1 Transform Practice

(a) We know that $\mathbb{E}[e^{sZ}] = M_Z(s) = \frac{a-3s}{s^2-6s+8}$. Since for $s = 0$, we must have that $\mathbb{E}[e^0] = 1 = \frac{a-0}{0-0+8}$. Solving for a, we get that $a = 8$.

(b) $\mathbb{E}[Z] = \frac{dM_Z(s)}{ds}\Big|_{s=0} = \frac{(s^2-6s+8)(-3)-(8-3s)(2s-6)}{(s^2-6s+8)^2}\Big|_{s=0} = \frac{24}{64} = \frac{3}{8}$

(c)

$$\begin{aligned}
\text{var}(Z) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{d^2M_Z(s)}{ds^2}\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{d}{ds}\left(\frac{(-3^2+18s-24)-16s+48+6s^2+18s}{(s^2-6s+8)^2}\right)\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{d}{ds}\left(\frac{3s^2-16s+24}{(s^2-6s+8)^2}\right)\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{(s^2-6s+8)^2(6s-16)-(3s^2-16s+24)(2(s^2-6s+8)(2s-6))}{(s^2-6s+8)^3}\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{(s^2-6s+8)(6s-16)-(3s^2-16s+24)(4s-12)}{(s^2-6s+8)^3}\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{(18s^3-16s^2-36s^2+96s+48s-128)-(12s^3-64s^2+96s-36s^2+192s-288)}{(s^2-6s+8)^3}\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{6s^3+48s^2-146s+160}{(s^2-6s+8)^3}\Big|_{s=0} - \left(\frac{3}{8}\right)^2 \\
&= \frac{160}{8^3} - \left(\frac{3}{8}\right)^2 \\
&= \frac{20}{8^2} - \left(\frac{3}{8}\right)^2 \\
&= \frac{11}{64}
\end{aligned}$$

2 Bounds for the Coupon Collector's Problem

(a) $\mathbb{P}(X > 2nH_n) \le \frac{\mathbb{E}[X]}{2nH_n} = \frac{nH_n}{2nH_n} = \frac{1}{2}$

(b) From Chebyshev's inequality, we have that $\mathbb{P}(|X - nH_n| \ge nH_n) \le \frac{\text{var}(X)}{(nH_n)^2}$. To find $\text{var}(X)$, we note that $X = \sum_{i=1}^n X_i$, where each X_i is a geometric random variable denoting the number of boxes needed to collect a new coupon after $i-1$ coupons have been collected. Since these are independent events, we can determine variance by $\text{var}(X) = \sum_{i=1}^n \text{var}(X_i)$. As a result, we have that

$$\begin{aligned}
\text{var}(X) < \sum_{i=1}^n \text{var}(X_i) &= \sum_{i=1}^n \frac{(1-p_i)}{p_i^2} \\
&= \sum_{i=1}^{n-1} \frac{i \cdot n}{(n-i)^2}
\end{aligned}$$

$$< n^2 \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{n^2\pi^2}{6}$$

Plugging this into Chebyshev's inequality above, we have that $\mathbb{P}(X > 2nH_n) \leq \mathbb{P}(|X - nH_n| > nH_n) \leq \frac{\frac{n^2\pi^2}{6}}{(nH_n)^2} = \frac{\pi^2}{6(\ln n)^2}$.

(c) Let X_i be the event that the ith coupon is not collected yet after $2nH_n$ boxes. Thus, the probability that $\mathbb{P}(X > 2nH_n) = \mathbb{P}\left(\bigcup_{i=1}^n X_i\right) \leq \sum_{i=1}^n \left(\frac{n-1}{n}\right)^{2nH_n} = \sum_{i=1}^n \left(1 - \frac{1}{n}\right)^{2nH_n} \leq n\left(\frac{1}{e}\right)^{2H_n} = n\left(\frac{1}{e}\right)^{2\ln n} = \frac{n}{n^2} = \frac{1}{n}$.

3 A Chernoff Bound for the Sum of Coin Flips

(a) We have that $\mathbb{P}(X \geq pn) = \mathbb{P}(e^{tX} \geq e^{tpn}) \leq e^{-tpn}M_X(t) = e^{-(tpn - \ln M_X(t))} = e^{-n(tp - \ln \mathbb{E}[e^{tX_1}])}$, where $M_X(t) = M_{X_1}(t)^n$.

(b) Differentiating with respect to t, we get that

$$\begin{aligned}
\frac{d}{dt}\left(e^{-n\left(tp - \ln \mathbb{E}\left[e^{tX_1}\right]\right)}\right) &= e^{-n\left(tp - \ln \mathbb{E}\left[e^{tX_1}\right]\right)} \cdot \left(-np + \frac{n}{\mathbb{E}\left[e^{tX_1}\right]} \cdot \frac{d}{dt}\left(\mathbb{E}[e^{tX_1}]\right)\right) \\
&= e^{-n\left(tp - \ln\left(1-q+qe^t\right)\right)} \cdot \left(-np + \frac{n}{1-q+qe^t} \cdot qe^t\right).
\end{aligned}$$

