = L[X \mid Z]$. X, $U(Y)$, and $V(Z)$ are all zero-mean. Observe that $V(Z)$ and Y are orthogonal. To see this, observe that Y is orthogonal to 1 (this is the statement that Y is zero-mean) and to Z, and hence to any affine function of Z (in particular, Y is orthogonal to $V(Z)$). A similar argument establishes that $U(Y)$ and Z are orthogonal as well. Now,

$$\begin{aligned}
\mathbb{E}[X - U(Y) - V(Z)] &= 0, \\
\mathbb{E}\big[\big(X - U(Y) - V(Z)\big)Y\big] &= \mathbb{E}[V(Z)Y] = 0, \\
\mathbb{E}\big[\big(X - U(Y) - V(Z)\big)Z\big] &= \mathbb{E}[U(Y)Z] = 0,
\end{aligned}$$

   since $X - U(Y)$ is orthogonal to Y and $X - V(Z)$ is orthogonal to Z. Therefore, $X - U(Y) - V(Z)$ is orthogonal to any linear function of 1, Y, and Z, and hence it is the LLSE of X given Y, Z.

   (b) Let $W = Z - L[Z \mid Y]$, so W and Y are orthogonal. From the previous part we know $L[X \mid Y] + L[X \mid W] = L[X \mid W, Y]$, so it remains to argue that $L[X \mid W, Y] = L[X \mid Y, Z]$. This is intuitively clear since (W, Y) and (Y, Z) are linear functions of each other.

2. **Noisy Guessing**

   Let X, Y, and Z be i.i.d. with the standard Gaussian distribution. Find $\mathbb{E}[X \mid X + Y, X + Z, Y - Z]$.

   Hint: Argue that the observation $Y - Z$ is redundant.

   Solution:

   Since $Y - Z = X + Y - (X + Z)$, we have

$$\mathbb{E}(X \mid X + Y, X + Z, Y - Z) = \mathbb{E}(X \mid X + Y, X + Z).$$

First, we calculate $\mathbb{E}(X \mid X+Y) = (X+Y)/2$ by symmetry. Also,

$$\mathbb{E}(X+Z \mid X+Y) = \mathbb{E}(X \mid X+Y) = \frac{X+Y}{2},$$

so the innovation is $X+Z-(X+Y)/2 = (X-Y+2Z)/2$. Thus,

$$\operatorname{cov}\Big(X, \frac{X-Y+2Z}{2}\Big) = \frac{1}{2},$$
$$\operatorname{var} \frac{X-Y+2Z}{2} = \frac{3}{2},$$

and so $\mathbb{E}(X \mid (X-Y+2Z)/2) = (X-Y+2Z)/6$. Hence,

$$\begin{aligned}\mathbb{E}(X \mid X+Y, X+Z) &= \frac{X+Y}{2} + \frac{X-Y+2Z}{6} = \frac{2}{3}X + \frac{1}{3}Y + \frac{1}{3}Z \\ &= \frac{1}{3}(X+Y+X+Z).\end{aligned}$$

3. **Joint Gaussian Probability**

Let $X \sim \mathcal{N}(1,1)$ and $Y \sim \mathcal{N}(0,1)$ be jointly Gaussian with covariance ρ. What is $\mathbb{P}(X > Y)$?

Solution:

Let $\bar{X} = X - 1$.

We can write $Y = \rho\bar{X} + \sqrt{1-\rho^2}Z$, where $Z \sim \mathcal{N}(0,1)$ is independent of $\bar{X}$. (To check that this is correct, observe that $\operatorname{cov}(\bar{X}, \rho\bar{X} + \sqrt{1-\rho^2}Z) = \rho$ and also $\operatorname{var}(\rho\bar{X} + \sqrt{1-\rho^2}Z) = \rho^2 + (1-\rho^2) = 1$ as required.)

So, $\mathbb{P}(X > Y) = \mathbb{P}(\bar{X} > Y - 1) = \mathbb{P}((1-\rho)\bar{X} - \sqrt{1-\rho^2}Z > -1)$. But

$$(1-\rho)\bar{X} - \sqrt{1-\rho^2}Z \sim \mathcal{N}\big(0, (1-\rho)^2 + 1 - \rho^2\big) = \mathcal{N}\big(0, 2(1-\rho)\big)$$

by independence so

$$\mathbb{P}(X > Y) = \mathbb{P}\Big(\mathcal{N}(0,1) > -\frac{1}{\sqrt{2(1-\rho)}}\Big) = \Phi\Big(\frac{1}{\sqrt{2(1-\rho)}}\Big).$$