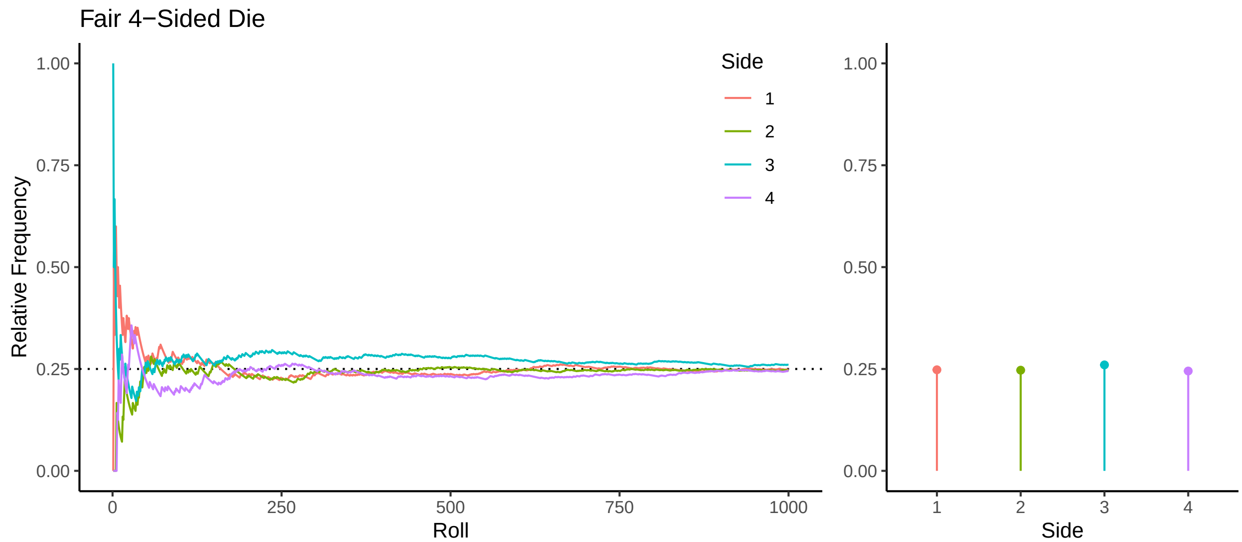
Friday, Sep 10

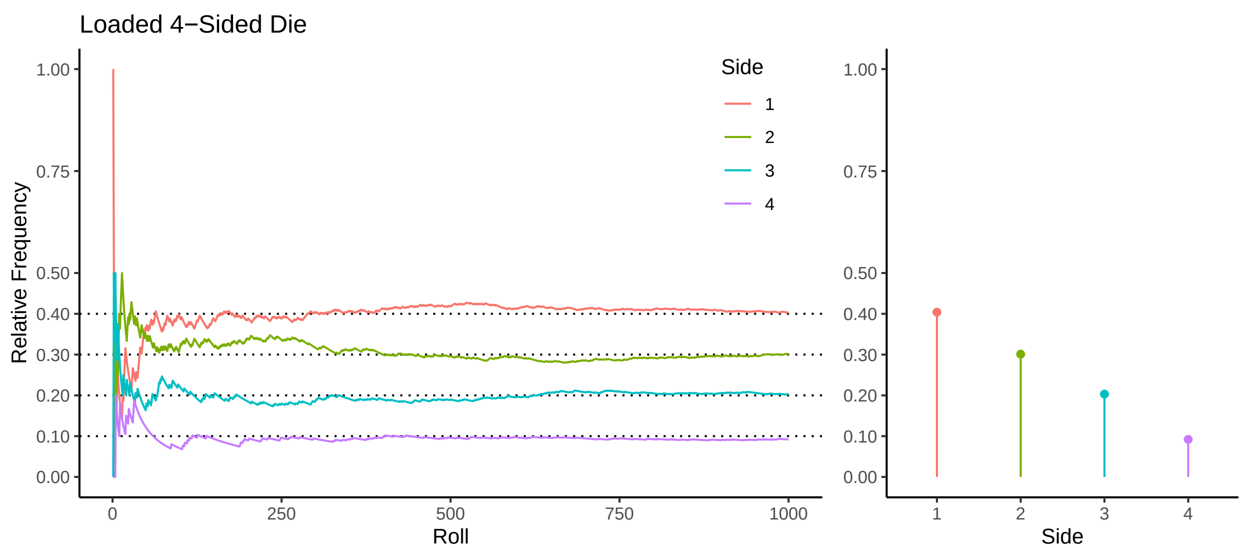
Probability and Relative Frequency

Probability is a measurement of the "likelihood" of an event as a number between 0 and 1. These measurements follow the mathematical rules of *probability theory*.

How can we connect probabilities with empirical observations? The *Law of Large Numbers* states that a *relative