Monotone scheme for diffusion type problems with heterogeneous anisotropic tensor on unstructured meshes

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Practical requirements for a discretization

- be locally conservative;
- preserve positivity of the differential solution;
- be applicable to unstructured, anisotropic, and severely distorted meshes;
- be applicable to heterogeneous full diffusion tensors;
- result in sparse systems with minimal number of non-zero entries.

Scheme description

Let us consider computational domain Ω and its partition $\mathcal{T}$ into triangles . The diffusion equation in the mixed form is

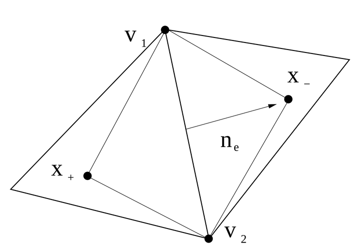
$$\operatorname{div} \mathbf{q} = f \qquad \mathbf{q} = -\mathbb{D} \operatorname{grad} C.$$

The conventional approach to the construction of cell-centered finite volume (FV) schemes is based on the cellwise application of the Green formula which results in the equation:

$$\sum_{e \subset \partial T} \mathbf{q}_e \cdot \mathbf{n}_e = \int_T f\,dx, \quad \forall T \in \mathcal{T}, \qquad \text{where } \mathbf{q}_e = \frac{\int_e \mathbf{q}\,ds}{|e|} \text{ and } |\mathbf{n}_e| = |e|. \tag{1}$$

Cell-centered finite volume discretizations assume that one degree of freedom C_T for C is assigned to each cell T. For each cell T a reference point $\mathbf{x}_T$ is defined (see the definition below) where C is approximated.

Flux definition



Let us consider an internal edge e connecting vertices $\mathbf{v}_1$ and $\mathbf{v}_2$. Points $\mathbf{x}_+$ and $\mathbf{x}_-$ are reference points which correspond to cells sharing the edge e. The two-point nonlinear flux approximation [C. LePotier, 2005] is derived on the basis of the Green formula and quadrature rules:

$$\mathbf{q}_e \cdot \mathbf{n}_e = A_e^+(C_{v_1}, C_{v_2})\, C_{x_+} - A_e^-(C_{v_1}, C_{v_2}) C_{x_-} \tag{2}$$

Interpolation

The values of C at vertices are involved in the flux coefficients in (2). For an approximation of C_{v_1} and C_{v_2} two interpolation techniques may be adopted.

- *linear* (preferable for smooth solutions). Let $\mathbf{x}_{T_j}$, $j = 1, 2, 3$ be three closest reference points to $\mathbf{v}_i$, such that triangle $\mathbf{x}_{T_1}\mathbf{x}_{T_2}\mathbf{x}_{T_3}$ contains $\mathbf{v}_i$. Then

$$C_{v_i} = \sum_{j=1}^{3} \lambda_j C_{\mathbf{x}_{T_j}} \qquad \text{where } \lambda_j \text{ are the barycentric coordinates of } \mathbf{v}_i.$$

- *inverse distance weighting* [D. Shepard, 1968] (preferable for nonsmooth solutions). The interpolation uses values $C_{\mathbf{x}_T}$ of all triangles sharing $\mathbf{v}_i$:

$$C_{v_i} = \sum_{T_j \ni \mathbf{v}_i} w_j C_{\mathbf{x}_{T_j}} \qquad \text{where } w_j = \frac{|\mathbf{v}_i - \mathbf{x}_{T_j}|^{-1}}{\sum_{T_j \ni \mathbf{v}_i} |\mathbf{v}_i - \mathbf{x}_{T_j}|^{-1}}.$$

Reference point position

For any triangle $T = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ and for any diffusion tensor $\mathbb{D}$ the point $\mathbf{x}_T$ is defined as follows :

$$\mathbf{x}_T = \frac{\sum_{i=1}^{3} \mathbf{v}_i |\mathbf{n}_{\alpha(i)}|_{\mathbb{D}}}{\sum_{i=1}^{3} |\mathbf{n}_{\alpha(i)}|_{\mathbb{D}}} \qquad \text{where } |\mathbf{n}_{\alpha(i)}|_{\mathbb{D}} = (\mathbb{D}\mathbf{n}_{\alpha(i)} \cdot \mathbf{n}_{\alpha(i)})^{1/2} \tag{3}$$

and $\mathbf{n}_{\alpha(i)}$ is the outward normal to the edge $e_{\alpha(i)}$ opposite to the vertex $\mathbf{v}_i$.

