UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

**Problem Set 5**
Fall 2017

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**Self-Graded Scores Due:** 5 PM, Monday, October 9, 2017
Submit your self-graded scores via the Google form:
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1. **Basketball**

   Michael misses shots with probability $1/4$, independently of other shots.

   (a) What is the expected number of shots that Michael will make before he misses three times?

   (b) What is the probability that the second and third time Michael makes a shot will occur when he takes his eigth and ninth shots, respectively?

   (c) What is the probability that Michael misses two shots in a row before he makes two shots in a row?

   **Solution:**

   (a) Note that missed shots are given by a Bernoulli process with parameter $p = 1/4$. The expected number of shots up to the third miss is the expected value of a Pascal random variable of order three, with parameter $1/4$, which is $3 \cdot 4 = 12$. Subtracting the number of failures, we have that the expected number of shots that Michael will make is $12 - 3 = 9$.

   (b) The event of interest is the intersection of the following three independent events: $A$, which is the event that there is exactly one make in the first seven shots, $B$, the event that the eighth shot is a make, and $C$, the event that the ninth shot is a make. Thus

$$\mathbb{P}(A) = \binom{7}{1}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right),$$

$$\mathbb{P}(B) = \mathbb{P}(C) = \frac{3}{4},$$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 7\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^3.$$

   (c) Let $B$ be the event that Michael misses two shots in a row before he makes two shots in a row. We can use $M$ (missed shot) and $S$ (successful shot) to indicate shots that he has missed or made, respectively. We thus have:

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(MM \cup SMM \cup MSMM \cup SMSMM \cup \cdots) \\
&= \mathbb{P}(MM) + \mathbb{P}(SMM) + \mathbb{P}(MSMM) + \cdots \\
&= \Big(\frac{1}{4}\Big)^2 + \frac{3}{4}\Big(\frac{1}{4}\Big)^2 + \frac{1}{4}\cdot\frac{3}{4}\Big(\frac{1}{4}\Big)^2 + \cdots \\
&= \Big[\sum_{i=0}^{\infty}\Big(\frac{1}{4}\cdot\frac{3}{4}\Big)^i\Big(\frac{1}{4}\Big)^2\Big] + \Big[\sum_{i=0}^{\infty}\Big(\frac{3}{4}\Big)^{i+1}\Big(\frac{1}{4}\Big)^i\Big(\frac{1}{4}\Big)^2\Big] \\
&= \frac{(1/4)^2}{1-(1/4)(3/4)} + \frac{(3/4)(1/4)^2}{1-(1/4)(3/4)} = \frac{7}{52}.
\end{aligned}$$

**2. Bus Arrivals at Cory Hall**

Starting at time 0, the F line makes stops at Cory Hall according to a Poisson process of rate $\lambda$. Students arrive at the stop according to an independent Poisson process of rate $\mu$. Every time the bus arrives, all students waiting get on.

(a) Given that the interarrival time between bus $i-1$ and bus $i$ is $x$, where $i$ is a positive integer $\geq 2$, find the distribution for the number of students entering the $i$th bus. (Here, $x$ is a given number, not a random quantity.)

(b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

(c) Find the distribution of the number of students getting on the next bus to arrive after 11:00 AM. (You can assume that time 0 was infinitely far in the past.)

**Solution:**

(a) We note that the student arrival process is independent of the bus arrival process and thus, the number of arrivals to the student arrival process in the interval of size $x$ is a Poisson random variable with parameter $\mu x$.

(b) We consider a related problem, where we would like to find the distribution of the number of students entering the $i$th bus. In fact, as we will see, the number of students entering the $i$th bus is independent of $i$, and thus, we may consider 9:30 AM to be the arrival time of the $(i-1)$th bus for any $i > 1$, and the result will be the same. We now find the distribution of the number of students entering the $i$th bus.

