Volatility-Managed Portfolio Performance

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Abstract

This paper studies the performance of volatility-managed portfolios. Using daily return data of the five Fama-French factor portfolios, covering the period from July 1963 to March 2023, we find that the realized-variance, GARCH and GJR-GARCH portfolios outperform the 1/N portfolio under the assumption of normally distributed error terms. Under Student's t-distributed error terms, the GAS portfolio outperforms the GARCH portfolio. But, none of the filters significantly outperforms any other filter under the assumption of generalized t-distributed error terms. Lastly, allowing for short-selling constraints does not lead to better portfolio construction performance in the mean-variance and minimum-variance portfolios.

1 Introduction

In this paper we aim to compare the performance of volatility-managed portfolios to the $1/N$ portfolio by using daily returns of the five Fama-French factor portfolios covering July 1963 to March 2023. Moreover, we analyze the use of the mean-variance portfolio (MVP) under different portfolio constraints for portfolio construction.

For managing volatility, we use several models: generalized autoregressive conditional heteroskedasticity (GARCH), assymetric GARCH (GJR-GARCH) and autoregressive score (GAS). We evaluate the effectiveness of these models based on portfolio performance and volatility forecast accuracy, using different risk-return measures such as the Sharpe ratio and Diebold-Mariano test, respectively. Finally, to examine the effect of the sample size on the portfolio construction performance of the different models, all volatility models are fitted using a rolling-window approach with two different sample sizes.

Moreira and Muir (2017) have shown that portfolios where assets are given a weight that is inversely proportional to their historical volatility outperform the $1/N$ portfolio. We will both replicate this study and use more advanced modelling techniques to forecast volatility. We add value by using the GAS filter for this purpose, which has not been used in the literature to date. Furthermore, DeMiguel et al. (2007) argue that the MVP does not outperform the $1/N$ portfolio. Nevertheless, Kirby and Ostdiek (2012) argue that this is due to the extreme target returns used in their optimization problems, which targets annual returns over 100 percent. Using Kirby and Ostdiek (2012) suggestion of targeting past $1/N$ portfolio returns instead, we check this statement using more recent data.

This paper is organised as follows. In section 2 we describe the data. The methodology is explained in section 3. After that, we present the results in section 4. This paper is concluded in section 5.

2 Data

We use daily percentage returns on the five Fama-French factors covering the period from July 1963 to March 2023, which can be found on Kenneth French's website. The five factors consist of excess market return (Mkt-RF, or 'market minus risk-free'), size (SMB, or 'small minus big'), value (HML, or 'high minus low'), momentum probability (RMW, or 'robust minus weak') and investment (CMA, or 'conservative minus aggressive'). In this way, all factors can be interpreted as long-short portfolios that do not require initial investment. Finally, we use the three-month US treasury bill as a proxy of the risk-free rate (RF).

Table 1 presents summary statistics for the daily percentage returns over the sample period. We observe that the average return and the standard deviation for the Mkt-RF factor is the highest, the latter of which can be interpreted as the average daily volatility, indicating that this factor has both the highest risk and highest return. The average return on the risk-free rate is the second highest. However, note that this factor does not represent a budget-neutral factor portfolio, contrary to the other factor portfolios. In other words, an investor must invest capital in order to go long on the risk-free rate.

Table 1 Summary statistics of daily percentage returns for the five Fama-French factors and risk-free rate, from 1963 to 2023

Statistic	N	Mean	St. Dev.	Min	Median	Max
Mkt-RF	15041	0.0268	1.0251	−17.4400	0.0500	11.3500
SMB	15041	0.0073	0.5414	−11.1900	0.0200	6.1700
HML	15041	0.0144	0.5817	−5.0000	0.0100	6.7400
RMW	15041	0.0137	0.3983	−3.0100	0.0100	4.5200
CMA	15041	0.0135	0.3780	−5.8700	0.0100	2.5300
RF	15041	0.0172	0.0126	0.0000	0.0180	0.0610

3 Methodology

In this section, we explain how we construct and evaluate the portfolios. Firstly, we explain the *benchmark* portfolio and how the volatility-managed portfolios are constructed. Secondly, we discuss the filters that are used to model the conditional volatility, under three different innovation distributions. Thirdly, we explain how the portfolio construction and volatility prediction performance of the models is evaluated. Finally, we outline the implementation of our methods.

3.1 Portfolio strategies

We take the perspective of a portfolio manager who is responsible for managing the risk of a portfolio of N assets. $r_t^{(i)}$ is the return in month t on asset i, $i \in N$. Furthermore, we assume that the portfolio manager has to decide the weight of each asset i at the start of each month t, $w_t^{(i)}$, such that the combination represents the portfolio P. In turn, these weights will determine the returns of portfolio P realized in month t by

$$r_t^P = \sum_{i=1}^{N} w_t^{(i)} r_t^{(i)}. \tag{1}$$

The weight given to each asset i at time t depends on the portfolio strategy that is considered, which we discuss next.

