EECS 126 Probability and Random Processes
Spring 2018 Kannan Ramchandran **HW 8**

1 Basketball II

The point difference between the Team A and Team B can be modeled as a continuous time Markov chain, with $2k+1$ states for point differences from $-k$ to k.

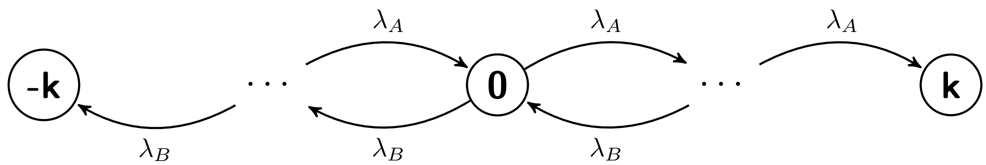

Figure 1: CTMC modeling point difference between Team A and Team B

We can convert this to a DTMC by normalizing the rates to probabilities by dividing by $\lambda_A + \lambda_B$. A self-loop for the recurrent states $-k$ and k must be added to maintain the balance equations for the resulting DTMC. Using $p = \frac{\lambda_A}{\lambda_A+\lambda_B}$ and $(1-p) = \frac{\lambda_B}{\lambda_A+\lambda_B}$ as a result, we get:

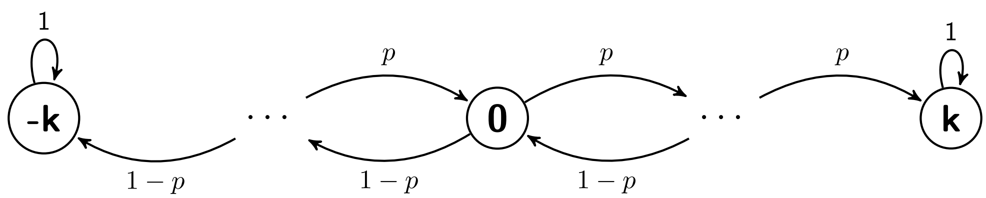

Figure 2: DTMC modeling point difference between Team A and Team B

We can now find $\alpha(0) = \mathbb{P}(T_k < T_{-k} \mid X_0 = 0)$ from the first step equations. We have $\alpha(-k) = 0$ and $\alpha(k) = 1$, and $\alpha(i) = (1-p)\alpha(i-1) + p\alpha(i+1)$, for $i = -k+1, \ldots, k-1$. This can be rewritten as $(1-p)(\alpha(i) - \alpha(i-1)) = p(\alpha(i+1) - \alpha(i))$, and letting $\delta_i = \alpha(i+1) - \alpha(i)$ and $\rho = \frac{1-p}{p}$, we have $\delta_i = \rho\delta_{i-1}$. Since $\sum_{i=-k}^{k-1} \delta_i = \alpha(k) - \alpha(-k) = 1$, we get $\sum_{i=0}^{2k-1} \delta_{-k}\rho^i = 1$ so that $\delta_{-k} = \frac{1}{\sum_{i=0}^{2k-1} \rho^i}$. We see that $\alpha(i) = \sum_{j=-k}^{i-1} \delta_j = \sum_{j=0}^{k+i-1} \delta_{-k}\rho^j = \frac{\sum_{j=0}^{k+i-1} \rho^j}{\sum_{j=0}^{2k-1} \rho^j}$. As such, we get $\alpha(0) = \frac{1-\rho^k}{1-\rho^{2k}} = \frac{1-\left(\frac{\lambda_B}{\lambda_A}\right)^k}{1-\left(\frac{\lambda_B}{\lambda_A}\right)^{2k}}$.

2 Taxi Queue

The number of passengers waiting can be modeled as a CTMC as follows.