We note that the student arrival process and the bus arrival process are independent Poisson processes, and we can thus consider the merged Poisson process with parameter $\lambda + \mu$. As we saw in discussion, each arrival for the combined process is a bus with probability $\lambda/(\lambda+\mu)$ and likewise each arrival for the combined process is a student with probability $\mu/(\lambda+\mu)$. The sequence of bus/student choices is an IID sequence, so starting immediately after bus $i-1$, the number of students before a bus arrival is a geometric random variable with parameter $\lambda/(\lambda+\mu)$. Thus, if we let $N_i$ give the number of students entering the $i$th bus, we see that:

$$\mathbb{P}(N_i = k) = \left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \frac{\lambda}{\lambda+\mu}.$$

Note that this is independent of $i$, so this also gives the number of students that will get on the next bus given that there was an arrival at 9:30 AM. What this problem is essentially telling us is that given the process started infinitely far in the past, if we pick some random time $t$, then the number of students arriving after $t$, but before the next bus has the same distribution as $N_i$.

(c) This is a slight variation on random incidence. We note that in part (b), we found the number of future student arrivals before the next bus. What we are looking for is the sum of the number of students waiting at 11:00 AM and the number of future student arrivals before the next bus. We see that by definition of the Poisson process, these are IID, so we may convolve their PMFs. Now, we find the PMF of the number of students waiting, $W$.

Note that since time 0 was infinitely far in the past, we consider $\{Z_i, i \in \mathbb{Z}\}$, the doubly infinite IID sequence of bus/student choices where $Z_i = 0$ if it the $i$th combined arrival is a bus, and $Z_i = 1$ if it was a student. We may index this sequence so that $-1$ is the index of the most recent combined arrival prior to 11:00 AM. We then see that if $Z_{-1} = 0$, then no customers are waiting at 11:00 AM. Also, if $Z_{-n} = 0$ and $Z_{-m} = 1$ for positive integers $m < n$, then $n$ customers are waiting. Since the $Z_i$ are IID, the distribution is also geometric with parameter $\lambda/(\lambda+\mu)$. In other words:

$$\mathbb{P}(W = n) = \left(\frac{\mu}{\lambda+\mu}\right)^n \cdot \frac{\lambda}{\lambda+\mu}.$$

Returning to the original problem, we now want:

$$\begin{aligned}\mathbb{P}(W + N_i = n) &= \sum_{m=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^m \cdot \frac{\lambda}{\lambda+\mu} \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{n-m} \cdot \frac{\lambda}{\lambda+\mu} \\ &= (n+1)\left(\frac{\mu}{\lambda+\mu}\right)^n \left(\frac{\lambda}{\lambda+\mu}\right)^2.\end{aligned}$$

**3. Poisson Process Warm-Up**

Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable $S_i$ denote the time of the $i$th arrival, where $i$ is a positive integer.

(a) Given $S_3 = s$, where $s > 0$, find the joint distribution of $S_1$ and $S_2$.

(b) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

(c) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

(d) Give an interpretation, in terms of a Poisson process with rate $\lambda$, of the following fact:

*If $N$ is a geometric random variable with parameter $p$, and $(X_i)_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with parameter $\lambda$, then $X_1 + \cdots + X_N$ has the exponential distribution with parameter $\lambda p$.*

**Solution:**

(a) We know the distribution of sum of IID exponential random variables is Erlang. So, since the inter-arrival times of Poisson process are exponentially distributed we have

$$f_{S_i}(s) = \frac{s^{i-1}\mathrm{e}^{-s}}{(i-1)!}\mathbb{1}\{s \ge 0\}.$$

$$\begin{aligned} f_{S_1,S_2|S_3}(s_1, s_2 \mid S_3 = s) &= \frac{f_{S_1,S_2,S_3}(s_1, s_2, s)}{f_{S_3}(s)} \\ &= \frac{\mathrm{e}^{-s_1}\mathrm{e}^{-(s_2-s_1)}\mathrm{e}^{-(s-s_2)}}{s^2\mathrm{e}^{-s}/2!}\mathbb{1}\{0 \le s_1 \le s_2 \le s\} \\ &= \frac{2}{s^2}\mathbb{1}\{0 \le s_1 \le s_2 \le s\}. \end{aligned}$$

Thus, $S_1$ and $S_2$ are uniformly distributed on the feasible region $\{0 \le s_1 \le s_2 \le s\}$.

