UC Berkeley
Department of Electrical Engineering and Computer Sciences

# Electrical Engineering 126: Probability and Random Processes

**Discussion 9**
Fall 2017

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## 1. Estimating an Exponential Distribution

(a) You draw a sample $X_1, \dots, X_n$ ($n$ is a positive integer) for the lifetime of a light bulb (assumed to be exponentially distributed). You have information from a trustworthy source that the rate of the exponential distribution satisfies $\lambda \ge 2$. Using Chebyshev's Inequality, what is the minimum $n$ required to construct a confidence interval for the mean lifetime of the light bulb? Your confidence interval must have tolerance at most $\varepsilon$ with confidence at least $1-\delta$ for parameters $\delta, \varepsilon > 0$. Also, you should precisely state what your estimate for the mean lifetime is.

(b) Due to budget constraints, you are only allowed to use 10000 samples. You must still maintain the $\varepsilon$ tolerance, but with such a large sample size, you feel justified in using the Central Limit Theorem. What is your new confidence? (Again, you may use the information from your trustworthy source. Express your answers in terms of $\Phi$, the CDF of the standard normal distribution.)

**Solution:**

(a) If we use $n^{-1}\sum_{i=1}^n X_i$ as an estimator, then we require

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^n X_i - \frac{1}{\lambda}\Big| \ge \varepsilon\Big) \le \delta.$$

By Chebyshev's Inequality, the above inequality will be satisfied so long as

$$\frac{\operatorname{var}(n^{-1}\sum_{i=1}^n X_i)}{\varepsilon^2} = \frac{\operatorname{var} X_1}{n\varepsilon^2} = \frac{1}{n\lambda^2\varepsilon^2} \le \delta.$$

From our trustworthy source, we have the lower bound $\lambda \ge 2$, so we are happy if we choose

$$n \ge \frac{1}{4\delta\varepsilon^2}.$$

(b) We now assume $n^{-1}\sum_{i=1}^n X_i \sim \mathcal{N}(\lambda^{-1}, n^{-1}\lambda^{-2})$. We now seek to bound the probability of an $\varepsilon$-deviation from the mean:

$$\begin{aligned}\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^n X_i - \frac{1}{\lambda}\Big| \ge \varepsilon\Big) &\approx \mathbb{P}\big(|\mathcal{N}(\lambda^{-1}, n^{-1}\lambda^{-2}) - \lambda^{-1}| \ge \varepsilon\big) \\ &= \mathbb{P}\big(|\mathcal{N}(0, n^{-1}\lambda^{-2})| \ge \varepsilon\big) \\ &= \mathbb{P}\big(|\mathcal{N}(0,1)| \ge \varepsilon\lambda\sqrt{n}\big) \le 2\Phi(-200\varepsilon).\end{aligned}$$

The confidence level is thus $1 - 2\Phi(-200\varepsilon)$.

2. **Random Telegraph Wave**

Let $\{N_t,\ t \geq 0\}$ be a Poisson process with rate $\lambda$ and define $X_t = X_0(-1)^{N_t}$ where $X_0 \in \{0, 1\}$ is a random variable independent of $N_t$.

(a) Does the process $X_t$ have independent increments?

(b) Calculate $\mathbb{P}(X_t = 1)$ if $\mathbb{P}(X_0 = 1) = p$.

(c) Assume that $p = 0.5$. Calculate $\mathbb{E}[X_{t+s}X_s]$ for $s, t \geq 0$.

**Solution:**

(a) No, the process does not have independent increments. According to the definition of independent increments, for any $0 < t_0 < t_1 < t_2$, we should have $X_{t_2} - X_{t_1}$ is independent of $X_{t_1} - X_{t_0}$. However, suppose $X_0 = 1$ and $X_{t_1} - X_{t_0} = 2$. This means that from $t_0$ to $t_1$, $X_t$ increases from $-1$ to 1. Then it is impossible to have $X_{t_2} - X_{t_1} = 2$ since $X_t \in \{-1, 1\}$ for all $t > 0$, when $X_0 = 1$.

(b) First we calculate $\mathbb{P}(N_t \text{ is even})$.

$$\begin{aligned}
\mathbb{P}(N_t \text{ is even}) &= \sum_{i=0,\ i \text{ is even}}^{\infty} \frac{(\lambda t)^i \mathrm{e}^{-\lambda t}}{i!} = \frac{\mathrm{e}^{-\lambda t}}{2}\Big(\sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-\lambda t)^i}{i!}\Big) \\
&= \frac{\mathrm{e}^{-\lambda t}}{2}(\mathrm{e}^{\lambda t} + \mathrm{e}^{-\lambda t}) = \frac{1 + \mathrm{e}^{-2\lambda t}}{2}.
\end{aligned}$$

$$\mathbb{P}(X_t = 1) = p\mathbb{P}(N_t \text{ is even}) = p\frac{1 + \mathrm{e}^{-2\lambda t}}{2}.$$

