UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Discussion 2
Fall 2017

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1. **Sum of Binomials**

   If $X \sim \text{Binomial}(n,p)$ and $Y \sim \text{Binomial}(m,p)$, X and Y are independent, show that $X+Y \sim \text{Binomial}(m+n,p)$.

   Solution:

   Let $0 \le k \le m+n$. We have,

$$\begin{aligned}
\mathbb{P}(X+Y=k) &= \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i) = \sum_{i=0}^{k} \mathbb{P}(X=i)\mathbb{P}(Y=k-i) \\
&= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\
&= p^k (1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \\
&= \binom{n+m}{k} p^k (1-p)^{n+m-k}
\end{aligned}$$

   (the last line is Vandermonde's identity).

2. **Linearity of Expectation, Independence**

   Let X_1, X_2, X_3 be discrete independent random variables with mean 0. Find $\mathbb{E}[(X_1+X_2)(X_2+X_3)(X_3+X_1)]$.

   Solution:

   $\mathbb{E}[(X_1+X_2)(X_2+X_3)(X_3+X_1)]$ contains terms of the form $\mathbb{E}(X_i^2 X_j), i \neq j$ and $\mathbb{E}(X_i X_j X_k), i \neq j \neq k$. Since X_1, X_2, X_3 are independent and zero mean, $\mathbb{E}(X_i^2 X_j) = \mathbb{E}(X_i^2)\mathbb{E}(X_j) = 0$, and from the same logic, $\mathbb{E}(X_i X_j X_k) = 0$. Hence, $\mathbb{E}[(X_1+X_2)(X_2+X_3)(X_3+X_1)] = 0$.

3. **Poisson Recursion**

   Suppose X is Poisson distributed with parameter λ. Show that $\mathbb{E}(X^n) = \lambda \mathbb{E}[(X+1)^{n-1}]$. Use this to compute $\mathbb{E}(X^3)$.

   Solution:

$$\begin{aligned}
\mathbb{E}(X^n) &= \sum_{i=0}^{\infty} i^n \mathrm{e}^{-\lambda} \lambda^i / i! = \sum_{i=1}^{\infty} i^n \mathrm{e}^{-\lambda} \lambda^i / i! = \sum_{i=1}^{\infty} i^{n-1} \mathrm{e}^{-\lambda} \lambda^i / (i-1)! \\
&= \sum_{j=0}^{\infty} (j+1)^{n-1} \mathrm{e}^{-\lambda} \lambda^{j+1} / j! = \lambda \mathbb{E}[(X+1)^{n-1}].
\end{aligned}$$

Now, $\mathbb{E}(X^3) = \lambda\mathbb{E}[(X+1)^2]$. Since X is Poisson, $\mathbb{E}(X) = \lambda$, $\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda$. Plugging in, $\mathbb{E}(X^3) = \lambda(\lambda^2 + 3\lambda + 1)$.