UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 4
Fall 2017

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1. **Uncorrelated & Independent**

   (a) If X and Y are uncorrelated, $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$.

   (b) If $X_1, \ldots, X_n$ are uncorrelated, $\text{var}(X_1 + \cdots + X_n) = \sum_{i=1}^{n} \text{var}(X_i)$.

   (c) Show that independent random variables are uncorrelated.

   (d) Find an example, where a pair of random variables are uncorrelated but not independent.

   Solution:

   (a) As X and Y are uncorrelated, $\text{cov}(X,Y) = 0$, hence, $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y) = \text{var}(X) + \text{var}(Y)$.

   (b) Observe the fact that if $X_1, \ldots, X_n$ are uncorrelated, $X_1 + \cdots + X_{n-1}$ is uncorrelated with X_n. Hence, $\text{var}(X_1 + \cdots + X_n) = \text{var}(X_1 + \cdots + X_{n-1}) + \text{var}(X_n)$. Iteratively, further applying this on $X_1, \ldots, X_{n-1}$ and so on, we get the result.

   (c) Since X and Y are independent, $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$, for any functions f, g. Now pick $f(X) = X - \mathbb{E}(X)$ and $g(Y) = Y - \mathbb{E}(Y)$. We have

   $$\text{cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0 \cdot 0 = 0.$$

   Hence they are uncorrelated.

   (d) Consider $X \sim \mathcal{N}(0,1)$ and $Y = ZX$, where $Z \in \{1, -1\}$ with probability $\{1/2, 1/2\}$ (Z is called a Rademacher random variable). Now since $\mathbb{E}(X) = 0$, $\mathbb{E}(X)\mathbb{E}(Y) = 0$. Also, $\mathbb{E}(XY) = \mathbb{E}(ZX^2) = 0$ (from the distribution of Z). So, X and Y are uncorrelated. But observe that Y is generated based on X, so Y cannot be independent of X.

2. **Second Moment Method**

   Consider a non-negative RV Y, with $\mathbb{E}(Y^2) < \infty$. Show that

   $$\mathbb{P}(Y > 0) \geq \frac{\mathbb{E}(Y)^2}{\mathbb{E}(Y^2)}.$$

   Hint: Use Cauchy-Schwarz on $Y \mathbb{1}_{\{Y>0\}}$.

   Solution:

Applying Cauchy-Schwarz on $Y\mathbb{1}_{\{Y>0\}}$,

$$\mathbb{E}(Y\mathbb{1}_{\{Y>0\}})^2 \leq \mathbb{E}(Y^2)\mathbb{E}(\mathbb{1}^2_{\{Y>0\}}) = \mathbb{E}(Y^2)\mathbb{P}(Y > 0)$$

where we use the fact that, since the indicator function is a $\{0,1\}$-valued function, squaring will not make a difference. Also, we claim that, for non-negative Y, $Y\mathbb{1}_{\{Y>0\}}$ equals Y. For $Y > 0$, $Y\mathbb{1}_{\{Y>0\}} = Y$, and for $Y = 0$, $Y\mathbb{1}_{\{Y>0\}} = 0 = Y$. Hence, the claim follows.

3. Conditioning on the Minimum of Uniforms

If X and Y are independent Uniform$[0,1]$, show that

$$\mathbb{E}(Y \mid \min\{X,Y\}) = \frac{1}{4} + \frac{3}{4}\min\{X,Y\}.$$

Solution:

We consider two cases: (i) $Y = \min\{X,Y\}$, i.e., $Y < X$, and (ii) $X = \min\{X,Y\}$, i.e., $X < Y$. Since X and Y have the same distribution, from symmetry, the occurrences of case (i) and (ii) are equiprobable with probability $1/2$. We compute $\mathbb{E}(Y \mid \min\{X,Y\})$ under these 2 cases.

Case (i): $\mathbb{E}(Y \mid \min\{X,Y\} = Y) = \mathbb{E}(Y \mid Y) = Y = \min\{X,Y\}$.

Case (ii): Since $X < Y$, $Y \sim$ Uniform$[X,1]$, hence

$$\mathbb{E}(Y \mid \min\{X,Y\} = X) = \frac{X+1}{2} = \frac{1+\min\{X,Y\}}{2}.$$

Combining everything,

$$\mathbb{E}(Y \mid \min\{X,Y\}) = \frac{1}{2}\min\{X,Y\} + \frac{1+\min\{X,Y\}}{4}.$$