Iterative solution

The substitution of flux (2) into equation (1) yields the nonlinear system:

$$\mathbf{A}(C)C = F. \tag{4}$$

We use the simplest Picard iterations: initial vector $C^0 \in \Re^{N_\mathcal{T}}$, $C^0 \geq 0$,

$$\text{solve} \quad \mathbf{A}(C^{k-1})C^k = F, \qquad k = 1, 2, \ldots \tag{5}$$

Matrix $\mathbf{A}(C^{k-1})$ has at most four non-zeros in each row.

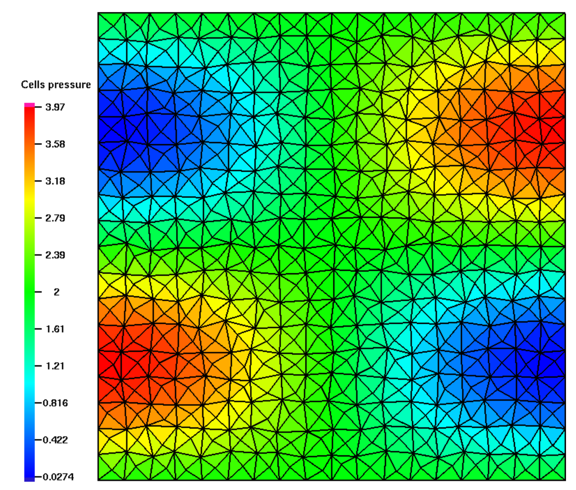
Theoretical result

Let $\mathbf{x}_{T_i}$ be defined by (3) and $F_{T_i} \geq 0$, $C^0_{T_i} \geq 0$, $i = 1, \ldots, N_\mathcal{T}$. Then all iterates C^k of method (5) are non-negative:

$$C^k_{T_i} \geq 0, \quad i = 1, \ldots, N_\mathcal{T},\ k = 1, 2, \ldots$$

The proof is based on the fact that if $\mathbf{x}_{T_i}$ are defined by (3) then A_e^+ and A_e^- in (2) are positive.

Numerical Test I: smooth solution



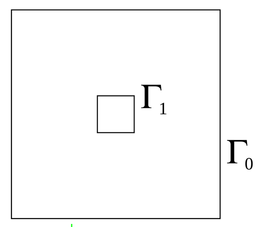
$$C(x, y) = 2cos(\pi x)sin(2\pi y) + 2$$

$$-\operatorname{div} \operatorname{grad} C + C = f \qquad \text{in } \Omega = [0;1]^2$$

h	$\|\|C - C_h\|\|_2$	$\|\|\mathbf{q} - \mathbf{q}_h\|\|_2$
1/18	$9.43e-3$	$3.25e-2$
1/36	$2.33e-3$	$8.48e-3$
1/72	$6.00e-4$	$2.73e-3$
1/144	$1.57e-4$	$9.17e-4$

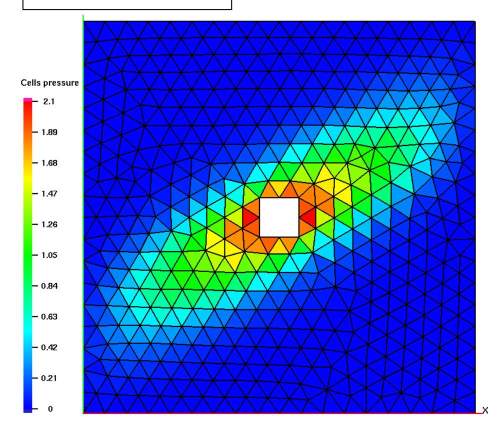
- Second order convergence in discrete L_2 norms for both C and $\mathbf{q}$ when the solution is smooth.