Optimizing with respect to t by setting this to 0, we get that

$$\begin{aligned}
\frac{nqe^t}{1-q+qe^t} &= np \\
qe^t &= p(1-q+qe^t) \\
qe^t(1-p) &= p(1-q) \\
e^t &= \frac{p(1-q)}{q(1-p)} \\
t &= \ln\left(\frac{p(1-q)}{q(1-p)}\right).
\end{aligned}$$

Plugging this into our bound, we have that

$$\begin{aligned}
\mathbb{P}(X \geq pn) \leq e^{-n(tp - \ln M_{X_1}(t))} &= e^{-n\left(\left(\ln\left(\frac{p(1-q)}{q(1-p)}\right)\right)p - \ln M_{X_1}\left(\ln\left(\frac{p(1-q)}{q(1-p)}\right)\right)\right)} \\
&= e^{-n\left(\left(\ln\left(\frac{p(1-q)}{q(1-p)}\right)\right)p - \ln\left(1-q+qe^{\ln\left(\frac{p(1-q)}{q(1-p)}\right)}\right)\right)} \\
&= e^{-n\left(p\left(\ln\frac{p}{q} - \ln\frac{1-p}{1-q}\right) - \ln\left(1-q+\frac{p(1-q)}{1-p}\right)\right)} \\
&= e^{-n\left(p\left(\ln\frac{p}{q} - \ln\frac{1-p}{1-q}\right) - \ln\frac{(1-q)(1-p)+p(1-q)}{(1-p)}\right)} \\
&= e^{-n\left(p\left(\ln\frac{p}{q} - \ln\frac{1-p}{1-q}\right) + \ln\frac{1-p}{(1-q)(1-p)+p(1-q)}\right)} \\
&= e^{-n\left(p\left(\ln\frac{p}{q} - \ln\frac{1-p}{1-q}\right) + \ln\frac{1-p}{(1-q)(1-p+p)}\right)} \\
&= e^{-n\left(p\ln\frac{p}{q} + (1-p)\ln\frac{1-p}{1-q}\right)} \\
&= e^{-nD(p \,\|\, q)}.
\end{aligned}$$

(c) Using the divergence inequality with the results from part (b), we have that $\mathbb{P}(X \geq (q+\epsilon)n) \leq e^{-nD((q+\epsilon)\ \|\ q)} \leq e^{-2n\epsilon^2}$. By symmetry, we must also have that $\mathbb{P}(X \leq (q-\epsilon)n) \leq e^{-2n\epsilon^2}$, where a value of $t < 0$ will be obtained in the optimization process instead.

(d) From part (c), since $\mathbb{P}(X \geq (q+\epsilon)n) \leq e^{-2n\epsilon^2}$ and $\mathbb{P}(X \leq (q-\epsilon)n) \leq e^{-2n\epsilon^2}$, it naturally follows that $\mathbb{P}(|X - qn| \geq \epsilon n) \leq 2e^{-2n\epsilon^2}$.

4 Decoding a Bit from a Noisy Signal

(a) To transform the B_is such that the results in Problem 3 can be applied, we must convert the B_is to a binomial representation. We are looking to bound the probability that the receiver cannot determine b correctly, which can only occur when $w|\sum_{i=1}^n B_i| \geq 1$. So, we would like to find a bound for $\mathbb{P}(|\sum_{i=1}^n B_i| \geq \frac{1}{w})$. Letting X denote a binomial random variable indicating the number of $+1$ bits sent, we can rewrite this as $\mathbb{P}(|X + (n-X)(-1)| \geq \frac{1}{w}) = \mathbb{P}(|2X - n| \geq \frac{1}{w}) = \mathbb{P}(|X - \frac{n}{2}| \geq \frac{1}{2w})$. In this form similar to that of Problem 3, we have that $\mathbb{P}(|X - \frac{n}{2}| \geq \frac{1}{2wn}n) \leq 2e^{-\frac{2n}{(2wn)^2}} = 2e^{-\frac{1}{2nw^2}}$.

(b) $1 - \mathbb{P}(|X - \frac{n}{2}| \geq \frac{1}{2w}) \geq 1 - 2e^{-\frac{1}{2nw^2}} \geq .999$. Solving for w, we get

$$\begin{aligned}
.001 &\geq 2e^{-\frac{1}{2nw^2}} \\
.0005 &\geq e^{-\frac{1}{2nw^2}} \\
\ln .0005 &\geq -\frac{1}{2nw^2} \\
2nw^2 &\leq -\frac{1}{\ln .0005} \\
w &\leq \sqrt{-\frac{1}{2n \ln .0005}}
\end{aligned}$$

(c) Chebyshev's inequality gives us $\mathbb{P}(|X - \frac{n}{2}| \geq \frac{1}{2w}) \leq \frac{\frac{n}{4}}{\frac{1}{(2w)^2}} = nw^2$. $1 - \mathbb{P}(|X - \frac{n}{2}| \geq \frac{1}{2w}) \geq 1 - nw^2 \geq .999$. Solving for w, we get $w \leq \sqrt{\frac{.001}{n}}$.

(d) Chebyshev's inequality gives a looser bound on the error probability and thus results in a stricter bound on the noise power. On the other hand, the Chernoff bound gives a tighter bound on the error probability and consequently allows for a higher threshold on the noise power.

5 [Bonus] Gaussian Tail Bounds

(a)

(b)

(c)