3.1.1 Benchmark portfolio

The *benchmark* portfolio is the $1/N$ portfolio, or equally-weighted (EW) portfolio, in which the portfolio manager's wealth is allocated proportionally to each of the N assets. In other words, at the start of month t , the manager gives each asset a weight $w_t^{(i)} = 1/N$ such that the monthly portfolio return is given by equation (1).

DeMiguel et al. (2007) compare this benchmark strategy with 14 other portfolio strategies and shows that the $1/N$ portfolio consistently outperforms, also across different subsamples. This is our main motivation to use the $1/N$ portfolio as benchmark. Finally, we assume that the portfolio manager re-balances its portfolio monthly such that each asset is given an equal weight at the beginning of the month.

3.1.2 Volatility-managed portfolio

Secondy, we consider volatility-managed portfolios in which the weights are inversely proportional to the expected variance over the holding period h in working days. Let the index t count days, such that the d-day-ahead expected variance for asset i is denoted by $(\hat{\sigma}_{t+d|t}^{(i)})^2$. Then, the *total expected variance* for the next h days is given by

$$\sum_{d=1}^{h}(\hat{\sigma}_{t+d|t}^{(i)})^2. \tag{2}$$

Therefore, at the start of each month, the portfolio manager predicts the total expected variance for the next h holding days, where we take $h = 22$. Now, equation (2) can be used to construct the weights for the volatility-managed portfolios by

$$w_t^{(i)} = \frac{1}{\sum_{d=1}^{h}(\hat{\sigma}_{t+d|t}^{(i)})^2}\left[\sum_{j=1}^{N}\frac{1}{\sum_{d=1}^{h}(\hat{\sigma}_{t+d|t}^{(j)})^2}\right]^{-1} \tag{3}$$

which can be used in equation (1) to get the monthly returns of portfolio P.

One approach we consider to manage volatility is to assume that the expected variance at each day $t+d$ for the coming month is the same as the average realized variance of that asset during the previous month, such that

$$\left(\sigma_{t+d|t}^{(i)}\right)^2 = \frac{1}{22}\sum_{j=0}^{21}\left(r_{t-j}^{(i)} - \frac{\sum_{k=0}^{21} r_{t-k}^{(i)}}{22}\right)^2, \quad \forall 1 \leq d \leq h. \tag{4}$$

This can be used together with equation (3) to construct the weights of the *realized variance* (RV) portfolio. However, note that we implicitly assume here that the volatility per asset is constant, which is not realistic. Therefore, in section 3.2, we consider different techniques to model and forecast volatility.

3.1.3 Markowitz's modern portfolios

The volatility-managed portfolios discussed in the previous section ignore information about the correlation between assets. Therefore, we may consider some of Markowitz (1952) modern portfolios. Specifically, we consider the mean-variance portfolio and the global minimum-variance portfolio.

Let $\boldsymbol{R_{t+1}}$ be the $N \times 1$ vector of risky asset returns at month $t+1$. Then, let $\boldsymbol{\mu} = \mathbb{E}_t[\boldsymbol{R_{t+1}}]$ and $\boldsymbol{\Sigma_t} = \mathbb{E}_t[(\boldsymbol{R_{t+1}} - \boldsymbol{\mu})(\boldsymbol{R_{t+1}} - \boldsymbol{\mu})']$ the covariance matrix of $\boldsymbol{R_{t+1}}$ conditional on the information in month t. Then, at the end of each month t, the portfolio manager solves the mean-variance portfolio (MVP) problem

$$\begin{aligned} \min_{\mathbf{w}_t} \quad & \mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t \\ \text{s.t.} \quad & \mathbf{w}_t'\boldsymbol{\mu} + (1 - \mathbf{w}_t'\mathbf{1})R_f = \mu_p, \end{aligned} \tag{5}$$

where $\mathbf{1}$ denotes a $N \times 1$ vector of ones, R_f the monthly risk-free rate and $\mathbf{w}_t$ a $N \times 1$ vector of portfolio weights for the risky assets. Thus, the weight attributed to the risk-free rate is $1 - \mathbf{w}_t'\mathbf{1}$. The variance-covariance matrix, $\boldsymbol{\Sigma}_t$, is estimated as

$$\hat{\sigma}_{ij,t} = \sum_{n=1}^{22} \left(r_{t-n}^{(i)} - \mu^{(i)} \right) \left(r_{t-n}^{(j)} - \mu^{(j)} \right), \tag{6}$$

where $r_t^{(i)}$ and $r_t^{(j)}$ denote the daily returns at the time t. $\mu^{(i)}$ is the average daily percentage return of asset i over the past $h = 22$ days and, thus, different from $\boldsymbol{\mu}$ which is the cumulative return over the past 22 days.

The MVP is one of the portfolios shown by DeMiguel et al. (2007) to not outperform the $1/N$ portfolio. It is argued that this under-performance is due to the estimation error, especially in the mean μ, hence also in the covariance matrix. Jagannathan and Ma (2003) argue that the problem of estimation errors can be alleviated by imposing short-selling constraints, even if those constraints are wrong. Therefore, we also examine the mean-variance optimization problem by adding the constraints $0 \leq w_{i,t} \leq 1$ and $-1 \leq w_{i,t} \leq 1$ separately.

