UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 7
Fall 2017

Self-Graded Scores Due: 5 PM, Monday, October 23, 2017
Submit your self-graded scores via the Google form:
https://goo.gl/forms/0XvcDrl4zHKU0kkI2.
Make sure you use your **SORTABLE NAME** on CalCentral.

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1. **Three-State Chain**

   Consider the Markov chain of Figure 1, where $a, b \in (0, 1)$.

   (a) Find the invariant distribution.

   (b) Calculate $\mathbb{P}(X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0)$.

   (c) Show that the Markov chain is aperiodic.

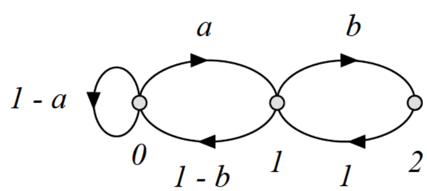


Figure 1: Markov chain for Problem 1.

Solution:

(a) To find the invariant distribution, we solve the following equations.

$$\pi(2) = b\pi(1), \;\; \pi(1) = a\pi(0) + \pi(2), \;\; \pi(0) + \pi(1) + \pi(2) = 1.$$

Then,

$$\pi(0) = \frac{1-b}{1-b+a+ab},$$
$$\pi(1) = \frac{a}{1-b+a+ab},$$
$$\pi(2) = \frac{ab}{1-b+a+ab}.$$

(b) By Markov property the probability is $a \times (1-b) \times (1-a) \times a = a^2(1-a)(1-b)$.

(c) The Markov chain is aperiodic since there is a self-loop.

2. Finite Random Walk

(a) Find the steady-state probabilities $\pi_0, \dots, \pi_{k-1}$ for the Markov chain in Figure 2. Here, k is a positive integer and $p \in (0,1)$. Express your answer in terms of the ratio $\rho = p/q$, where $q = 1-p$. Pay particular attention to the special case $\rho = 1$.

(b) Find the limit of π_0 as k approaches infinity; give separate answers for $\rho < 1$, $\rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.

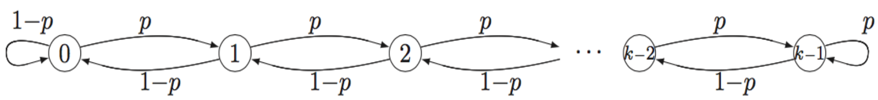


Figure 2: Markov chain for Problem 2.

Solution:

(a) Without loss of generality, we can consider $\rho \leq 1$. This is because if $\rho > 1$, we can flip the chain and get a new chain with $\rho \leq 1$. Consider the invariant distribution, using the abbreviation $q = 1-p$, we have

$$\begin{aligned} \pi_0 &= q\pi_0 + q\pi_1 \\ \pi_j &= p\pi_{j-1} + q\pi_{j+1}; \quad \text{for } j = 1, \dots, k-2 \\ \pi_{k-1} &= p\pi_{k-2} + p\pi_{k-1}. \end{aligned}$$

Simplifying the first equation, we get $p\pi_0 = q\pi_1$, i.e., $\pi_1 = \rho\pi_0$. Substituting $q\pi_1$ for $p\pi_0$ in the second equation, we get $\pi_1 = q\pi_1 + q\pi_2$. Simplifying the second equation, then, we get $\pi_2 = \rho\pi_1$. We can then use induction. Using the inductive hypothesis $\pi_j = \rho\pi_{j-1}$ (which has been verified for $j = 1, 2$) on $\pi_j = p\pi_{j-1} + q\pi_{j+1}$, we get

$$\pi_{j+1} = \rho\pi_j \quad \text{for } j = 1, \dots, k-2.$$

Combining these equations, $\pi_j = \rho\pi_{j-1}$ for $j = 1, \dots, k-1$, so $\pi_j = \rho^j\pi_0$.

$$\pi_0\left(\sum_{j=0}^{k-1}\rho^j\right) = 1 \quad \text{so} \qquad \pi_0 = \frac{1-\rho}{1-\rho^k}; \quad \pi_j = \rho^j\frac{1-\rho}{1-\rho^k}.$$

For $\rho = 1$, $\rho^j = 1$ and $\pi_j = 1/k$ for $j = 0, \dots, k-1$.

