UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Midterm 1 Review

Fall 2017

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1. Compact Arrays

Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \dots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

$$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

After making the array compact, it now looks like

$$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

Let i be a fixed positive integer in $\{1, \dots, n\}$. Suppose that the ith entry of the array is non-zero (for this question, assume that the array is indexed starting from 1). After making the array compact, the ith entry has been moved to index X. Calculate $\mathbb{E}[X]$ and $\operatorname{var} X$.

Solution:

Let X_j be the indicator that the jth entry of the original array is 0, for $j \in \{1, \dots, i-1\}$. Then, the ith entry is moved backwards $\sum_{j=1}^{i-1} X_j$, positions, so

$$\mathbb{E}[X] = i - \sum_{j=1}^{i-1} \mathbb{E}[X_j] = i - \frac{i-1}{10} = \frac{9i+1}{10}.$$

The variance is also easy to compute, since the X_j are independent. Then, $\operatorname{var} X_j = (1/10)(9/10) = 9/100$, so

$$\operatorname{var} X = \operatorname{var}\Big(i - \sum_{j=1}^{i-1} X_j\Big) = \sum_{j=1}^{i-1} \operatorname{var} X_j = \frac{9(i-1)}{100}.$$

2. Graphical Density

Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

(a) Find A and sketch f_X, f_Y, and $f_{X \mid X+Y \le 3}$.

(b) Find $\mathbb{E}[X \mid Y = y]$ for $1 \le y \le 3$ and $\mathbb{E}[Y \mid X = x]$ for $1 \le x \le 4$.

(c) Find $\operatorname{cov}(X, Y)$.

Solution:

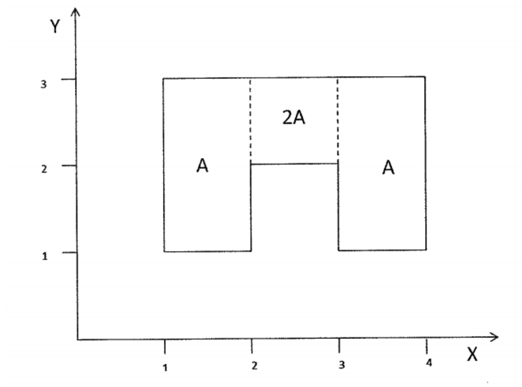

Figure 1: Joint density of X and Y.

(a) The integration over the total shown area should be 1 so $2A+2A+2A=1$ so $A=1/6$. We find the densities as follows. X is clearly uniform in intervals $(1,2)$, $(2,3)$, and $(3,4)$. The probability of X being in any of these intervals is $2A=1/3$ so

$$f_X(x) = \frac{1}{3}\mathbb{1}\{1 \leq x \leq 4\}.$$

Y is uniform in intervals $(1,2)$ and $(2,3)$. The probability of the first interval is $1/3$ and the probability of being in second one is $2/3$. So

$$f_Y(y) = \frac{1}{3}\mathbb{1}\{1 \leq y \leq 2\} + \frac{2}{3}\mathbb{1}\{2 < y \leq 3\}.$$

Finally, given that $X+Y \leq 3$, (X,Y) is chosen randomly in the triangle constructed by $(1,1),(1,2),(2,1)$. Thus,

$$f_{X|X+Y\leq 3}(x) = \int_1^{3-x} 2\,\mathrm{d}y = 2(2-x)\mathbb{1}\{1 \leq x \leq 2\}.$$

Sketching the densities is then straightforward.

(b) Given any value of $y \in [1,3]$, X has a symmetric distribution with respect to the line $x=2.5$. Thus, $\mathbb{E}[X \mid Y=y] = 2.5$ for all y, $1 \leq y \leq 3$. To calculate $\mathbb{E}[Y \mid X=x]$, we consider two cases:

(a) $2 \leq x \leq 3$, then $\mathbb{E}[Y \mid X=x] = 2.5$,

(b) $1 \leq x < 2$ or $3 < x \leq 4$, then $\mathbb{E}[Y \mid X=x] = 2$.