Numerical Test II: anisotropic solution



$$-\operatorname{div} \mathbb{D} \operatorname{grad} C = 0 \quad \text{in } \Omega, \quad C = 0 \text{ on } \Gamma_0, \quad C = 2 \text{ on } \Gamma_1.$$

$$\mathbb{D} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

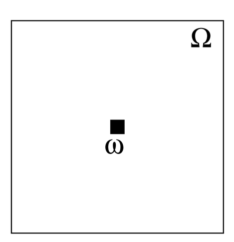


$$k_1 = 100 \qquad k_2 = 1 \qquad \theta = \pi/6$$

h	$\|\|C - C_h\|\|_\infty$	$\|\|C - C_h\|\|_2$
1/18	$1.11e-0$	$9.70e-2$
1/36	$8.66e-1$	$5.11e-2$
1/72	$6.04e-1$	$2.53e-2$
1/144	$3.38e-1$	$1.22e-2$

- First order convergence in the discrete L_2 norm for C variable when the solution is nonsmooth and highly anisotropic.

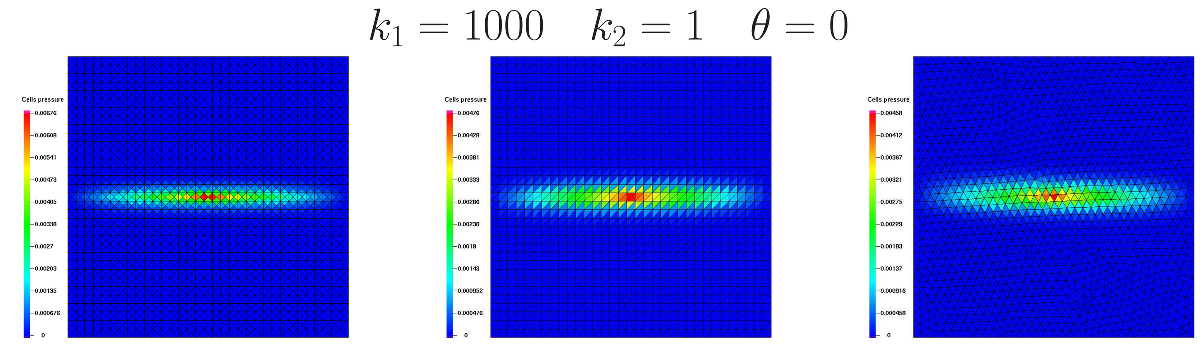
Numerical Test III: positivity



$$-\operatorname{div} \mathbb{D} \operatorname{grad} C = f \quad \text{in } \Omega, \qquad C = 0 \text{ on } \partial\Omega \qquad \Omega = [0,1]^2$$

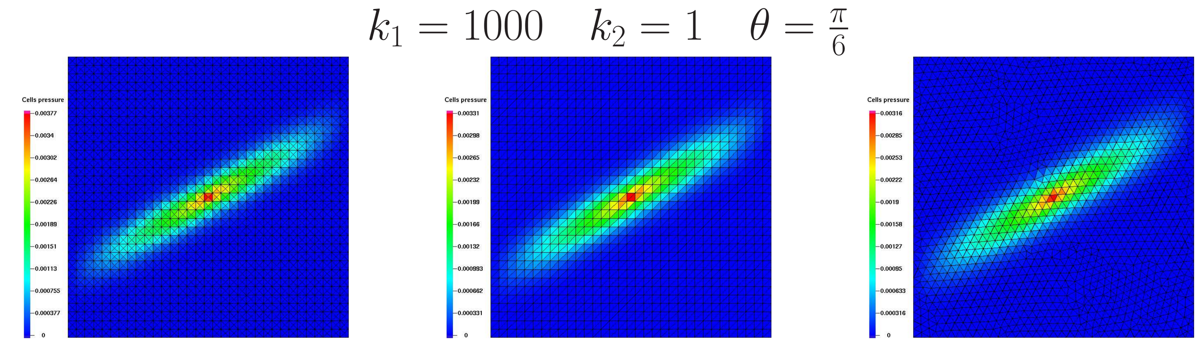
$$f = \begin{cases} \frac{1}{|\omega|} & \text{in } \omega \\ 0 & \text{otherwise} \end{cases} \qquad \omega = [0.5 - h/2, 0.5 + h/2]^2$$