(b) For state 0,

$$\lim_{k\to\infty}\pi_0 = \begin{cases} \lim_{k\to\infty}\dfrac{1-\rho}{1-\rho^k} = 1-\rho, & \text{for } \rho < 1, \\ \lim_{k\to\infty}\dfrac{1}{k} = 0, & \text{for } \rho = 1. \end{cases}$$

For state $k-1$, the analogous result is

$$\lim_{k\to\infty} \pi_{k-1} = \begin{cases} \lim\limits_{k\to\infty} \rho^{k-1}\dfrac{1-\rho}{1-\rho^k} = 0, & \text{for } \rho < 1, \\ \lim\limits_{k\to\infty} \dfrac{1}{k} = 0, & \text{for } \rho = 1. \end{cases}$$

3. **Fly on a Graph**

A fly wanders around on a graph G with vertices $V = \{1, \ldots, 5\}$, shown in Figure 3.

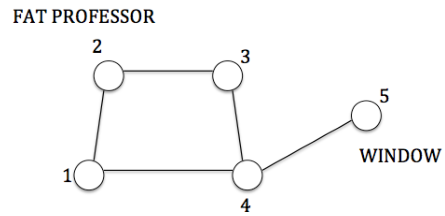


Figure 3: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node i at time n, then it chooses one of its neighbors j of i uniformly at random, and then wanders to node j at time $n+1$. For times $n = 0, 1, 2, \ldots$, let X_n be the fly's position at time n. Argue that $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very fat, so he/she doesn't move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node i at time n, it chooses uniformly from all neighbors of node i except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let Y_n be the fly's position at time n. Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains.

Solution:

(a) Given the position of the fly at time n, the distribution of the position of the fly at time $n+1$ is conditionally independent of the previous positions of the fly before n. Therefore, $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain. We can get the probability transition matrix

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

According to $\pi P = \pi$, we get the invariant distribution

$$\pi = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \end{bmatrix}.$$

(b) Let p be the probability that the fly escapes through the window supposing that it starts at node 1. According to symmetry, starting from node 3, the probability that the fly escapes through the window is also p. Let q be the probability that the fly escapes through the window supposing that it starts at node 4. We have

$$\begin{aligned} p &= \frac{1}{2}(0+q), \\ q &= \frac{1}{3}(1+p+p). \end{aligned}$$

Then we get $p = 1/4$.

(c) No, it is not a Markov chain. According to the definition of the process

$$\mathbb{P}(Y_{n+1} = 1 \mid Y_n = 4, Y_{n-1} = 1) = 0,$$

while

$$\mathbb{P}(Y_{n+1} = 1 \mid Y_n = 4, Y_{n-1} = 3) = 0.5.$$

Therefore, given Y_n, Y_{n+1} and Y_{n-1} are not conditionally independent. Then the process $\{Y_n,\ n \in \mathbb{N}\}$ is not a Markov chain.

4. Product of Rolls of a Die

A fair die with labels (1 to 6) is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

Solution:

We model this process as a Markov chain with 3 states. The states correspond to the outcome of the last roll. If the last outcome is 1 or 5, it is useless for getting a product of 12, and we say that the Markov chain is in state s_1. If the last outcome is one of 2, 3, 4, or 6, the outcome is useful, and we say that the Markov chain is in state s_2. If the product of the last two rolls is 12, we say that the Markov chain is in state s_3. Then the probability transition matrix is

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let T_i be the expected number of rolls that is needed to get to state s_3, starting from state s_i, $i = 1, 2$. Then we have

$$T_1 = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2,$$
$$T_2 = 1 + \frac{1}{3}T_1 + \frac{1}{2}T_2.$$

Solving the equations, we get $T_1 = 10.5$ and $T_2 = 9$. Then the expected number of rolls is

$$T = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2 = 10.5.$$

5. **Ant**

An ant is walking on the non-negative integers. At each step, the ant moves forward one step with probability $p \in (0, 1)$, or slides back down to 0 with probability $1 - p$. What is the average time it takes for the ant to get to n, where n is a positive integer, starting from state 0?