(c) Since $\mathbb{E}[X \mid Y=y] = \mathbb{E}[X]$ we have

$$\begin{aligned}\mathbb{E}[XY] &= \int_1^3 \mathbb{E}[XY \mid Y=y] f_Y(y)\,\mathrm{d}y = \int_1^3 y f_Y(y)\,\mathbb{E}[X]\,\mathrm{d}y \\ &= \mathbb{E}[X]\,\mathbb{E}[Y].\end{aligned}$$

So the covariance is 0.

3. **Office Hours**

In an EE 126 office hour, students bring either a difficult-to-answer question with probability $p = 0.2$ or an easy-to-answer question with probability $1-p = 0.8$. A GSI takes a random amount of time to answer a question, with this time duration being exponentially distributed with rate $\mu_D = 1$ (questions per minute)—where D denotes "difficult"—if the problem is difficult, and $\mu_E = 2$ (questions per minute)—where E denotes "easy"—if the problem is easy.

(a) You visit office hours and find a GSI answering the question of another student. Conditioned on the fact that the GSI has been busy with the other students question for $T > 0$ minutes, let q be the conditional probability that the problem is difficult. Find the value of q.

(b) Conditioned on the information above, find the expected amount of time you have to wait from the time you arrive until the other students question is answered.

(c) Now suppose two GSIs share a room and the professor is holding office hours in a different room. Both GSIs in the shared room are busy helping a student, and each has been answering questions for $T > 0$ minutes (there are no other students in the room). The amount of time the professor takes to answer a question is exponentially distributed with rate $\lambda = 6$ regardless of the difficulty. Supposing that the professor's room has two students (one of whom is being helped), in which room should you ask your question?

Solution:

(a) Let X be the random amount of time to answer a question and Z the indicator that the problem being answered is difficult. We have:

$$\mathbb{P}(X > t \mid Z = 0) = \mathrm{e}^{-\mu_E t}$$
$$\mathbb{P}(X > t \mid Z = 1) = \mathrm{e}^{-\mu_D t}$$

for $t \geq 0$. Thus, we have:

$$\mathbb{P}(X > t) = p\mathrm{e}^{-\mu_D t} + (1-p)\mathrm{e}^{-\mu_E t} = 0.2\mathrm{e}^{-t} + 0.8\mathrm{e}^{-2t}.$$

We are interested in $q = \mathbb{P}(Z = 1 \mid X > T)$. Using Bayes Rule, we have:

$$q = \mathbb{P}(Z = 1 \mid X > T) = \frac{\mathbb{P}(Z = 1, X > T)}{\mathbb{P}(X > T)} = \frac{p\mathrm{e}^{-\mu_D T}}{p\mathrm{e}^{-\mu_D T} + (1-p)\mathrm{e}^{-\mu_E T}}$$
$$= \frac{1}{1 + 4\mathrm{e}^{-T}}.$$

(b) We are interested in $\mathbb{E}[X - T \mid X > T]$. Thus, we have:

$$\begin{aligned}\mathbb{E}[X - T \mid X > T] &= \mathbb{E}[X - T \mid X > T, Z = 0]\mathbb{P}(Z = 0 \mid X > T) \\ &\quad + \mathbb{E}[X - T \mid X > T, Z = 1]\mathbb{P}(Z = 1 \mid X > T) \\ &= (1-q)\frac{1}{\mu_E} + q\frac{1}{\mu_D} = \frac{1+q}{2}.\end{aligned}$$

(c) Let X_1 and X_2 be the amount of time that the two GSIs still need to take to answer their questions. The amount time to wait for the GSIs is $\min\{X_1, X_2\}$. Let X_3 be the amount of time that the professor needs to take to finish the two students' questions. Thus,

$$\begin{aligned}
\mathbb{E}[\min\{X_1, X_2\}] &= \frac{q^2}{2\mu_D} + \frac{2q(1-q)}{\mu_D + \mu_E} + \frac{(1-q)^2}{2\mu_E} \\
&= \frac{6q^2 + 8q(1-q) + 3(1-q)^2}{12}, \\
\mathbb{E}[X_3] &= \frac{2}{\lambda} = \frac{1}{3}.
\end{aligned}$$

We equate the two equations to see:

$$6q^2 + 8q(1-q) + 3(1-q)^2 = 4.$$

Solving gives $q = \sqrt{2} - 1$ and $\mathrm{e}^{-T} = \sqrt{2}/4 = 2^{-3/2}$. Therefore, if $T < (3/2)\ln 2$, you should choose the GSI room, and otherwise choose the professor's room.

