1 Miscellaneous Review

(a) The probability that only one of the events occurs is equivalent to the probability of the union of the events, but without their intersection. In other words, we are looking for $\mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

(b) From the definition of independence of events, we know that if A is independent of itself, it must be the case that $\mathbb{P}(A \cap A) = \mathbb{P}(A) = \mathbb{P}(A) \cdot \mathbb{P}(A)$. We then get $\mathbb{P}(A) = (\mathbb{P}(A))^2$, which only holds for $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

(c) Consider flipping two fair coins. Let A denote the event that the first coin is heads and let B denote the event that the second coin is heads. Let C be the event that the total number of heads obtained from flipping the two coins is 1. In this case, we have that $\mathbb{P}(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(B)$. We also have that $\mathbb{P}(A \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(C)$, and similarly so for $\mathbb{P}(B \cap C)$. However, these events are not mutually independent as $\mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$.

(d) Let B_1 denote the event that the first ball was blue and let B_2 denote the event that the second ball was blue. We are interested in finding $\mathbb{P}(B_1 \mid B_1 \cup B_2)$. This is equivalent to

$$\begin{aligned}
\frac{\mathbb{P}(B_1 \cap (B_1 \cup B_2))}{\mathbb{P}(B_1 \cup B_2)} &= \frac{\mathbb{P}(B_1)}{\mathbb{P}(B_1 \cup B_2)} \\
&= \frac{\mathbb{P}(B_1)}{\mathbb{P}(B_1) + \mathbb{P}(B_2) - \mathbb{P}(B_1 \cap B_2)} \\
&= \frac{\mathbb{P}(B_1)}{\mathbb{P}(B_1) + \left(\mathbb{P}(B_1) \cdot \mathbb{P}(B_2 \mid B_1) + \mathbb{P}\left(B_1^{\complement}\right) \cdot \mathbb{P}\left(B_2 \mid B_1^{\complement}\right)\right) - \mathbb{P}(B_1) \cdot \mathbb{P}(B_2 \mid B_1)} \\
&= \frac{\mathbb{P}(B_1)}{\mathbb{P}(B_1) + \mathbb{P}\left(B_1^{\complement}\right) \cdot \mathbb{P}\left(B_2 \mid B_1^{\complement}\right)} \\
&= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3}} \\
&= \frac{3}{4}
\end{aligned}$$

2 Passengers on a Plane

Let A_i be the event that the ith passenger is in their assigned seat. We are then trying to find $\mathbb{P}\left(\left(\bigcup_{i=1}^{N} A_i\right)^{\complement}\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right)$. From the inclusion-exclusion principle, we have that $\mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{k=1}^{N} (-1)^{k+1} \sum_{\substack{I \subseteq \{1,\ldots,N\} \\ |I|=k}} \mathbb{P}\left(\bigcap_{i \in I} A_i\right)$, so that we are looking for

$$\mathbb{P}\left(\left(\bigcup_{i=1}^{N} A_i\right)^{\complement}\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right)$$

$$= 1 - \sum_{k=1}^{N} (-1)^{k+1} \sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I|=k}} \mathbb{P}\left(\bigcap_{i \in I} A_i\right)$$
$$= 1 - \sum_{k=1}^{N} (-1)^{k+1} \left(\frac{\binom{N}{k}}{\prod_{i=0}^{k-1}(N-i)}\right)$$
$$= 1 - \sum_{k=1}^{N} (-1)^{k+1} \left(\frac{\frac{N!}{(N-k)!k!}}{\prod_{i=0}^{k-1}(N-i)}\right)$$
$$= 1 - \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k!}$$
$$= 1 + \sum_{k=1}^{N} \frac{(-1)^{k}}{k!}$$
$$= \sum_{k=0}^{N} \frac{(-1)^{k}}{k!}$$

When we are interested in this probability as $N \to \infty$, we get $\lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k}{k!} = e^{-1} \approx .3679$, using the Taylor series expansion $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