(b) By part (a), $S_2$ is the maximum of two uniform random variables between $0$ and $s$. Thus, if $0 \le x \le s$,

$$F_{S_2|S_3=s}(x) = \mathbb{P}(S_2 \le x \mid S_3 = s) = \left(\frac{x}{s}\right)^2$$

and

$$f_{S_2|S_3=s}(x) = \frac{2x}{s^2}\mathbb{1}\{0 \le x \le s\}.$$

Finally,

$$\mathbb{E}[S_2 \mid S_3 = s] = \int_0^s \frac{2x^2}{s^2}\,\mathrm{d}x = \frac{2s}{3}.$$

(c) By the memoryless property, $\mathbb{E}[S_3 \mid N_1 = 2] = 1 + \mathbb{E}[S_1] = 2$.

(d) Consider a Poisson process with rate $\lambda$ and split the process by keeping each arrival with probability $p$, independently of the other arrivals. In the original process, the inter-arrival times are IID exponential random variables with parameter $\lambda$, and $X_1 + \cdots + X_N$ represents the amount of time until the first arrival we keep. By Poisson splitting, we know that the split process is a Poisson process with rate $\lambda$, and so the time until the first arrival is an exponential random variable with parameter $\lambda p$.

4. **Illegal U-Turns**

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate $\lambda$. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let $N$ be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let $n$ be a positive integer $\geq 2$. Find the conditional expectation of the time elapsed between police cars $n-1$ and $n$, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.

**Solution:**

(a) The random variable $N$ is equal to the number of successive interarrival intervals that are smaller than $\tau$. Interarrival intervals are independent and each one is smaller than $\tau$ with probability $1 - e^{-\lambda\tau}$ . So $\mathbb{P}(N = k) = e^{-\lambda\tau}(1-e^{-\lambda\tau})^k$, and $N$ is a shifted geometric random variable with parameter $p = e^{-\lambda\tau}$ that starts from 0. Thus, $\mathbb{E}[N] = 1/p - 1 = e^{\lambda\tau} - 1$.

(b) Let $T_n$ be the $n$th interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and $n$ is less than or equal to $\tau$. So we want to compute

$$\mathbb{E}[T_n \mid T_n < \tau] = \frac{\int_0^\tau t\lambda e^{-\lambda t}\,dt}{\int_0^\tau \lambda e^{-\lambda t}\,dt}.$$

Using integration by part for the integral in the numerator, we find that the answer is

$$= \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

(c) You make the U-turn at time $T = T_1 + T_2 + \cdots + T_N + \tau$ and $T_i \leq \tau$ for $i \in \{1, \ldots, N\}$. Then, using Parts (a) and (b),

$$\begin{aligned}
\mathbb{E}[T] &= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n)\,\mathbb{E}[T_1 + \cdots + T_N \mid N = n] \\
&= \tau + \sum_{n=0}^{\infty} \mathbb{P}(N = n) n\,\mathbb{E}[T_i \mid T_i \leq \tau] \\
&= \tau + (e^{\lambda\tau} - 1) \times \frac{1/\lambda - (\tau + 1/\lambda)e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.
\end{aligned}$$

**5. Expected Squared Arrival Times**

Let $(N(t), t \geq 0)$ be a Poisson process with arrival instants $(T_n, n \in \mathbb{N})$, where $0 < T_1 < T_2 < \cdots$. Find $\mathbb{E}(\sum_{k=1}^{3} T_k^2 \mid N(1) = 3)$.

**Solution:**

Conditioned on $\{N(1) = 3\}$, the three points of the Poisson Process are independently and uniformly distributed in the interval $[0, 1]$. ($T_1$ is the least of the locations, then $T_2$ and finally $T_3$.) Thus if $U_1$, $U_2$ and $U_3$ are independent uniform random variables within $[0, 1]$, we can write:

$$\mathbb{E}\Big(\sum_{k=1}^{3} T_k^2 \,\Big|\, N(1) = 3\Big) = \mathbb{E}\Big(\sum_{k=1}^{3} U_k^2\Big)$$

Now,

$$\mathbb{E}(U_i^2) = \frac{1}{3}$$

for $i = 1, 2, 3$. So, $\mathbb{E}(\sum_{k=1}^{3} T_k^2 \mid N(1) = 3) = 1$.

Note: One can also find the result by successive integration with proper limit points. Although $T_1$, $T_2$, $T_3$ are ordered random variable, it is not used in this problem as we are interested in the sum of expectations.