frequency* will tend to "approach" (in some sense) the *probability* of an event as the number of observations increases.

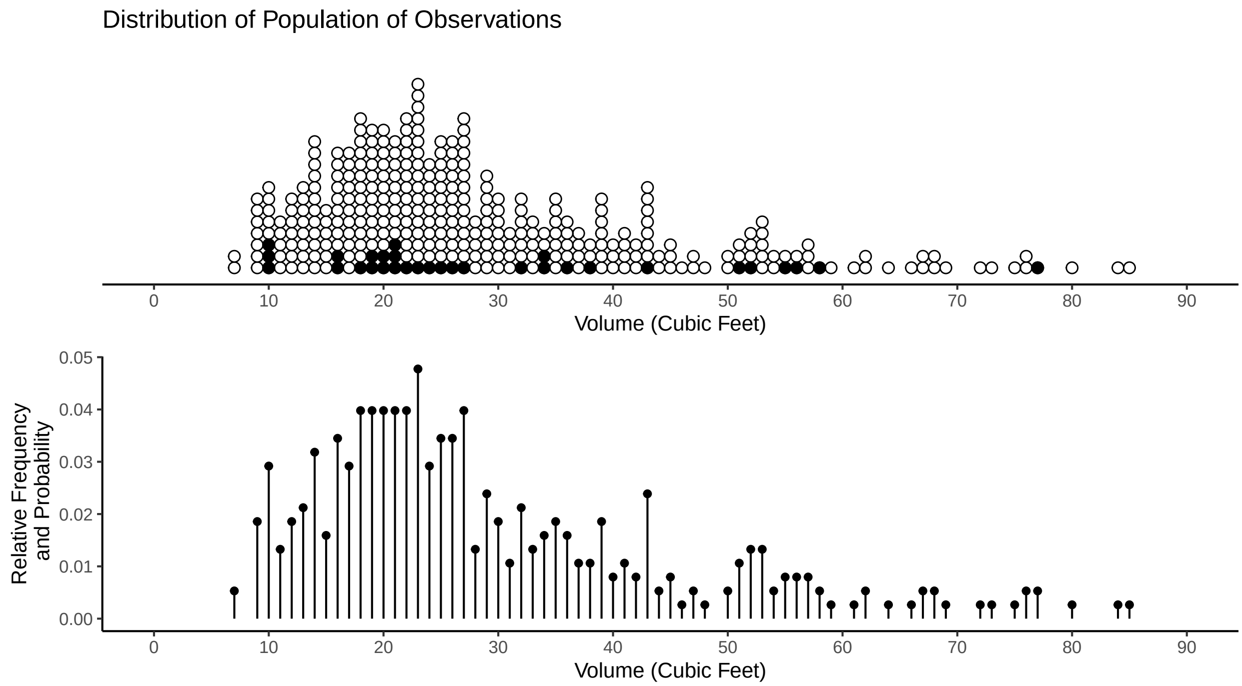
Example: Consider rolling a 4-sided die many times and looking at the distribution of the sides.





In a *survey* the relative frequencies for the distribution of the *population* of observations become probabilities if we select units *at random*.