On the other hand, Kirby and Ostdiek (2012) argue that this under-performance of the MVP with respect to the EW portfolio is only due to setting extreme target returns, μ_p. To illustrate, DeMiguel et al. (2007) target annualized returns over 100 percent, leading to both excess risk-taking and magnifying the existing estimation errors. Kirby and Ostdiek (2012) show that the MVP outperforms the EW portfolio when the MVP is implemented by targeting the conditional expected return of the $1/N$ portfolio. Using more recent data, we investigate this statement by setting the target return in the MVP each month equal to the cumulative returns on the EW and market factor portfolio over the past 22 days, respectively.

Finally, we also consider the global minimum-variance portfolio (GMVP) to solve the problem of estimation errors in the sample covariance matrix. It ignores the (extreme) target return μ_p, and only focuses on risk. More specifically, for the GMVP the portfolio manager solves the optimization problem in equation (5) by relaxing the MVP constraint but ensuring that the portfolio weights of the risky assets sum to one.

3.2 Volatility modelling

In Section 3.1.2, we examined the portfolio manager's approach to constructing volatility-managed portfolios. Additionally, we described the realized variance (RV) portfolio and its limitation of assuming constant volatility for each asset. To this extent, we propose three distinct models, referred to as filters, for estimating conditional variance. By employing these filters, we can forecast the total expected variance for the upcoming month. Subsequently, the weights for each asset $i \in N$ can be derived using Equation (3).

3.2.1 GARCH

First, we consider the generalized autoregressive conditional heteroskedasticity (GARCH) model of order (1,1). Under the assumption of the dynamic model for daily returns $r_t = \mu + \sigma_t \epsilon_t$, we use the representation given by

$$\hat{\sigma}_{t+1}^2 = (1 - \alpha - \beta)\bar{\sigma}^2 + \alpha \hat{\sigma}_t^2 \hat{\epsilon}_t^2 + \beta \hat{\sigma}_t^2, \tag{7}$$

where $\hat{\sigma}_{t+1}^2$ is known at time t and $\bar{\sigma}^2$ is the unconditional expectation of $\hat{\sigma}_t^2$, or

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}. \tag{8}$$

Furthermore, to ensure that variable on the right-hand side is nonnegative, we impose $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$. Besides that, stationarity is achieved by imposing $\alpha + \beta < 1$, which will be necessary to derive a closed form expression for the unconditional expectation.

More specifically, the total expected variance for the next h holding period days can be presented as

$$\sum_{d=1}^{h} \hat{\sigma}_{t+d|t}^2 = h\bar{\sigma}^2 + \frac{1 - (\alpha + \beta)^h}{1 - \alpha - \beta} \left(\hat{\sigma}_{t+1}^2 - \bar{\sigma}^2\right). \tag{9}$$

3.2.2 GJR-GARCH

The GARCH filter in the previous section assumes that positive and negative shocks $\hat{\epsilon}_t$ have the same effect on the conditional variance $\hat{\sigma}_{t+1}^2$. The asymmetric filter by Glosten et al. (1993) (GJR-GARCH) assumes that the effects of these shocks also depends on whether the shock is positive or negative. The GJR-GARCH filter is given by

$$\hat{\sigma}_{t+1}^2 = \omega + (\alpha_{pos} \mathbf{1}_{\hat{\epsilon}_t \geq 0} + \alpha_{neg} \mathbf{1}_{\hat{\epsilon}_t < 0}) \hat{\sigma}_t^2 \hat{\epsilon}_t^2 + \beta \hat{\sigma}_t^2, \tag{10}$$

where $\mathbf{1}_A$ equals 1 when A is true and 0 otherwise. Furthermore, in order for the right-hand side to be positive, we restrict the parameters such that $\alpha_{pos} \geq 0$, $\alpha_{neg} \geq 0$, $\omega > 0$ and $\beta \geq 0$. To ensure stationarity, we set $\alpha_{pos}/2 + \alpha_{neg}/2 + \beta < 1$. Finally, the unconditional expectation and the total expected variance of the GJR-GARCH model is the same as in equation (8) and (9), respectively, with α substituted by $\alpha_{pos}/2 + \alpha_{neg}/2$.

