UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Discussion 14
Fall 2017

---

1. **Frogs**

Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

(a) Find the stationary distribution for $(X_t)_{t\geq 0}$.

(b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

Solution:

(a) Let the states $S = \{0,1,2,3\}$ be the number of frogs in the sun. The Markov chain has $\lambda_0 = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_3 = 3$, $\mu_2 = 2$, and $\mu_1 = 1$. Here λ_i and μ_i are the rates of jumping forward and backward from state $i \in S$, respectively. Using detailed balance, we compute the stationary distribution to be

$$\pi = \frac{1}{27}\begin{bmatrix}1 & 6 & 12 & 8\end{bmatrix}.$$

(b) The individual frogs follow independent Markov chains, each with stationary distribution

$$\pi = \frac{1}{3}\begin{bmatrix}2 & 1\end{bmatrix}.$$

The probability of being in state $i \in S$ is therefore

$$\mathbb{P}(X_t = i) = \binom{3}{i}\left(\frac{1}{3}\right)^{3-i}\left(\frac{2}{3}\right)^{i}, \qquad i \in S.$$

2. **Lazy Server**

Customers arrive at a service facility at the times of a Poisson process of rate λ. The service facility has infinite capacity. There is an infinitely powerful but lazy server who visits the service facility at the times of a Poisson process of rate μ. The Poisson process of server visits is independent of the Poisson process of arrival times of the customers. When the server visits the facility she instantaneously serves all the customers that are currently waiting in the facility and then immediately leaves (until her next visit).

Thus, for instance, at any time, any customers that are waiting in the service facility would only be those that arrived after the most recent visit of the server.

(a) Show that the system admits a well-defined stationary regime for all values of the parameters $\lambda > 0$ and $\mu > 0$.

(b) Find the mean number of customers waiting in the system at any given time in the stationary regime.

Solution:

(a) We can model the queue length as a continuous-time Markov chain on the state space $\mathcal{S} := \mathbb{N}$. For each $i \in \mathcal{S} \setminus \{0\}$, the rate at at which a customer arrives is λ, and the rate at which the server arrives is μ, so the rates are $q(i, i+1) = \lambda$, $q(i, 0) = \mu$. Together with $q(0, 1) = \lambda$, we have completely specified the chain. For $j \in \mathcal{S} \setminus \{0\}$, the balance equation reads $\pi(j-1)\lambda - \pi(j)(\lambda + \mu) = 0$, or

$$\pi(j) = \Big(\frac{\lambda}{\lambda + \mu}\Big)^j \pi(0).$$

Thus, summing over $j \in \mathcal{S} \setminus \{0\}$, we obtain

$$\sum_{j \in \mathbb{N}} \pi(j) = \sum_{j \in \mathbb{N}} \Big(\frac{\lambda}{\lambda + \mu}\Big)^j \pi(0) = \frac{1}{1 - \lambda/(\lambda + \mu)} \pi(0) = \frac{\lambda + \mu}{\mu} \pi(0),$$

so by taking $\pi(0) = \mu/(\lambda + \mu)$ we find that

$$\pi(j) := \frac{\mu}{\lambda + \mu} \Big(\frac{\lambda}{\lambda + \mu}\Big)^j, \qquad j \in \mathcal{S}$$

is a stationary distribution.

(b) If X is a random variable with $\mathbb{P}(X = j) = \pi(j)$ for $j \in \mathcal{S}$, then we see that

$$X + 1 \sim \text{Geometric}\Big(\frac{\mu}{\lambda + \mu}\Big).$$

Thus, $\mathbb{E}[X] = (\lambda + \mu)/\mu - 1 = \lambda/\mu$. One way to understand this is that $1/\mu$ is the mean time that a customer spends in the system.

3. Queueing MDP

Consider a queue with Poisson arrivals with rate λ. The queue can hold N customers (N is a positive integer). The service times are i.i.d. Exponential(μ). When a customer arrives, you can choose to pay him $c > 0$ so that he does not join the queue. You also pay c when a customer arrives at a full queue. You want to decide when to accept customers to minimize the cost of rejecting them, plus the cost of the average waiting time they spend in the queue.

Formulate the problem as a Markov decision problem. For simplicity, consider a total discounted cost. That is, if x_t customers are in the system at time $t \geq 0$, then the waiting cost during $[t, t + \epsilon]$ is $\mathrm{e}^{-\beta t} x_t \epsilon$, for $\epsilon > 0$. Similarly, if you reject a customer at time t, then the cost is $c\mathrm{e}^{-\beta t}$. Write the dynamic programming equations.

Solution:

After discretizing time into ϵ slots, let $V(x, y)$ denote the value function for $(x, y) \in \mathcal{S} = \{0, \ldots, N\} \times \{0, 1\}$. The state space consists of tuples, where the first component is the number of customers currently in the queue, and the second component indicates whether or not a new customer arrives at the queue.

The discount factor is $\gamma := 1 - \beta\epsilon$. (Note: For simplicity, we will assume that ϵ is sufficiently small so that all terms of order $O(\epsilon^2)$ can be ignored.)

The dynamic programming equations are

$$\begin{aligned} V(x, 0) &= x + \gamma\big[(1 - \lambda\epsilon - \mu\epsilon)V(x, 0) + \lambda\epsilon V(x, 1) + \mu\epsilon V\big((x - 1)^+, 0\big)\big], \\ V(x, 1) &= c\mathbb{1}\{x = N\} + \min\{V_{\mathrm{a}}(x), V_{\mathrm{r}}(x)\}. \end{aligned}$$

The first equation reflects the cost of the customers waiting in the queue, and the expected discounted cost-to-go depending on whether no arrivals and no services occur, an arrival occurs, or a service occurs. The second equation reflects the cost of an arriving customer if the queue is full, and the expected discounted cost-to-go based on the two possible actions. Here, V_{a} is the action of accepting the new customer into the queue, and V_{r} is the action of rejecting the new customer from the queue. Now, we write down the equations for $V_{\mathrm{a}}(x)$ and $V_{\mathrm{r}}(x)$:

$$\begin{aligned} V_{\mathrm{a}}(x) &= V(\min\{x + 1, N\}, 0), \\ V_{\mathrm{r}}(x) &= c + V(x, 0). \end{aligned}$$