UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 1

Fall 2017

Self-Graded Scores Due: 5 PM, Monday, September 11, 2017
Submit your self-graded scores via the Google form:
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1. **Events**

   (a) Show that the probability that exactly one of the events A and B occurs is $\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

   (b) If A is independent of itself, show that $\mathbb{P}(A) = 0$ or 1.

   Solution:

   (a) The probability of the event that exactly one of A and B occur is

$$\begin{aligned}\mathbb{P}(A \cap B^{\mathsf{c}}) + \mathbb{P}(A^{\mathsf{c}} \cap B) &= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).\end{aligned}$$

   (b) $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$, so $\mathbb{P}(A) = \mathbb{P}(A)^2$; this implies that $\mathbb{P}(A) \in \{0, 1\}$.

2. **Joint Occurrence**

   You know that, at least one of the events A_r (for $r \in \{1, \dots, n\}$, $n \in \mathbb{Z}_{\geq 2}$) is certain to occur but certainly no more than two occur. Show that if the probability of occurrence of any single event is p, and the probability of joint occurrence of any two distinct events is q, we have $p \geq 1/n$ and $q \leq 2/n$.

   Solution:
   Since $1 = \mathbb{P}(\bigcup_{r=1}^n A_r) \leq \sum_{r=1}^n \mathbb{P}(A_r) = np$, we see that $p \geq 1/n$. Since no more than two events can occur, the inclusion-exclusion formula truncates after we consider pairwise intersections of events, and thus $1 = \mathbb{P}(\bigcup_{r=1}^n A_r) = \sum_{r=1}^n \mathbb{P}(A_r) - \sum_{(r_1,r_2)\in I} \mathbb{P}(A_{r_1} \cap A_{r_2})$, where $I = \{(i, j) \in \{1, \dots, n\}^2 : i < j\}$. Thus, $1 = np - \binom{n}{2}q$, or $\binom{n}{2}q = np - 1 \leq n - 1$ since $p \leq 1$. Writing $\binom{n}{2} = n(n-1)/2$, we see that $q \leq 2/n$.

   There is actually an even better solution. Notice that the events $\{A_i \cap A_j : (i, j) \in I\}$ are pairwise disjoint, so by countable additivity, $1 \geq \mathbb{P}(\bigcup_{(i,j)\in I}(A_i \cap A_j)) = \sum_{(i,j)\in I} \mathbb{P}(A_i \cap A_j) = \binom{n}{2}q$, so $q \leq \binom{n}{2}^{-1} = 2/(n(n-1))$.

3. **Coin Flipping & Symmetry**

Alice and Bob have $2n+1$ fair coins (where $n \in \mathbb{Z}_{>0}$), each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that, after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

Hint: Use symmetry before diving into long calculations.

Solution:

If we let Ω be the sample space consisting of all possible $2n+1$ tosses, then Ω is a uniform probability space by assumption. Define the events

$$\begin{aligned} A &= \{\text{there are more heads in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}, \\ B &= \{\text{there are more tails in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}. \end{aligned}$$

By symmetry, $\mathbb{P}(A) = \mathbb{P}(B)$, and we note that $A \cup B = \Omega$ since it is impossible for the first $n+1$ tosses to have more heads *and* more tails than the last n tosses. So, $\mathbb{P}(A) + \mathbb{P}(B) = 1$ and hence $\mathbb{P}(A) = 1/2$.

4. **Passengers on a Plane**

There are N passengers in a plane with N assigned seats ($N \in \mathbb{Z}_{>0}$), but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.

Hint: Use the inclusion-exclusion principle.

Solution:

First, let us calculate the probability that at least one passenger sits in his or her assigned seat using inclusion-exclusion. Let A_i, $i = 1, \ldots, N$, be the event that passenger i sits in his or her assigned seat. We first add the probabilities of the single events (of which there are N), and the probability of each event is $(N-1)!/N!$ (indeed there are $(N-1)!$ ways to permute the remaining passengers once a specific passenger is fixed, and $N!$ total permutations, so the probability is $(N-1)!/N!$); next, we subtract the probabilities of the pairwise intersections of events (of which there are $\binom{N}{2}$), and the probability of each event is $(N-2)!/N!$ (there are $(N-2)!$ ways to permute the passengers other than the fixed two); continuing on, we see that

$$\mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{j=1}^{N} (-1)^{j+1} \binom{N}{j} \frac{(N-j)!}{N!} = \sum_{j=1}^{N} (-1)^{j+1} \frac{1}{j!}.$$

Now, the event that no passenger sits in his or her assigned seat is the complement of the event just discussed:

$$1 - \mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) = 1 - \sum_{j=1}^{N} (-1)^{j+1} \frac{1}{j!} = \sum_{j=0}^{N} \frac{(-1)^j}{j!}.$$

Taking the limit as $N \to \infty$, the expression converges to $\sum_{j=0}^{\infty}(-1)^j/j!$, and using the expression for the power series of the exponential function, we conclude that the probability converges to $\exp(-1) \approx 0.368$.