3.2.3 GAS

Depending on the assumed distribution for the error terms, ϵ_t, alternative filters may be more suitable than the GARCH(1,1) filter. Specifically, when the error term follows a Student's t-distribution or a generalized t-distribution, a generalized autoregressive score (GAS) model becomes a more appropriate choice. Unlike the GARCH(1,1) filter, which incorporates $\hat{\epsilon}_t^2$, the GAS model accounts for the possibility of large values in the implied error term, $\hat{\epsilon}_t^2$, due to fat-tailed distributions. Robustness to outliers is a notable advantage of the GAS model over the GARCH(1,1) filter. This alternative filter is formulated as

$$\hat{\sigma}_{t+1}^2 = \bar{\sigma}^2 + \alpha \left[\frac{\nu^{-1} + 1}{\nu^{-1} + K(\nu, \gamma)^{-\gamma} / |\hat{\epsilon}|_t^{\gamma}} - 1 \right] \hat{\sigma}_t^2 + (\alpha + \beta)(\hat{\sigma}_t^2 - \bar{\sigma}^2), \quad \nu \geq 2, \tag{11}$$

where $\bar{\sigma}^2$ is the same as in equation (8) and the parameter-restrictions are the same as for the GARCH(1,1) filter. Equation (2) can still be used to calculate the total expected variance.

Moreover, the GAS literature suggests that under the assumptions of a Student's t-distribution of the error terms, ϵ_t, the GAS filter in equation (11) should be restricted by imposing $\gamma = 2$. Besides that, under the assumption of a Generalized t-distribution of the error terms, literature suggests the use of the GAS filter in equation (11). Finally, the two distributions mentioned in this section are restricted to have unit variance and the formulations can be found in the Appendix.

Our main objective is to check whether the proposed filters by the GAS literature have indeed better portfolio construction and volatility prediction performance. For this purpose, we use evaluation measures, which will be discussed in the next section.

3.3 Evaluation

3.3.1 Portfolio performance

The portfolio performance will be evaluated using risk-return measures. First, we calculate the Sharpe and Sortino ratios for each portfolio by using their monthly returns. Furthermore, we calculate the skewness of the monthly portfolio returns and examine a measure of the Value-at-Risk (VaR), an industry-standard metric fore measuring downside risk. Specifically, we focus on the 95% VaR and 99% VaR, representing the respective quantiles of negative monthly returns. Our approach employs historical VaR, whereby we rank the monthly portfolio returns and determine the corresponding quantile.

3.3.2 Volatility predictions

To evaluate the forecasting accuracy of each model regarding forcasting the total expected variances, we develop a measure of the mean squared prediction error (MSPE) and use the Diebold-Mariano test to test the null hypothesis of equal MSPEs between the different models. However, in order to calculate the MSPE values, we need to have the true total variances for each month, which we do not observe. Therefore, Franses et al. (2014) suggest using data that is sampled more frequently than the time series of interest. Since we are interested in the monthly variances, this means that we can use daily returns in that month to approximate the monthly true variance. More specifically, for each month, we estimate the true total expected variance by summing over the first 22 squared daily returns in deviation from the average daily return in that month.

3.4 Implementation

In this section, we discuss how we estimate the volatility models in section 3.2. All our models are estimated using Maximum Likelihood Estimation (MLE) and under the assumption that the true model for the daily returns is: $r_t = \mu + \sigma_t \epsilon_t$. The log-likelihood functions can easily be derived under the assumption of a specific distribution for ϵ_t. As mentioned before, we consider the normal, Student's t-, and generalized t-distribution.

Finally, we estimate our models using a rolling window approach of 2500 observations, unless otherwise stated. For illustration, this means that we construct the first portfolio for the month August of 1973. We fit the volatility models using the preceding 2500 (or 1000) daily observations, up to and including July 31 in 1973. With these models, we forecast the total expected variance for August 1973 to estimate the portfolio weights, and calculate our monthly portfolio returns for that month. Then we repeat the above procedure by forecasting the total expected variance for September 1973, by fitting volatility models with 2500 preceding observations up to and including August 31 in 1973. All in all, we repeat this procedure till we have reached the end of our data sample. To be consistent, we used the same forecast sample (i.e., August 1983 till March 2023) for the Markowitz (1952) portfolios that were described in section 3.1.3.

4 Results

In this section we discuss performance of all portfolios for the forecasting period from August 1973 until March 2023. We examine whether: (i) volatility-managed portfolios outperform the $1/N$ portfolio; (ii) GMVP outperforms MVP; and (iii) GAS filter outperforms GARCH and GJR-GARCH under the fat-tailed distributions.

4.1 Portfolio performance: $1/N$ and realized variance

The out-of-sample performance of the $1/N$ portfolio is shown in Table 2. We observe that the $1/N$ portfolio outperforms the single-factor portfolios in terms of Sharpe and Sortino ratios, indicating that the risk-adjusted returns are higher. Both ratios of the EW portfolio are similar, indicating that the distribution of returns is relatively symmetric, which is confirmed by the skewness of -0.339. The EW portfolio also outperforms all single factor portfolios in terms of VaR: We expect the EW portfolio to suffer a loss of 1.78% and 3.73% (or more) coming month with a probability of 5% and 1% respectively, under the assumption that the distribution of returns remains unchanged. All single factor portfolios are expected to incur significantly higher losses at the same confidence level.

