1 Markov Chains with Countably Infinite State Space

(a) We observe that $\pi(1) = \frac{3}{4}\pi(1) + \frac{1}{2}\pi(1)$. Solving for $\pi(2)$, we see that $\pi(2) = \frac{\pi(1)}{2}$. We now inductively prove our inductive hypothesis $\pi(i) = \frac{\pi(1)}{i}$ for $i \geq 2$, starting from this base case. In our inductive step, we have that:

$$\begin{aligned}
\pi(i) &= \frac{i-1}{2(i-1)+2}\pi(i-1) + \frac{1}{2i+2}\pi(i) + \frac{1}{2}\pi(i+1) \\
\pi(i+1) &= 2\left(\frac{2i+1}{2i+2}\pi(i) - \frac{i-1}{2(i-1)+2}\pi(i-1)\right) \\
&= 2\left(\frac{2i+1}{2i+2}\frac{1}{i}\pi(1) - \frac{i-1}{2(i-1)+2}\frac{1}{i-1}\pi(1)\right) \\
&= 2\pi(1)\left(\frac{2i+1}{2(i+1)i} - \frac{i+1}{2i(i+1)}\right) \\
&= \frac{1}{i+1}\pi(1)
\end{aligned}$$

Since $\sum_{i=1}^{\infty}\pi(i) = \pi(1)\sum_{i=1}^{\infty}\frac{1}{i}$, the sum diverges for $\pi(1) > 0$ and is equal to 0 for $\pi(1) = 0$, and as such is not a valid probability and thus stationary distribution over the Markov chain.

(b) For $i = 1$, we have that

$$\begin{aligned}
\pi(i) = \pi(1) &= (1-\lambda)\pi(1) + \mu\pi(2) \\
&= (1-\lambda)\left(1 - \frac{\lambda}{\mu}\right) + \mu\frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \\
&= 1 - \frac{\lambda}{\mu}
\end{aligned}$$

For $i > 1$, we have that

$$\begin{aligned}
\pi(i) &= \lambda\pi(i-1) + (1-\lambda-\mu)\pi(i) + \mu\pi(i+1) \\
&= \lambda\left(\frac{\lambda}{\mu}\right)^{i-2}\left(1 - \frac{\lambda}{\mu}\right) + (1-\lambda-\mu)\left(\frac{\lambda}{\mu}\right)^{i-1}\left(1 - \frac{\lambda}{\mu}\right) + \mu\left(\frac{\lambda}{\mu}\right)^{i}\left(1 - \frac{\lambda}{\mu}\right) \\
&= \left(\frac{\lambda}{\mu}\right)^{i-2}\left(1 - \frac{\lambda}{\mu}\right)\left(\lambda + (1-\lambda-\mu)\left(\frac{\lambda}{\mu}\right) + \mu\left(\frac{\lambda}{\mu}\right)^{2}\right) \\
&= \left(\frac{\lambda}{\mu}\right)^{i-1}\left(1 - \frac{\lambda}{\mu}\right)
\end{aligned}$$

Furthermore,

$$\sum_{i=1}^{\infty}\pi(i) = \sum_{i=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{i-1}\left(1 - \frac{\lambda}{\mu}\right)$$

$$= \sum_{i=1}^{\infty} \left(\left(\frac{\lambda}{\mu}\right)^{i-1} - \left(\frac{\lambda}{\mu}\right)^{i-2} \right)$$
$$= 1$$

So that $\pi(i) = \left(\frac{\lambda}{\mu}\right)^{i-1}\left(1 - \frac{\lambda}{\mu}\right)$ is a statinoary distribution of the Markov chain.

2 Choosing Two Good Movies

(a) $\beta(S) = \frac{31}{6}$

(b) $\beta(x) = \begin{cases} 1 + \frac{1}{2}\beta(x) + \frac{1}{5}\int_{2.5}^{5} \beta(i)\mathrm{d}i, & x \le 2.5 \\ 1 + \frac{7.5-2x}{5}\int_{x}^{7.5-x} \beta(i)\mathrm{d}i + \frac{x}{5}\beta(x), & x > 2.5 \end{cases}$

3 Customers in a Store

(a) $\mathbb{P}[S_1 < S_2] = \frac{\lambda_1}{\lambda_1+\lambda 2}$, when considering the arrival times of the merged process $PP(\lambda_1 + \lambda_2)$.

(b) For the merged process $PP(\lambda_1 + \lambda_2)$, we get that $\mathbb{P}[N_1 = 6] = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^6}{6!}$.

(c)

$$\mathbb{P}[S_1 = 4 \mid S = 6] = \frac{\mathbb{P}[S_1 = 4, S = 6]}{\mathbb{P}[S = 6]}$$
$$= \frac{\frac{\lambda_1^4 e^{-\lambda_1}}{4!}\frac{\lambda_2^2 e^{-\lambda_2}}{2!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^6}{6!}}$$
$$= \binom{6}{4}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^4\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$$

4 Arrival Times of a Poisson Process

(a) From the memoryless property, we have that $\mathbb{E}[S_3 \mid N_1 = 2] = 1 + \mathbb{E}[S_1] = 1 + 1 = 2$.

(b) $f_{S_1,S_2|S_3}(s_1, s_2, s) = \frac{f_{S_1,S_2,S_3}(s_1,s_2,s)}{f_{S_3}(s)} = \frac{2}{s^2}$

(c) $\mathbb{E}[S_2 \mid S_3 = s] = \frac{2s}{3}$

5 Bus Arrivals at Cory Hall

(a) $N \sim \text{Poisson}(\mu x)$.

(b) $\mathbb{P}[N = n] = \left(\frac{\mu}{\lambda+\mu}\right)^n \frac{\lambda}{\lambda+\mu}$

(c) $\mathbb{P}[N = n] = \left(\frac{\lambda^2\mu^n}{(\lambda+\mu)^{n+2}}\right)$

6 [Bonus] Choosing Two Good Movies (cont.)