4. **Exponential Fun**

(a) Let X_1 and X_2 be i.i.d. exponential random variables with parameter λ. Compute the density of $X_1 + X_2$.

(b) Now, for a positive integer n, let $X_1, \ldots, X_n$ be i.i.d. exponential random variables with parameter λ and $S := \sum_{i=1}^{n} X_i$. The density of S is given by the n-fold convolution of the exponential distribution with itself. Compute this density.

(c) Using the above result, consider now the random sum $X_1 + \cdots + X_N$, where N is a geometric random variable with parameter p. Compute the density of $X_1 + \cdots + X_N$.

Solution:

(a) We compute the density to be, for $x \geq 0$,

$$\begin{aligned}
f_{X_1+X_2}(x) &= \int_{-\infty}^{\infty} f_{X_1}(s) f_{X_2}(x-s)\,\mathrm{d}s = \int_0^x \lambda \mathrm{e}^{-\lambda s} \cdot \lambda \mathrm{e}^{-\lambda(x-s)}\,\mathrm{d}s \\
&= \lambda^2 \mathrm{e}^{-\lambda x} \int_0^x \mathrm{d}s = \lambda^2 x \mathrm{e}^{-\lambda x}.
\end{aligned}$$

(b) Let $f_n(x)$ denote the density of $X_1 + \cdots + X_n$. We prove by induction that

$$f_n(x) = \frac{\lambda^n x^{n-1} \mathrm{e}^{-\lambda x}}{(n-1)!}, \qquad x > 0.$$

The case for $n = 1$ is trivial. We compute the convolution:

$$\begin{aligned}
f_n(x) &= \int_0^{\infty} f_{n-1}(s) f(x-s)\,\mathrm{d}s = \int_0^{\infty} \frac{\lambda^{n-1} s^{n-2} \mathrm{e}^{-\lambda s}}{(n-2)!} \lambda \mathrm{e}^{-\lambda(x-s)}\,\mathrm{d}s \\
&= \frac{\lambda^n \mathrm{e}^{-\lambda x}}{(n-2)!} \int_0^{\infty} s^{n-2}\,\mathrm{d}s = \frac{\lambda^n x^{n-1} \mathrm{e}^{-\lambda x}}{(n-1)!}.
\end{aligned}$$

(c) Let f_N denote the density of $X_1 + \cdots + X_N$. We condition on N, to obtain

$$\begin{aligned} f_N(x) &= \sum_{n=1}^{\infty} f_n(x)\mathbb{P}(N = n) = \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} \mathrm{e}^{-\lambda x}}{(n-1)!} \cdot p(1-p)^{n-1} \\ &= \lambda p \mathrm{e}^{-\lambda x} \sum_{n=1}^{\infty} \frac{(\lambda x(1-p))^{n-1}}{(n-1)!} = \lambda p \mathrm{e}^{-\lambda x} \mathrm{e}^{\lambda x(1-p)} \\ &= \lambda p \mathrm{e}^{-\lambda p x}, \qquad x > 0. \end{aligned}$$

We have obtained another exponential distribution with parameter λp.

5. **Galton-Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}[\xi] = \mu$ and $\operatorname{var} \xi = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year. Compute $\mathbb{E}[X_n]$ and $\operatorname{var} X_n$. [*Hint*: For the variance, you will need to consider the case when $\mu = 1$ separately from the case when $\mu \neq 1$.]

Solution:

Note first that $X_0 = N$, so $\mathbb{E}[X_0] = N$ and $\operatorname{var} X_0 = 0$.

Condition on X_{n-1}, the number of people in the previous year. One has

$$\begin{aligned} \mathbb{E}[X_n] &= \mathbb{E}[\mathbb{E}(X_n \mid X_{n-1})] = \mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{X_{n-1}} \xi_i \,\middle|\, X_{n-1}\right)\right] = \mathbb{E}\big[X_{n-1}\,\mathbb{E}[\xi]\big] \\ &= \mu\,\mathbb{E}[X_{n-1}]. \end{aligned}$$

By recursion, we find $\mathbb{E}[X_n] = \mu^n N$.

