UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 11

Fall 2017

Self-Graded Scores Due: 5 PM, Monday, December 4, 2017
Submit your self-graded scores via the Google form:
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1. **Continuous-Time Markov Chains: Introduction**

Consider the continuous-time Markov chain with state space $\{1, 2, 3, 4\}$ and the rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(a) Find the stationary distribution π.

(b) Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

(c) Again assume the chain starts in state 1. What is the expected amount of time until the chain is in state 4?

Solution:

(a) Solving $\pi Q = 0$, we find the stationary distribution π is

$$\pi = \begin{bmatrix} \frac{3}{38} & \frac{7}{38} & \frac{9}{38} & \frac{1}{2} \end{bmatrix}.$$

(b) The distribution of the time the chain remains in state 1 before jumping is exponential with rate 3, so the expected amount of time it needs to wait in state 1 before jumping is $1/3$.

(c) This problem can be solved by writing a first jump equation. Let $b(i)$ denote the mean time needed to reach state 4 from state $i \in \{1, 2, 3, 4\}$. Then $b(4) = 0$, and for $i = 1, 2, 3$, we have

$$b(i) = \frac{1}{q(i)} + \sum_{j \in \{1,2,3,4\} \setminus \{i\}} \frac{q(i, j)}{q(i)} b(j).$$

If we delete the 4th row and column of the rate matrix Q and consider the submatrix corresponding to the remaining rows and columns, we get

$$\tilde{Q} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}.$$

Let $\bar{b} = \begin{bmatrix} b(1) & b(2) & b(3) \end{bmatrix}^\mathsf{T}$. Then the equation for $b(\cdot)$ can be written as

$$\tilde{Q}\bar{b} = -\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}.$$

Solving this yields $\bar{b} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$, which gives that the expected time starting from state 1 until the chain is in state 4 is 1.

2. $M/M/2$ **Queue**

A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. Where there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ.

(a) Argue that the queue length is a Markov chain.

(b) Draw the state transition diagram.

(c) Find the minimum value of μ so that the queue is positive-recurrent (i.e., admits a stationary distribution) and solve the balance equations.

Solution:

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when k ($k = 1, 2$) customers are being served, the completion of their service is independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

(b) It is shown in the following figure.

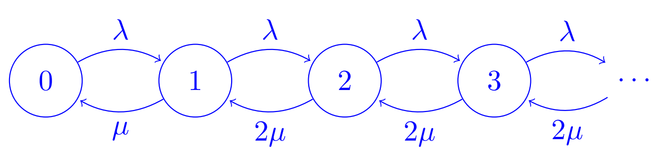

Figure 1: Markov chain for a queue with two servers.

(c) We write the flow conservation. Thus,

$$\pi(1) = \frac{\lambda}{\mu}\pi(0)$$

$$\pi(i+1) = \frac{\lambda}{2\mu}\pi(i), \qquad i \in \mathbb{Z}_+.$$

Thus,

$$\pi(i) = \Big(\frac{\lambda}{\mu}\Big)\Big(\frac{\lambda}{2\mu}\Big)^{i-1}\pi(0).$$

Also, we know that

$$\sum_{i=0}^{\infty}\pi(i) = 1.$$

Thus,

$$\pi(0)\Big[1 + \sum_{i=1}^{\infty}\Big(\frac{\lambda}{\mu}\Big)\Big(\frac{\lambda}{2\mu}\Big)^{i-1}\Big] = 1.$$

The series converges if $\lambda < 2\mu$. So the minimum value of μ for positive recurrence is $\mu > \lambda/2$. Then, solving the equation we have

$$\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}.$$

Then,

$$\pi(i) = \frac{2\mu-\lambda}{2\mu+\lambda}\Big(\frac{\lambda}{\mu}\Big)\Big(\frac{\lambda}{2\mu}\Big)^{i-1}, \qquad i \in \mathbb{Z}_+.$$

3. **Links**

Consider the graph shown in Figure 2. There are two parallel directed paths from the source node S to the destination node D. One of these is the path comprised of the two successive directed links 1 and 2, going through the intermediate node I. The other is the direct path comprised of the directed link 3. At any time, each of the links is in one of two state: *on* or *off*. Link

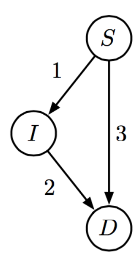

Figure 2: Source to destination links.

i switches between its on and off states at rate λ_i, $i = 1, 2, 3$, independently over links. The rates here refer to the rate in exponential distribution. Thus the state of each link can be modeled as a two-state continuous-time Markov chain. We will say that S is connected to D at any time iff there is a path from S to D comprised of on links.

(a) What is the stationary probability that S is connected to D?

(b) Assume that the process is in stationarity. Condition on S being connected to D at time 0, and let δ_t, for $t > 0$, denote the conditional probability that S is not connected to D at time t. What is $\frac{\mathrm{d}}{\mathrm{d}t}\delta_t\Big|_{t=0}$ (i.e., the first derivative of δ_t at time 0)?

(c) Assume that the process is in stationarity. Condition on S being not connected to D at time 0. What is the conditional mean time it takes for S to be connected to D?

Solution:

(a) There are 8 possible configurations for the network, each of them equiprobable in stationarity. In precisely 5 of these configuration S is connected to D. Hence the stationary probability that S is connected to D is 5/8.