$k_1 = 1000 \quad k_2 = 1 \quad \theta = 0$



$C_{min} = 0.$ $C_{min} = 0.$ $C_{min} = 0.$

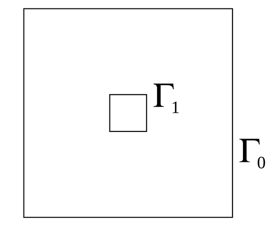
$k_1 = 1000 \quad k_2 = 1 \quad \theta = \frac{\pi}{6}$



$C_{min} = 0.$ $C_{min} = 0.$ $C_{min} = 0.$

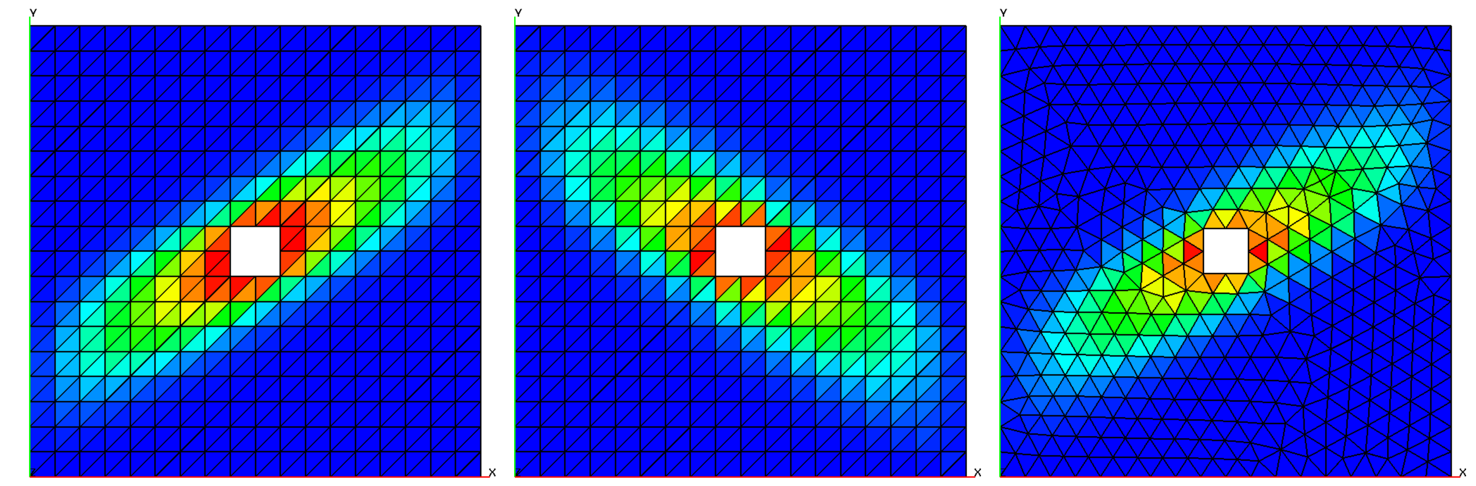
- The solution remains non-negative on different types of meshes and for different directions of anisotropy.
- Both RT_0 and P_1 methods produce negative C_{min} in all cases.

Overshooting



$$\begin{aligned} -\operatorname{div} \mathbb{D} \operatorname{grad} C &= 0 && \text{in } \Omega && \Omega = [0,1]^2 \\ C &= 0 && \text{on } \Gamma_0 \\ C &= 2 && \text{on } \Gamma_1 \end{aligned}$$

- In the case of highly anisotropic solution overshoots may occur.
- The overshoot values are sensitive to the choice of an interpolation technique.
- Overshooting depends on a mutual orientation of an anisotropic solution and mesh edges.