Solution:

Let $\beta(m)$ be the average time to reach the n starting from m, $m \in \{0, 1, 2, \ldots, n\}$. The FSE are as follows:

$$\beta(m) = 1 + p\beta(m+1) + (1-p)\beta(0), \text{ for } m \in \{0, 1, \ldots, n-1\}$$
$$\beta(n) = 0$$

Make the substitutions

$$a = \frac{1}{p},$$
$$b = -\frac{1}{p} - \frac{(1-p)\beta(0)}{p}$$

and notice that $\beta(m+1) = a\beta(m) + b$. Thus, we have:

$$\beta(m) = a^m\beta(0) + \frac{1-a^m}{1-a}b$$

Since $\beta(n) = 0$, we have the equation:

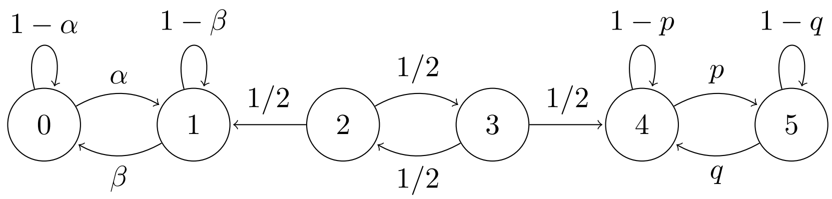
$$a^n\beta(0) + \frac{1-a^n}{1-a}b = 0$$

Thus,

$$\beta(0) = \frac{1-p^n}{p^n - p^{n+1}}.$$

6. **Reducible Markov Chain**

Consider the following Markov chain, for $\alpha, \beta, p, q \in (0, 1)$.



(a) What are all of the communicating classes? (Two nodes x and y are said to belong to the same communicating class if x can reach y and y can reach x through paths of positive probability.) For each communicating class, classify it as recurrent or transient.

(b) Given that we start in state 2, what is the probability that we will reach state 0 before state 5?

(c) What are all of the possible stationary distributions of this chain? (Note that there is more than one.)

(d) Suppose we start in the initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0,1]$. Does the distribution of the chain converge, and if so, to what?

Solution:

(a) The communicating classes are $\{0,1\}$ (recurrent), $\{4,5\}$ (recurrent), and $\{2,3\}$ (transient).

(b) Let T_0 and T_5 denote the time it takes to reach states 0 and 5 respectively. (Note that exactly one of T_0 and T_5 will be finite.) We are looking to compute $\mathbb{P}_2(T_0 < T_5)$, and we can set up hitting equations:

$$\begin{aligned} \mathbb{P}_2(T_0 < T_5) &= \frac{1}{2} + \frac{1}{2}\mathbb{P}_3(T_0 < T_5), \\ \mathbb{P}_3(T_0 < T_5) &= \frac{1}{2}\mathbb{P}_2(T_0 < T_5). \end{aligned}$$

Thus, $\mathbb{P}_2(T_0 < T_5) = 2/3$.

(c) First we observe that no stationary distribution can put positive probability on a transient state, so the stationary distribution is supported on the states $\{0,1,4,5\}$. Next, if we restrict our attention to only the states $\{0,1\}$, then we have an irreducible Markov chain with stationary distribution

$$\pi_1 := \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \end{bmatrix},$$

and similarly, if we restrict our attention to only the states $\{4,5\}$, then again we have an irreducible Markov chain with stationary distribution

$$\pi_2 := \frac{1}{p+q}\begin{bmatrix} q & p \end{bmatrix}.$$

Any stationary distribution for the entire chain must be some convex combination of these two stationary distributions. Explicitly, the stationary distributions are of the form

$$\pi = \begin{bmatrix} \frac{c\beta}{\alpha+\beta} & \frac{c\alpha}{\alpha+\beta} & 0 & 0 & \frac{(1-c)q}{p+q} & \frac{(1-c)p}{p+q} \end{bmatrix} \tag{1}$$

for some $c \in [0,1]$.

(d) Indeed the distribution will converge, even though we do not have irreducibility. The intuition is as follows. The probability will leak out of the transient states $\{2,3\}$ until all of the probability mass is supported on the recurrent states. The two recurrent classes can each be considered to be an irreducible aperiodic Markov chain and so the probability mass which enters a recurrent class will settle into equilibrium. To aid us in finding the limiting distribution, we can use the results of Part (b). With probability γ, we start in state 2, and with a further probability $2/3$ we end up in the recurrent class $\{0,1\}$. By symmetry, the probability that we end up in $\{0,1\}$ starting form state 3 is $1/3$. Thus, the total probability mass which settles into the recurrent class $\{0,1\}$ is $2\gamma/3+(1-\gamma)/3 = 1/3+\gamma/3$. Then, the probability mass settling in the recurrent class $\{4,5\}$ is $2/3-\gamma/3$. Therefore, the chain converges to the stationary distribution in (1) with $c = 1/3+\gamma/3$.