As we computed above, $\mathbb{E}(X_n \mid X_{n-1}) = \mu X_{n-1}$. The conditional variance is $\operatorname{var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$. Then, we have

$$\operatorname{var} X_n = \mathbb{E}[\sigma^2 X_{n-1}] + \operatorname{var}(\mu X_{n-1}) = \sigma^2 \mu^{n-1} N + \mu^2 \operatorname{var} X_{n-1}.$$

First, suppose that $\mu = 1$. Then, the recurrence simplifies to $\operatorname{var} X_n = \sigma^2 N + \operatorname{var} X_{n-1}$, which means that the variance increases linearly:

$$\operatorname{var} X_n = \sigma^2 N n.$$

For $\mu \neq 1$, the solution to the recurrence is obtained by finding a pattern after a few iterations:

$$\begin{aligned} \operatorname{var} X_n &= \sigma^2 \mu^{n-1} N + \mu^2 \operatorname{var} X_{n-1} = \sigma^2 \mu^{n-1} N + \sigma^2 \mu^n N + \mu^4 \operatorname{var} X_{n-2} \\ &= \cdots = \sigma^2 \mu^{n-1} N \sum_{k=0}^{n-1} \mu^k = \sigma^2 \mu^{n-1} N \frac{1-\mu^n}{1-\mu} \end{aligned}$$

We have used the formula for a finite geometric series.

6. **Combining Transforms**

Let X, Y, and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.

(b) Find the transform of $X + Y$.

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U.

Solution:

Note that the moment generating functions for X, Y, and Z are

$$\begin{aligned} M_X(s) &= \frac{3}{4} + \frac{1}{4}\mathrm{e}^s, \\ M_Y(s) &= \frac{3}{3-s}, \text{ for } s < 3, \text{ and} \\ M_Z(s) &= \mathrm{e}^{5(\mathrm{e}^s - 1)}. \end{aligned}$$

(a) By direct substitution of $5Z + 1$ in the expectation,

$$M_{5Z+1}(s) = \mathbb{E}[\mathrm{e}^{s(5Z+1)}] = \mathrm{e}^s\,\mathbb{E}[\mathrm{e}^{s(5Z)}] = \mathrm{e}^s M_Z(5s) = \mathrm{e}^s \mathrm{e}^{5(\mathrm{e}^{5s}-1)}.$$

(b) Since X and Y are independent,

$$M_{X+Y}(s) = M_X(s)M_Y(s) = \left(\frac{3}{4} + \frac{1}{4}\mathrm{e}^s\right)\frac{3}{3-s}, \qquad \text{for } s < 3.$$

(c) We can use the total expectation theorem to find the transform of U.

$$\begin{aligned} M_U(s) &= \mathbb{P}(X=1)\,\mathbb{E}[\mathrm{e}^{sU} \mid X=1] + \mathbb{P}(X=0)\,\mathbb{E}[\mathrm{e}^{sU} \mid X=0] \\ &= \mathbb{P}(X=1)\,\mathbb{E}[\mathrm{e}^{s(1\cdot Y + 0\cdot Z)} \mid X=1] \\ &\quad + \mathbb{P}(X=0)\,\mathbb{E}[\mathrm{e}^{s(0\cdot Y + 1\cdot Z)} \mid X=0] \\ &= \mathbb{P}(X=1)\,\mathbb{E}[\mathrm{e}^{sY} \mid X=1] + \mathbb{P}(X=0)\,\mathbb{E}[\mathrm{e}^{sZ} \mid X=0]. \end{aligned}$$

But X and Y are independent so

$$\mathbb{E}[\mathrm{e}^{sY} \mid X=1] = \mathbb{E}[\mathrm{e}^{sY}] = M_Y(s)$$

and

$$\mathbb{E}[\mathrm{e}^{sZ} \mid X=0] = \mathbb{E}[\mathrm{e}^{sZ}] = M_Z(s).$$

Therefore,

$$\begin{aligned} M_U(s) &= \frac{1}{4}M_Y(s) + \frac{3}{4}M_Z(s) \\ &= \frac{1}{4}\cdot\frac{3}{3-s} + \frac{3}{4}\cdot\mathrm{e}^{5(\mathrm{e}^s-1)} \qquad \text{for } s < 3. \end{aligned}$$