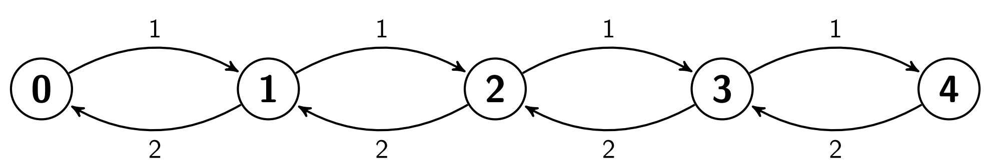

Figure 3: CTMC modeling the number of passengers waiting

The balance equations give us

$$\pi(0) = 2\pi(1)$$

$$\pi(1) = 2\pi(2)$$
$$\pi(2) = 2\pi(3)$$
$$\pi(3) = 2\pi(4)$$

Combining these with our normalization requirement $\sum_{i=0}^{4} \pi(i) = 1$, we have that $\pi(0) = \frac{16}{31}, \pi(1) = \frac{8}{31}, \pi(2) = \frac{4}{31}, \pi(3) = \frac{2}{31}, \pi(4) = \frac{1}{31}$. Since the expected waiting time for John's position in the queue to decrease is $\frac{1}{2}$, we have that his expected waiting time given that he joins the queue is $\frac{\frac{16}{31}\frac{1}{2}+\frac{8}{31}1+\frac{4}{31}\frac{3}{2}+\frac{2}{31}2}{\frac{16+8+4+2}{31}} = \frac{13}{15}$.

3 Two-Server System

We can model the number of broken servers as CTMC.

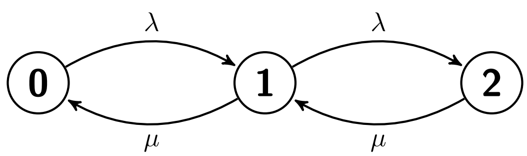

Figure 4: CTMC modeling the number of broken servers

The balance equations give us

$$\pi(2)\mu = \pi(1)\lambda$$
$$\pi(1)\mu = \pi(0)\lambda$$

Combining these with our normalization requirement, $\sum_{i=0}^{2} \pi(i) = 1$, we have that $\left(\frac{\lambda}{\mu}\right)^2 \pi(0) + \frac{\lambda}{\mu}\pi(0) + \pi(0) = 1$, so that $\pi(0) = \frac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2}$.

4 Poisson Queues

(a) The queue-length is a Markov chain since the arrivals are independent of the queue-lengths and the service rate only depends on the current queue-length.

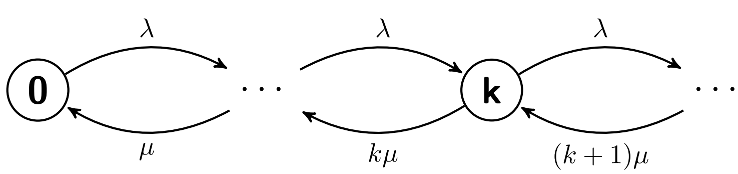

Figure 5: CTMC modeling the customers being served

(b) From the balance equations we have that $\lambda\pi(0) = \mu\pi(1)$, so that $\pi(1) = \frac{\lambda}{\mu}\pi(0)$, and $\pi(i+1) = \frac{\lambda}{(i+1)\mu}\pi(i)$, so $\pi(i) = \frac{\lambda^i}{i!\mu^i}\pi(0)$. From the our normalization constraint, $\sum_{i=0}^{\infty} \pi(i) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!\mu^i}\pi(0) = e^{\frac{\lambda}{\mu}}\pi(0) = 1$, we get that $\pi(0) = e^{-\frac{\lambda}{\mu}}$, so that $\pi(i) = \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!} e^{-\frac{\lambda}{\mu}} \sim \text{Poisson}\left(\frac{\lambda}{\mu}\right)$.

Thus, since the invariant distribution is Poisson $\left(\frac{\lambda}{\mu}\right)$, the chain is positive recurrent for all finite values λ & μ.

5 [Bonus] Jukes-Cantor Model