(c) If $X_0 = 0$, obviously there is $\mathbb{E}[X_{t+s}X_s] = 0$ for all $s, t \geq 0$. For $X_0 = 1$, we have

$$\begin{aligned}
\mathbb{P}(X_{t+s}X_s = 1) &= \mathbb{P}(N_{t+s} - N_s \text{ is even}) = \mathbb{P}(N_t \text{ is even}) \\
&= \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(X_{t+s}X_s = -1) &= \mathbb{P}(N_{t+s} - N_s \text{ is odd}) = \mathbb{P}(N_t \text{ is odd}) \\
&= \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t}).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathbb{E}[X_{t+s}X_s] &= \frac{1}{2}\mathbb{E}[X_{t+s}X_s \mid X_0 = 1] \\
&= \frac{1}{2}\Big[\frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) - \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t})\Big] = \frac{1}{2}\mathrm{e}^{-2\lambda t}.
\end{aligned}$$

3. **Markov Chains Meet Linear Algebra**

Consider the transition matrix:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find $P^n$, for each positive integer $n$.
*Hint:* This can be done without any math.

(b) Find the distinct eigenvalues of $P$ along with their multiplicities.

(c) Can you write $P = U\Lambda U^{-1}$ for some diagonal matrix $\Lambda$ and invertible matrix $U$?

**Solution:**

(a) One way to do this, is to find $P^2$, then $P^3$, and keep multiplying by $P$ until you have $P^n$. We will show a different way, by visualizing the Markov chain: Note that $P^n$ is the transition matrix after $n$ steps. In this

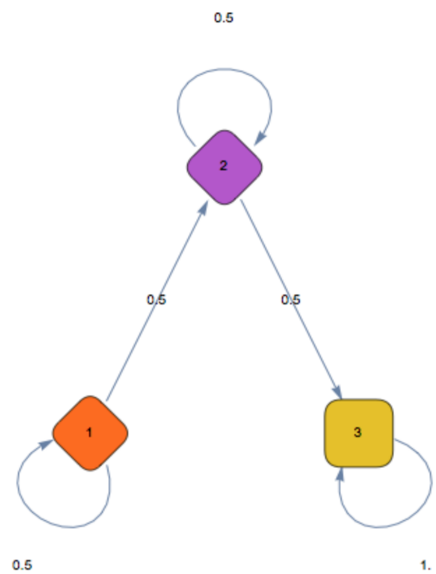


Figure 1: Markov chain for Problem 3.

sense, $P_{1,1}$ is the probability of going from $1 \to 1$ in $n$ steps. So $P_{i,j}$ is the probability of going from $i \to j$ in $n$ steps. Thus, note that $P_{1,1} = 2^{-n}$, since it must transition to itself every step. Similarly, $P_{2,2} = 2^{-n}$. Now, note that $P_{1,2} = n2^{-n}$ since this implies out of $n$ transitions, exactly 1 of them was from $1 \to 2$, and the other $n-1$ were staying in the same state (either 1 or 2). The probability of every path is $2^{-n}$, hence the transition probability is $n2^{-n}$. Now, note that $P_{3,3} = 1$ as before. We may fill in the rest of the entries, noting that all below diagonal entries are 0, and the remaining above diagonal entries must make the rows sum to 1, so we get the final matrix:

$$P^n = \begin{bmatrix} 2^{-n} & n2^{-n} & 1-(n+1)2^{-n} \\ 0 & 2^{-n} & 1-2^{-n} \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Note that $P$ is an upper triangular matrix, so $P - \lambda I$ is also upper triangular and its determinant is the product of its diagonal entries:

$$\det(P - \lambda I) = \left(\frac{1}{2} - \lambda\right)^2 (1 - \lambda).$$

Thus, $\lambda = 1$ is an eigenvalue of multiplicity 1 and $\lambda = 1/2$ is an eigenvalue of multiplicity 2.

(c) We find the right eigenvectors. Note that clearly, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$ is a right eigenvector corresponding to $\lambda = 1$. Now, let's examine some $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\mathsf{T}$, a right eigenvector of eigenvalue $1/2$. It must be that:

$$\begin{aligned}
&\frac{1}{2}v_1 + \frac{1}{2}v_2 = \frac{1}{2}v_1 \\
&\frac{1}{2}v_2 + \frac{1}{2}v_3 = \frac{1}{2}v_2 \\
&v_3 = \frac{1}{2}v_3
\end{aligned}$$

Thus, we can see that $v_2 = v_3 = 0$ and $v_1 = 1$, where we can scale this vector by any scalar and obtain another solution. Thus, the dimension of the eigenspace corresponding to eigenvalue $1/2$ is 1, and the original matrix $P$ is not diagonalizable since the sum of the dimensions of its eigenspaces is $2 < 3$.