(b) Among the 5 possible configurations that S is connected to D, 3 configurations have jumping rate to disconnect state λ_3, 1 configuration has jumping rate to disconnect state $\lambda_1 + \lambda_2$. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t\Big|_{t=0} = \frac{3}{5}\lambda_3 + \frac{1}{5}(\lambda_1 + \lambda_2).$$

(c) Define

$$\tau = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}, \qquad u_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}, \qquad u_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}.$$

Let 0, 1, and 2 denote the three states where S is not connected to D, being respectively the state where no link is on, the state where only link 1 is on, and the state where only link 2 is on. Let α_0, α_1, and α_2 denote the respective mean times it takes for S to be connected to D when starting from states 0, 1, and 2. One gets the equations:

$$\begin{aligned}
\alpha_0 &= \tau + u_1\alpha_1 + u_2\alpha_2 \\
\alpha_1 &= \tau + u_1\alpha_0 \\
\alpha_2 &= \tau + u_2\alpha_0
\end{aligned}$$

These may be solved to give

$$\alpha_0 = \tau\frac{1 + u_1 + u_2}{1 - u_1^2 - u_2^2}.$$

From this, we can easily get α_1 and α_2. The overall time of interest is

$$\frac{1}{3}(\alpha_0 + \alpha_1 + \alpha_2).$$

4. **Poisson Queues**

A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue ($k \in \mathbb{N}$), k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive-recurrent and find the invariant distribution.

Solution:

(a) The queue length is a MC as customer arrivals are independent of the current number of customers in the queue. Also, the departures only depend on the current number of customers being served. Next, even when k customers are being served, the completion of their service is independent of one another. Finally, even if one of the k customers has been completely served, the other customer has the same service time distribution as before as the exponential distribution is memoryless.

The only non-zero transition rates are

$$\begin{aligned} Q(k, k+1) &= \lambda, & k &\in \mathbb{N}, \\ Q(k, k-1) &= k\mu, & k &\in \mathbb{Z}_+. \end{aligned}$$

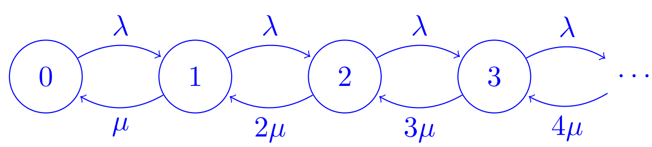

Figure 3: Markov chain of a memoryless queue with infinitely many servers.

(b) By flow conservation equations,

$$\pi(k)Q(k, k+1) = \pi(k+1)Q(k+1, k), \qquad k \in \mathbb{N}.$$

Thus,

$$\pi(k) = \pi(0)\Big(\frac{\lambda}{\mu}\Big)^k \frac{1}{k!}.$$

Let $\rho = \lambda/\mu$. Then, $\pi(0)\sum_{k=0}^{\infty} \rho^k/k! = 1$. Thus, $\pi(0) = \mathrm{e}^{-\rho}$ and the MC is positive-recurrent for all finite λ and μ.

5. **Taxi Queue**

Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

Solution:

Consider a continuous time Markov chain with states $X \in \{0, 1, 2, 3, 4\}$ which denotes the number of people waiting. For $n = 0, 1, 2, 3$, the transitions from n to $n+1$ have rate 1, and the transitions from $n+1$ to n have rate 2. The balance equations are then

$$\pi(n) = \frac{1}{2}\pi(n-1), \qquad n = 1, 2, 3, 4.$$

Using the above equations and $\sum_{i=0}^{4} \pi(i) = 1$ we find that $\pi(i) = 2^{-i}\pi(0)$ and $\pi(0) = 16/31$. Since the expected waiting time for a new taxi is 0.5, the expected waiting time of John given that he joins the queue can be computed as follows.

$$\mathbb{E}[T] = \frac{\pi(0) \times 0.5 + \pi(1) \times 1 + \pi(2) \times 1.5 + \pi(3) \times 2}{\pi(0) + \pi(1) + \pi(2) + \pi(3)} = \frac{13}{15}.$$

The denominator represents the fact that we are conditioning on the event that John joins the queue.

6. **Two-Server System**

A company has two servers (the second server is a backup in case the first server fails, so only one server is ever used at a time). When a server is running, the time until it breaks down is exponentially distributed with rate μ. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is exponentially distributed with rate λ. Find the long-run probability that no servers are operational.

Solution:

The idea is to model the number of operational servers as a continuous-time Markov chain on the state space $\{0, 1, 2\}$. By thinking about the infinitesimal transition probabilities (which are simply the rates of the exponential distributions), we have the following matrix:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda + \mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}.$$

Now, we write down the balance equations.

$$\begin{aligned} \lambda\pi(0) &= \mu\pi(1) \\ (\lambda + \mu)\pi(1) &= \lambda\pi(0) + \mu\pi(2) \\ \mu\pi(2) &= \lambda\pi(1) \\ 1 &= \pi(0) + \pi(1) + \pi(2) \end{aligned}$$

We eliminate $\pi(2)$ with $\pi(2) = (\lambda/\mu)\pi(1)$. Plugging this into the second and fourth equations, we have

$$\begin{aligned} \mu\pi(1) &= \lambda\pi(0), \\ 1 &= \pi(0) + \Big(1 + \frac{\lambda}{\mu}\Big)\pi(1). \end{aligned}$$

We next eliminate $\pi(1)$ with $\pi(1) = (\lambda/\mu)\pi(0)$. Plugging this into the second equation above, we have

$$\pi(0) = \frac{1}{1 + \lambda/\mu + (\lambda/\mu)^2}.$$

This is the long-run probability that we will be in state 0, i.e. there are no operational servers.