Table 2 Performance measures factor, equally-weighted and volatility-managed portfolios

	Sharpe ratio	Sortino ratio	Skewness	VaR 95%	VaR 99%
Mkt-RF	0.4567	0.4091	-0.5292	-7.7305	-11.8800
SMB	0.1791	0.1843	0.1517	-4.2593	-7.3257
HML	0.3458	0.3707	0.2366	-4.1293	-7.5173
RMW	0.4733	0.5070	0.4061	-2.8280	-5.6703
CMA	0.8850	0.6235	0.5981	-2.4676	-4.3786
EW	0.8850	0.8395	-0.3394	-1.7789	-3.7312
RV	0.9016	0.9713	0.2495	-1.4329	-2.5467
GARCH[1]	0.8966	0.9693	0.3701	-1.4047	-2.5663
GARCH[2]	0.8855	0.9579	0.3227	-1.3970	-2.6370
GJR-GARCH[1]	0.8924	0.9599	0.3486	-1.4686	-2.5510
GJR-GARCH[2]	0.8900	0.9488	0.2788	-1.3971	-2.6845
tGARCH[2]	0.8757	0.9484	0.3160	-1.4110	-2.6268
GAS[2]	0.8790	0.9488	0.3124	-1.3808	-2.5403

[1] Using rolling estimation window of $N = 1000$ observations.
[2] Using rolling estimation window of $N = 2500$ observations.

The portfolio performance measures of the realized-variance (RV) portfolio are also presented in Table 2. The RV portfolio exhibits the highest risk-adjusted returns, as measured by the Sharpe and Sortino ratios. Moreover, the RV portfolio also comes in at a lower VaR than the EW portfolio, implying a loss of only 1.43% and 2.55% (or more) at a 5% and 1% confidence level, respectively. Considering all the presented performance measures, we can conclude that the RV portfolio outperforms the EW portfolio, confirming the findings of Moreira and Muir (2017).

4.2 Portfolio performance: GARCH and GJR-GARCH

In this section, we discuss the portfolio construction performance of the GARCH and GJR-GARCH models under the assumption of normally distributed errors ϵ_t. We adopt a rolling window approach of 1000 and 2500 observations, respectively, to examine the impact of varying sample sizes on the results. However, Table 2 shows that these concerns have minimal influence on the reported performance measures.

Our findings reveal that the GARCH portfolio outperforms the *benchmark* portfolio in terms of risk-adjusted returns. Although the differences in Sharpe ratios are negligible, the Sortino ratio for the GARCH filter is significantly higher. This result aligns with our hypothesis that volatility-managed portfolios, characterized by positively skewed returns, effectively mitigate downside risk and yield superior performance compared to the equally-weighted (EW) portfolio. Additionally, it is noteworthy that the RV and GARCH portfolios exhibit nearly identical performance, as indicated by the VaR measures.

To further investigate whether positive and negative shocks have a different effect on portfolio construction performance we turn to the GJR-GARCH filter. The results in Table 2 shows that the overall performance of the GJR-GARCH mirrors the performance of the regular GARCH filter for both sample sizes. We observe a slightly lower, positive skew of returns compared to GARCH which manifests in both a marginally lower Sortino ratio and higher VaR at both confidence levels. Therefore, we can conclude that treating positive and negative shocks in the volatility models differently does not noticably influence the portfolio construction performance.

4.3 Portfolio performance: tGARCH and GAS

In Table 2 the performance measures are presented for the GARCH and GAS model under the assumption of the Student's t-distributed errors (see section 3.2.3 for exact formulation). We refer to the GARCH model under this assumption as tGARCH. We observe that the GAS filter has better risk-return measures than the tGARCH filter, although the difference is small. Both models outperform the EW portfolio on all measures except the Sharpe ratio. However, they have worse portfolio construction performance compared to the RV portfolio.

Furthermore, the GAS filter does not provide the most accurate volatility forecasts for all factors, compared to the tGARCH filter. This can be observed in Table 3, which shows the Diebold-Mariano test statistics and MSPE values for these models. Only for the SMB factor does the GAS model provide statistically significant better forecasts. Interestingly, Table 2 shows that the SMB factor has the lowest Sharpe and Sortino ratios, and skewness. This, combined with the average returns presented in Table 1, imply that the returns of the Sortino ratio are relatively stable around 0%. All in all, only for daily returns relatively close to 0% can the GAS model provide significantly better volatility forecasts. Based on this finding, and given the similar portfolio construction performance of both filters, we conclude that the GAS filter is the most appropriate model to use under the assumption of Student's t-distributed errors.

To further test the robustness of the normality assumption, we again implement a GAS and GARCH model but assume a more general fat-tailed distribution for the errors: the generalized t-distribution. However, Table 4 shows that portfolios do not manage to decisively outperform the other filters across all portfolio measures. Therefore, we can conclude that all three filters perform equally well under the assumption of generalized t-distributed errors.