C_{max}	Interpolation	
k1/k2	I_1	I_2
10	1.96	1.96
1000	1.98	1.98

C_{max}	Interpolation	
k1/k2	I_1	I_2
10	1.97	1.97
1000	2.95	2.05

C_{max}	Interpolation	
k1/k2	I_1	I_2
10	1.99	1.99
1000	2.75	2.14

I_1 – linear interpolation

I_2 – inverse distance weighting interpolation

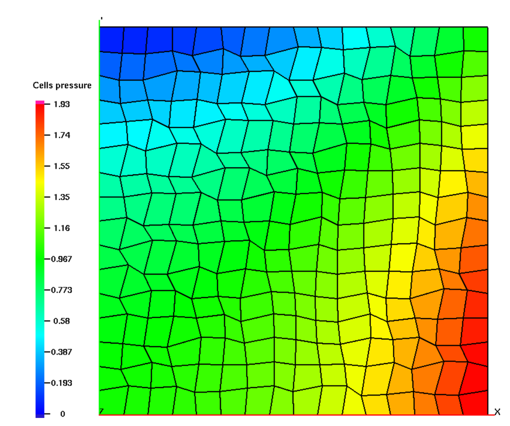
- Inverse distance weighting interpolation is more stable in the case of highly anisotropic solutions.
- L_2 norm of the overshoot error $(max\{C_h - 2\,,\,0\})$ behaves like $O(h)$.

Extension to quadrilateral meshes

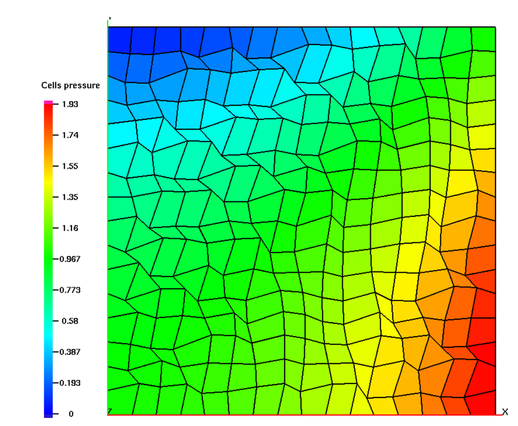
- $\mathcal{Q}$ is a comformal quadrilateral mesh with shape regular cells
  $$diam(Q)/\rho(Q) \leq const \quad \forall Q \in \mathcal{Q}.$$
- $\mathbf{x}_Q \in Q$ is a point separated from ∂Q for which Q is star-shaped. For convex or slightly non-convex cells $\mathbf{x}_Q$ may be the mass center.
- $\mathbb{D}$ is isotropic and heterogeneous on $\mathcal{Q}$
- Each edge may be artificially extended so that coefficients A_e^+ and A_e^- in (2) are positive. Hence the scheme is monotone.

$$C(x, y) = x^2 - y^2 + 1$$

$$-\operatorname{div} \operatorname{grad} C + C = f \qquad \text{in } \Omega = [0;1]^2$$



less distorted mesh



more distorted mesh

h	$\|\|C - C_h\|\|_2$	$\|\|\mathbf{q} - \mathbf{q}_h\|\|_2$
1/16	$9.87e-4$	$2.70e-3$
1/32	$2.55e-4$	$7.23e-4$
1/64	$6.38e-5$	$2.27e-4$
1/128	$1.59e-5$	$7.59e-5$

h	$\|\|C - C_h\|\|_2$	$\|\|\mathbf{q} - \mathbf{q}_h\|\|_2$
1/16	$4.14e-3$	$3.18e-2$
1/32	$2.44e-3$	$2.17e-2$
1/64	$1.33e-3$	$1.88e-2$
1/128	$6.99e-4$	$1.16e-2$

- The convergence rate is affected by the distortion degree.
- Matrix $\mathbf{A}(C^{k-1})$ from (5) has at most five nonzeros in each row.