3 Joint Occurrence

We are given that at least one of the events A_r is certain to occur. In other words, $\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) = 1$. By the union bound, $\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) \leq \sum_{r=1}^{n} \mathbb{P}(A_r)$. Since we are also given that the probability of occurrence of any single event is p, each $\mathbb{P}(A_r) = p$, and thus $\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) \leq np$. Taking this in conjunction with the first equality, we get that $p \geq \frac{1}{n}$. We also must have that $\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) \geq \sum_{\substack{R \subseteq \{1,\dots,n\} \\ |R|=2}} \mathbb{P}\left(\bigcap_{r \in R} A_r\right)$. This inequality holds as the probability at least one event occurs must be greater than or equal to the probability of exactly 2 events occurring. Since we are also given that the probability of joint occurrence of any two distinct events is q, we have that $\mathbb{P}\left(\bigcup_{r=1}^{n} A_r\right) \geq \binom{n}{2} \cdot q = \frac{n!}{(n-2)!2!} \cdot q = \frac{n(n-1)}{2} \cdot q$. Using the first equality once again, we obtain $q \leq \frac{2}{n(n-1)}$.

4 Expanding the NBA

Let A_i denote the event that the ith city to be interviewed is the best city and let B_i denote the event that the ith city is selected. We are then looking for $\sum_{i=1}^{N} \mathbb{P}(A_i \cap B_i) = \sum_{i=1}^{N} \mathbb{P}(A_i) \cdot \mathbb{P}(B_i \mid A_i)$. However, if the best city is included in the first m cities, for $i <= m$, we must necessarily have that $\mathbb{P}(B_i \mid A_i) = 0$, as the first m cities are all rejected. As a result, our sum can be adjusted accordingly to $\sum_{i=m+1}^{N} \mathbb{P}(A_i) \cdot \mathbb{P}(B_i \mid A_i)$. The probability $\mathbb{P}(A_i)$ that the ith city is the best city is simply $\frac{1}{N}$, while the probability $\mathbb{P}(B_i \mid A_i)$ that the ith city is selected given that it is the best city is contingent on whether or not the relatively best city of the first $i-1$ cities is within the first m cities that are initially all rejected. In this manner, the ith city must necessarily be selected as it would be the first city better than the previous $i-1$ cities. Thus we have that $\mathbb{P}(B_i \mid A_i) = \frac{m}{i-1}$, as this is the probability that the relatively best city of the first $i-1$ cities is within the first m cities. Plugging these new values into our summation, we obtain $\sum_{i=m+1}^{N} \frac{1}{N} \cdot \frac{m}{i-1} = \frac{m}{N} \sum_{i=m+1}^{N} \frac{1}{i-1} \approx \frac{m}{N} (\ln N - \ln (m-1))$. Differentiating with respect to m to obtain the optimal value of m, we have

$$\begin{aligned}
\frac{m}{N}(-\frac{1}{m-1}) + \frac{1}{N}(\ln N - \ln (m-1)) &= 0 \\
-\frac{m}{m-1} - \ln (m-1) &= -\ln N \\
\ln m &\approx \ln N - 1 \\
m &\approx \frac{N}{e}
\end{aligned}$$

where we have approximated $m - 1 \approx m$.

5 Superhero Basketball

Let A denote the event that there was at least one tie and Superman scored the first point, and let B denote the event that there was at least one tie and Captain America scored the first point. Our event of interest then is $(A \cup B)^{\complement}$, so we would like to find $\mathbb{P}((A \cup B)^{\complement}) = 1 - \mathbb{P}(A \cup B) = 1 - \mathbb{P}(A) - \mathbb{P}(B)$. Since Captain America won the game, there must have been at least one tie if Superman scored the first point. Thus, $\mathbb{P}(A) = \frac{\binom{n+m-1}{m-1}}{\binom{n+m}{m}} = \frac{\frac{(n+m-1)!}{(m-1)!n!}}{\frac{(n+m)!}{m!n!}} = \frac{1}{\frac{n+m}{m}} = \frac{m}{n+m}$. By symmetry, we also have that $\mathbb{P}(B) = \mathbb{P}(A)$, since any sequence of shots leading up to a tie starting with Superman scoring the first point can be flipped so that Captain America scores the first point instead. Thus, we have $\mathbb{P}((A \cup B)^{\complement}) = 1 - \mathbb{P}(A) - \mathbb{P}(B) = 1 - 2 \cdot \mathbb{P}(A) = 1 - 2 \cdot \frac{m}{n+m} = \frac{n-m}{n+m}$.

6 [Bonus] Tournament Probabilistic Proof