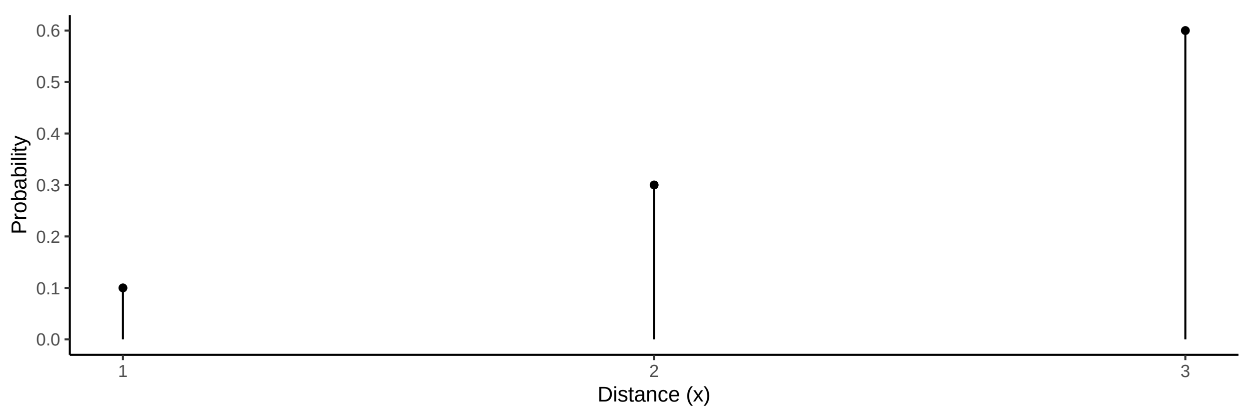
Example: Consider a survey of tree volume.



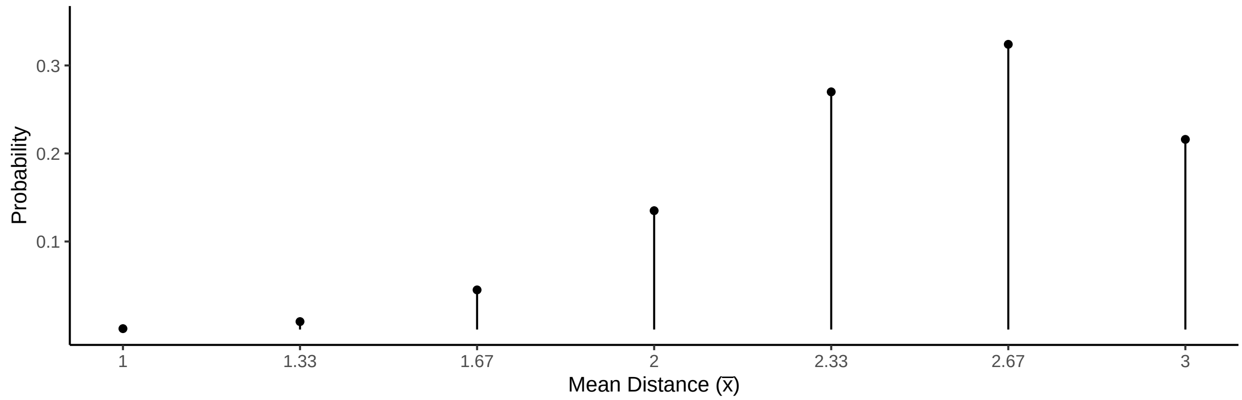
In an *experiment* the probabilities are determined by the underlying process that produces the observations.

Example: Suppose we are studying the distance that a toy trebuchet will throw a projectile.