Table 3 MSPE and Diebold-Mariano test statistics of tGARCH and GAS

	Mkt-RF	SMB	HML	RMW	CMA
GAS	2401.5867	105.6448	101.0156	19.5108	12.9771
tGARCH	2129.3158	118.2860	101.4561	19.2394	13.6650
DM-statistic	0.9862	-2.3129	-0.1252	0.4740	-1.8193
p-values	0.3244	0.0211	0.9004	0.6357	0.0694

Table 4 Portfolio construction performance for different filters under the assumption of a generalized t-distribution

	Sharpe ratio	Sortino ratio	Skewness	VaR 95%	VaR 99%
GAS Student's t	0.8755	0.9392	0.2644	-1.3859	-2.5242
GAS Generalized t	0.8790	0.9479	0.2830	-1.3715	-2.5537
GARCH(1,1)	0.8764	0.9489	0.2994	-1.3800	-2.6224

4.4 Portfolio performance: MVP and GMVP

The performances of the MVP and GMVP portfolios are presented in Table 5. The MVP using last month's EW portfolio returns as the target outperforms the MVP using excess market return as the target in all performance measures. Additionally, the MVP using $1/N$ returns as the target outperforms the GMVP and the $1/N$ portfolio, which supports the findings of Kirby and Ostdiek (2012) with more recent data.

Table 5 shows that imposing a no short-selling constraint ($0 \leq w_{i,t} \leq 1$) leads to worse portfolio performance compared to having no weight constraints, contradicting the findings of Jagannathan and Ma (2003) mentioned in section 3.1.3. However, restricting the weights to be between -1 and 1 for each asset improves the MVP models' portfolio performance, addressing the issue of assigning unrealistic weights to certain assets. Hence, we can conclude that setting constraints, including allowing short-selling, enhances the MVP performance.

Table 5 Performance measures minimum-variance and mean-variance portfolios

	Sharpe Ratio	Sortino Ratio	Skewness	VaR 95%	VaR 99%
GMVP	1.0668	1.1041	0.1403	-1.1928	-2.2033
MVP target: Mkt-RF					
No Weight Constraint	0.4874	0.5088	0.1881	-3.9371	-8.4910
Weight Constraint	0.5703	0.5993	0.1445	-3.4977	-7.6632
No Short-selling	0.2377	0.3185	-0.8041	-4.6131	-9.8072
MVP target: 1\N					
No Weight Constraint	1.5801	1.6866	0.7909	-0.8523	-1.8145
Weight Constraint	1.5931	1.7094	0.8324	-0.8366	-1.7541
No Short-selling	1.5510	1.6333	0.7017	-0.8954	-1.8739

5 Conclusion

We have analyzed the portfolio construction performance of three volatility models under various error-distribution assumptions. The main conclusions are presented in this section.

First we find that, under the assumption of a Student's t-distribution for the error-terms, the portfolio construction performance of the corresponding GAS filter in equation (11) with $\gamma = 2$ is slightly better than the GARCH(1,1) filter in equation (7). However, the GAS filter only provides significantly better volatility forecasts for the SMB factor portfolio, which has the lowest daily returns and are less volatile.

Second, under the assumption of the generalized t-distribution for the errors, there is not a specific filter that outperforms the other filters based on their portfolio construction performance. Which is contrary to our expectations, since the GAS filter is specifically constructed for this distribution.

Third, we have shown that the MVP performs better when we target the past $1/N$ returns instead of the high excess returns of the market. Thus, we have shown that the findings of Kirby and Ostdiek (2012) also hold for more recent data. Furthermore, we conclude that both the GMVP or setting short-selling constraints in the MVP optimization problem are not able ot circumvent the estimation errors in the MVP optimization. This contradicts the findings of Jagannathan and Ma (2003).

Finally, further research is required to understand the causes for the comparable performance of the GARCH and two GAS filters under generalized t-distributed errors.

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A Distributions

We estimate the conditional volatility models under the assumption of three different distributions for the error terms, ϵ_t.

A.1 t-distribution

Suppose the shocks ε_t in equation (3.1) are Student's t-distributed with degrees of freedom ν and unit variance (note that ν is just a parameter, hence need not be integer-valued). Then the PDF of ε_t reads

$$p(\varepsilon_t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\nu/2\right)\sqrt{\pi(\nu-2)}}\left(1+\frac{\varepsilon_t^2}{\nu-2}\right)^{-(\nu-1)/2}, \qquad \nu > 2 \tag{12}$$

where $\Gamma(\cdot)$ is the Gamma function.

A.2 Generalized t-distribution

The probability density function (PDF) of the generalized t-distribution with unit variance is

$$p(\varepsilon_t) = \frac{\gamma}{2\nu^{1/\gamma}}\frac{K(\nu,\gamma)}{B(1/\gamma,\nu/\gamma))}\left(1+\frac{K(\nu,\gamma)^\gamma}{\nu}|\varepsilon_t|^\gamma)\right)^{-(\nu+1)/\gamma}, -\infty < \varepsilon_t < \infty, \nu > 2 \tag{13}$$

where $\nu > 2$ and $\gamma > 0$ are shape parameters, $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ denotes the beta function, $\Gamma(x)$ is the gamma function, and

$$K(\nu,\gamma) = \nu^{\frac{1}{\gamma}}\sqrt{\frac{\Gamma\left(\frac{3}{\gamma}\right)\Gamma\left(\frac{\nu-2}{\gamma}\right)}{\Gamma\left(\frac{1}{\gamma}\right)\Gamma\left(\frac{\nu}{\gamma}\right)}}, \qquad \nu > 2 \tag{14}$$