5. **Variance**

If $X_1, \dots, X_n$, where $n \in \mathbb{Z}_{>0}$, are i.i.d. random variables with zero-mean and unit variance, compute the variance of $(X_1 + \cdots + X_n)^2$. You may leave your answer in terms of $\mathbb{E}[X_1^4]$, which is assumed to be finite.

Solution:

Notice that $\mathbb{E}[(X_1 + \cdots + X_n)^2] = \operatorname{var}(X_1 + \cdots + X_n) + \mathbb{E}[X_1 + \cdots + X_n]^2 = \sum_{i=1}^{n} \operatorname{var} X_i = n$, where in the last step we used linearity of variance because of the assumed independence of the random variables, and also the fact that they are zero-mean. Next we compute $\mathbb{E}[(X_1 + \cdots + X_n)^4] = \mathbb{E}[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} X_i X_j X_k X_l]$, but notice that for any term in the summation $X_i X_j X_k X_l$, if any of the indices appears exactly once (suppose it is i), then $\mathbb{E}[X_i X_j X_k X_l] = \mathbb{E}[X_i]\,\mathbb{E}[X_j X_k X_l] = 0$ by independence. Hence, the only terms that survive are terms of the form $\mathbb{E}[X_i^4]$ and $\mathbb{E}[X_i^2 X_j^2]$ for distinct indices $i, j \in \{1, \dots, n\}$. There are exactly n terms of the form $\mathbb{E}[X_i^4]$ as i ranges over $1, \dots, n$. To count the second type of term, note that there are $\binom{n}{2}$ ways to choose the two indices i and j, and $\binom{4}{2} = 6$ ways to permute i and j. To illustrate, note that the term $X_1^2 X_2^2$ can arise in one of 6 ways, as $X_1X_1X_2X_2$, $X_1X_2X_1X_2$, $X_1X_2X_2X_1$, $X_2X_1X_1X_2$, $X_2X_1X_2X_1$, and $X_2X_2X_1X_1$. Thus:

$$\begin{aligned}
\mathbb{E}[(X_1 + \cdots + X_n)^4] &= n\,\mathbb{E}[X_1^4] + 6\binom{n}{2}\,\mathbb{E}\big[X_1^2 X_2^2\big] \\
&= n\,\mathbb{E}[X_1^4] + 3n(n-1)\,\mathbb{E}\big[X_1^2\big]\,\mathbb{E}\big[X_2^2\big] \\
&= n\,\mathbb{E}[X_1^4] + 3n(n-1).
\end{aligned}$$

So, the variance is

$$\begin{aligned}
\operatorname{var}\big((X_1 + \cdots + X_n)^2\big) &= \mathbb{E}[(X_1 + \cdots + X_n)^4] - \mathbb{E}[(X_1 + \cdots + X_n)^2]^2 \\
&= n\,\mathbb{E}[X_1^4] + 3n(n-1) - n^2.
\end{aligned}$$

6. **Poisson Practice**

Suppose X is a Poisson random variable with parameter λ. Find:

(a) $\mathbb{E}[X^2]$.

(b) $\mathbb{P}(X \text{ is even})$. (*Hint*: Use the Taylor series expansion of e^x.)

Solution:

(a) First compute

$$\begin{aligned}
\mathbb{E}[X(X-1)] &= \sum_{x=2}^{\infty} x(x-1)\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=2}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!} = \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^{-\lambda}}{(x-2)!} \\
&= \lambda^2.
\end{aligned}$$

Hence, $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \lambda^2 + \lambda$.

(b) Note that

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\begin{aligned}
\mathbb{P}(X \text{ is even}) &= \sum_{k=0}^{\infty} \mathbb{P}(X = 2k) = \sum_{k=0}^{\infty} \frac{\lambda^{2k} \mathrm{e}^{-\lambda}}{(2k)!} \\
&= \frac{\mathrm{e}^{-\lambda}}{2} \Big(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \Big) = \frac{\mathrm{e}^{-\lambda}}{2} (\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}) \\
&= \frac{1 + \mathrm{e}^{-2\lambda}}{2}.
\end{aligned}
$$

To explain the second line, note that the odd terms cancel out and the even terms are counted twice.