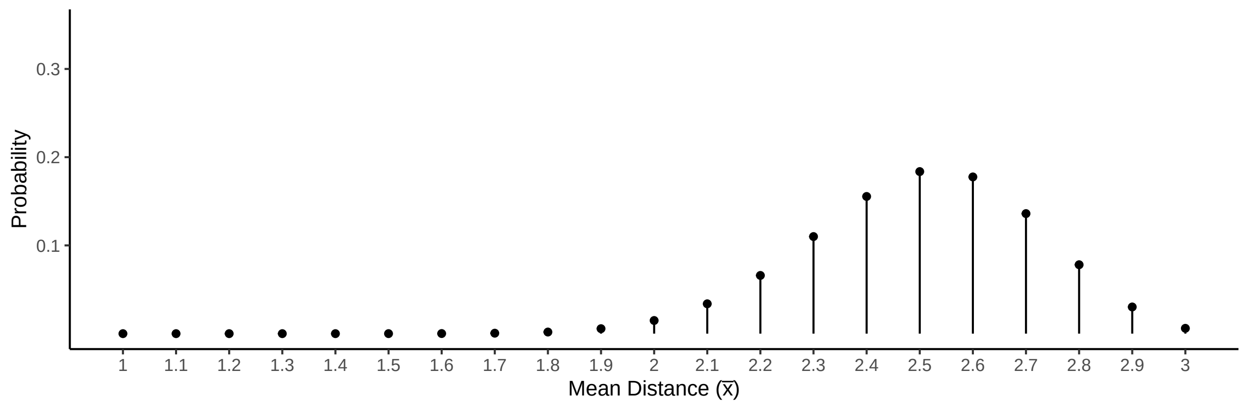
First consider observing the distance (x) of *one* throw.



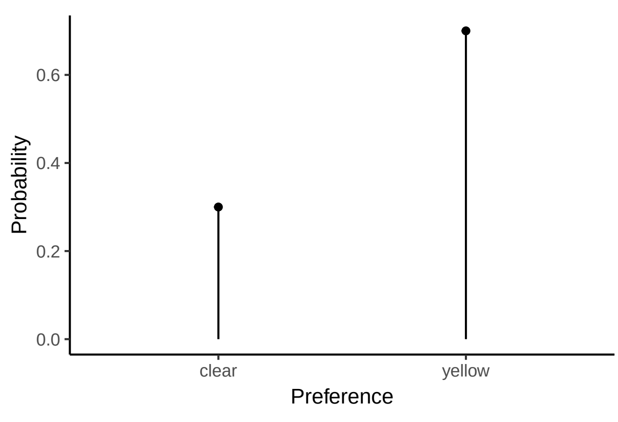
Now consider observing the *mean* distance (i.e., $\bar{x}$) of a sample of $n = 3$ throws.

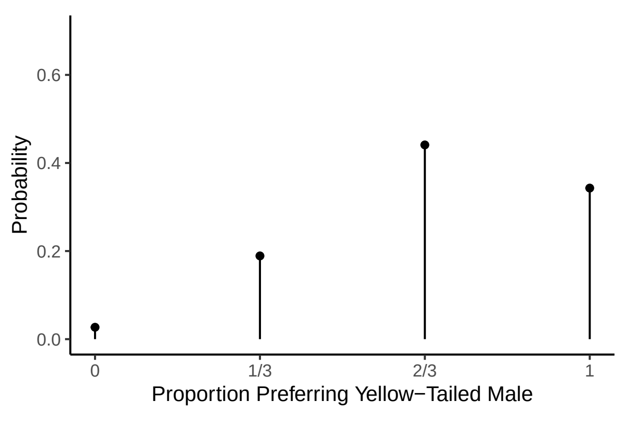


Now consider observing the *mean* distance (i.e., $\bar{x}$) of a sample of $n = 10$ throws.



Example: Suppose we are studying the "preference" of female platies for males with clear versus yellow tails. Consider observing (a) the apparent preference from one observation and (b) the *proportion* of observations out of $n = 3$ observations where the yellow-tailed male is preferred.





Random Variables and Probability Distributions

A **random variable** occurs when we assign values to an *event*. An *event* corresponds to a particular *outcome* of a random process. Distance, mean distance, preference, and proportion preferring yellow-tailed male are all *random variables* in the examples above. Random variables can be *quantitative* or *categorical*.