B Sharpe and Sortino ratio

$$\text{Sharpe ratio} := \frac{\sqrt{12}\bar{r}}{\sqrt{\frac{\sum_{t=1}^T (r_t-\bar{r})^2}{T-1}}} \qquad \text{Sortino ratio} := \frac{\sqrt{12}\bar{r}}{\sqrt{\frac{\sum_{t=1}^T (r_t-\bar{r})^2 \mathbf{1}_{r_t<\bar{r}}}{\left(\sum_{t=1}^T \mathbf{1}_{r_t<\bar{r}}\right)-1}}}$$

where r_t is the monthly return at time $t = 1, 2, ..., T$, $\bar{r} = \sum_{t=1}^T r_t/T$, and $\mathbf{1}_A$ is 1 when A is true and zero otherwise.

C Additional Figures

Figure 1: Daily percentage returns for the factor portfolios

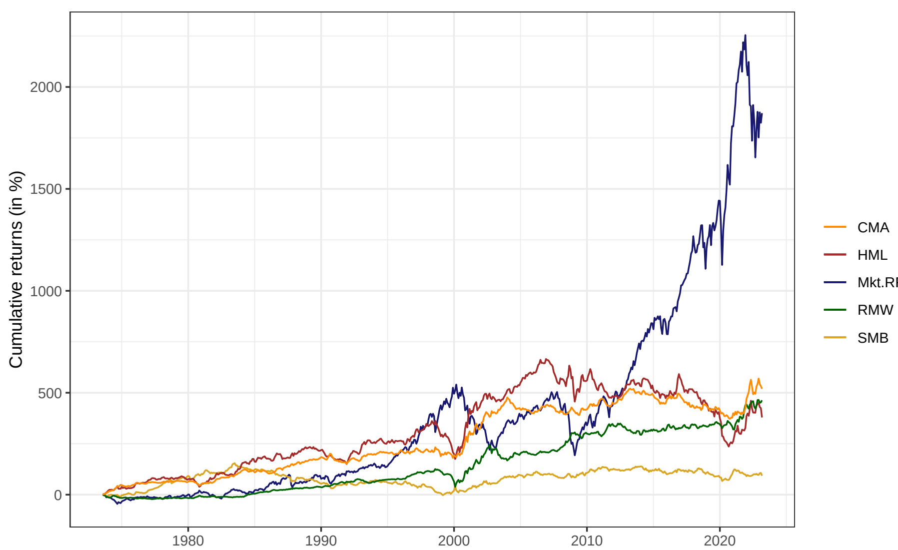

Figure 2: Cumulative returns for the five Fama-French factor portfolios in the period 1973-08 up to and including 2023-03

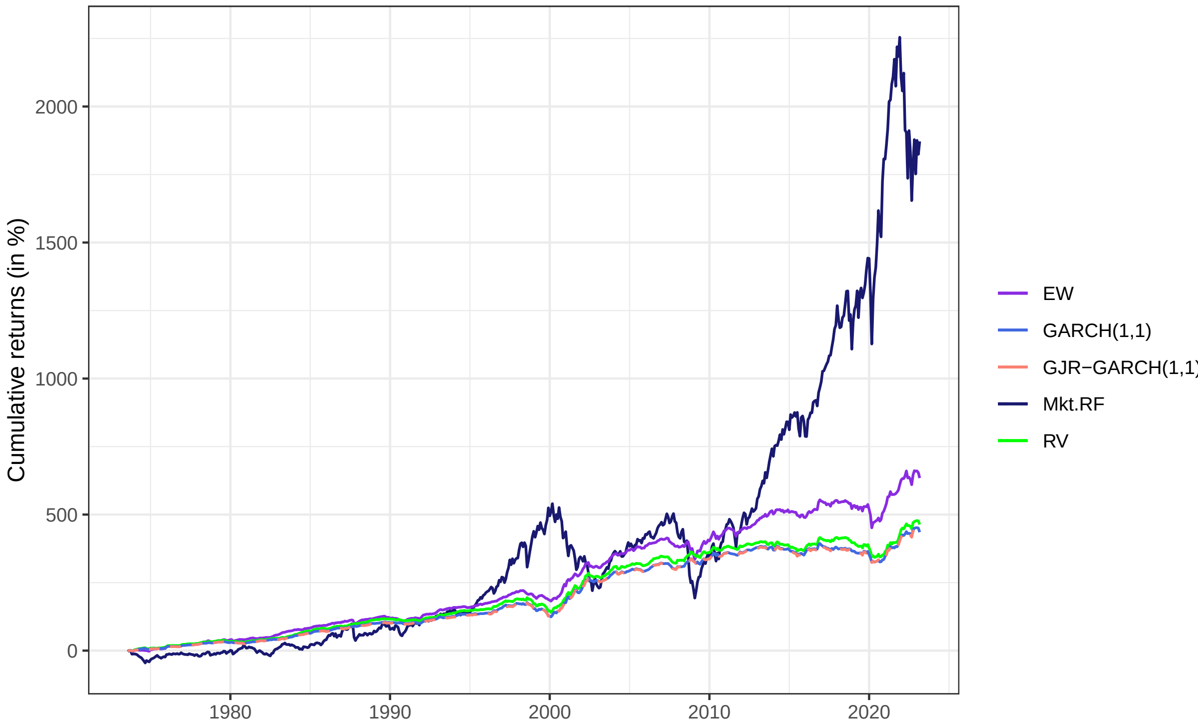

Figure 3: Cumulative returns of the EW, RV, GARCH(1,1) and GJR-GARCH model with an estimation sample of N = 2500, for the period 1973-08 up to 2023-03

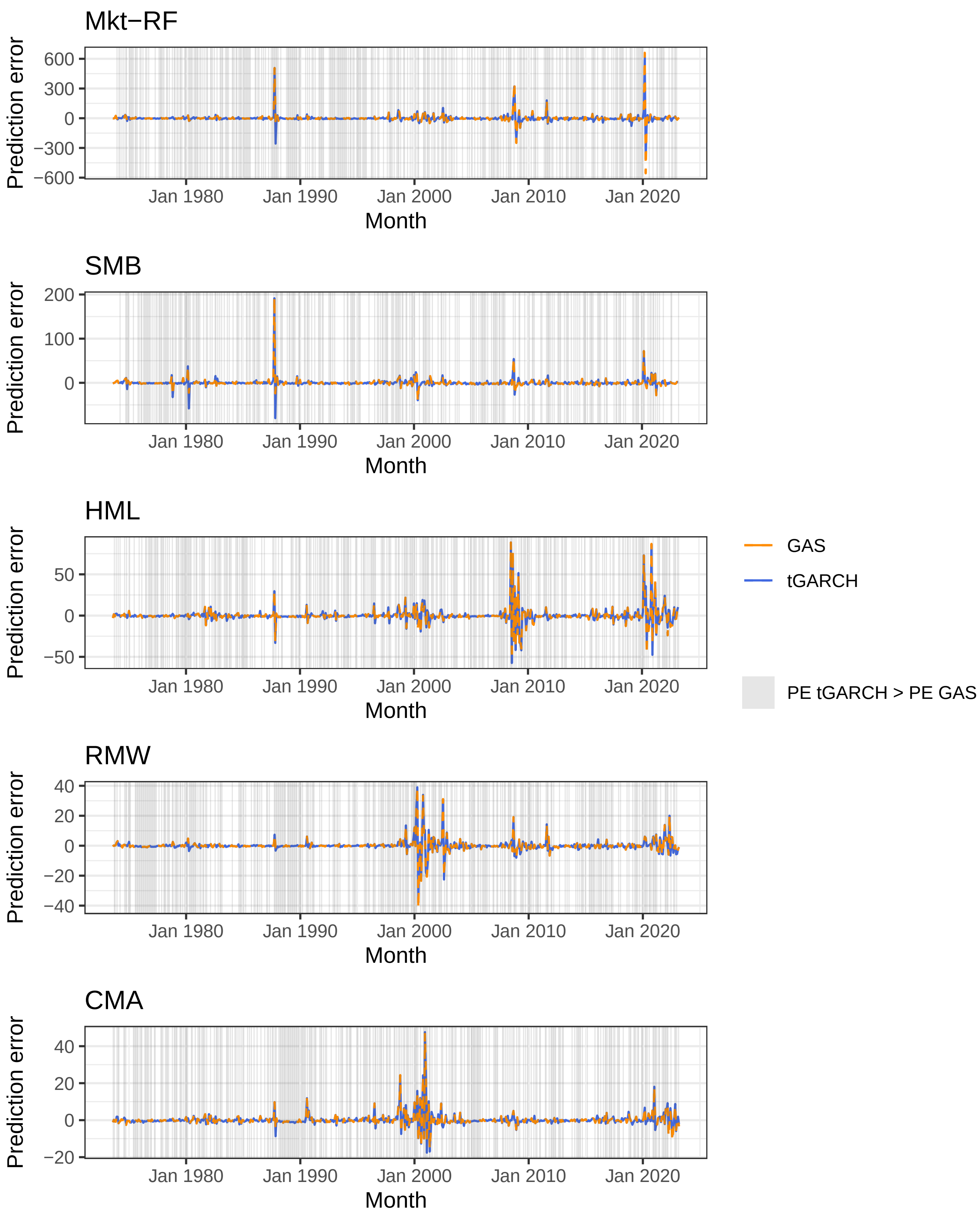

Figure 4: Prediction errors for the total expected variance forecasts of tGARCH and GAS filter with respect to the 'true' total expected variance