Types of *Quantitative* Random Variables:

1. **Discrete**. A random variable is *discrete* if the possible values are *countable*.
2. **Continuous**. A random variable is *continuous* if the possible values are *not countable*.

The **probability distribution** of a *discrete* random variable consists of (a) the *possible values* of the random variable and (b) their *probabilities*. The distribution can be shown using a plot (as shown earlier) or a table (as shown below).

Example: Here are the probability distributions of one observation of the distance a trebuchet throws (x), and the mean distance in a sample of $n = 3$ throws ($\bar{x}$).

x	$P(x)$
1	0.1
2	0.3
3	0.6

$\bar{x}$	$P(\bar{x})$
1.00	0.001
1.33	0.009
1.67	0.045
2.00	0.135
2.33	0.270
2.67	0.324
3.00	0.216

Example: Here are the probability distributions of one observation of female platy preference (x), and the proportion of observations out of $n = 3$ where the yellow-tailed male is preferred ($\hat{p}$).

x	$P(x)$
clear	0.3
yellow	0.7

$\hat{p}$	$P(\hat{p})$
0	0.027
1/3	0.189
2/3	0.441
1	0.343

Two Important Probability Distributions in Statistical Inference

1. The probability distribution of a *single observation* (a **population distribution**).
2. The probability distribution of a *statistic* (a **sampling distribution**).

Mean of a Random Variable (Discrete Case)

The mean of a *discrete* random variable is

$$\mu = \sum_x xP(x),$$

where x denotes a value of the random variable and $P(x)$ denotes the probability of that value.[1] Note that the x below the summation sign here indicates that we sum over all values of x.

The Law of Large Numbers implies that as the number of observations of a random variable increases, their mean ($\bar{x}$) will tend to "approach" (in some sense) μ.

Example: Consider the probability distribution of an observation of a single throw of the trebuchet (a *population distribution*).

x	$P(x)$
1	0.1
2	0.3
3	0.6

We can confirm that the mean of the random variable x is $\mu = 2.5$ m.

Example: Consider the probability distribution of the proportion of female platies that prefer the yellow-tailed male from a sample $n = 3$ observations (a *sampling distribution*).

$\hat{p}$	$P(\hat{p})$
0	0.027
1/3	0.189
2/3	0.441
1	0.343

We can confirm that the mean of the random variable $\hat{p}$ is $\mu = 0.7$.

[1] We can say that "μ is the mean of the probability distribution of the random variable" or, more simply, "μ is the mean of the random variable." Similarly we can say that "σ is the standard deviation of a probability distribution" or that "σ is the standard deviation of a random variable.

Variance of a Random Variable (Discrete Case)

The *variance* of a *discrete* random variable is

$$\sigma^2 = \sum_x (x-\mu)^2 P(x),$$

and the standard deviation is

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)}.$$

Example: Consider the probability distribution of an observation of a single throw of the trebuchet (a *population distribution*).

x	$P(x)$
1	0.1
2	0.3
3	0.6

Recall that the mean of the random variable x is $\mu = 2.5$ m. We can confirm that the standard deviation of x is $\sigma \approx 0.67$ m.

Example: Consider the probability distribution of the mean distance of a sample of $n = 3$ throws of the trebuchet (a *sampling distribution*).

$\bar{x}$	$P(\bar{x})$
1.00	0.001
1.33	0.009
1.67	0.045
2.00	0.135
2.33	0.270
2.67	0.324
3.00	0.216

The mean of $\bar{x}$ is $\mu = 2.5$ m. We can confirm that the standard deviation of $\bar{x}$ is $\sigma \approx 0.39$ m.

$$\sigma = \sqrt{(1-2.5)^2 \times 0.001 + (1.33-2.5)^2 \times 0.009 + (1.67-2.5)^2 \times 0.045 + \cdots + (3-2.5)^2 \times 0.216} \approx 0.39.$$