Monday, Sep 27

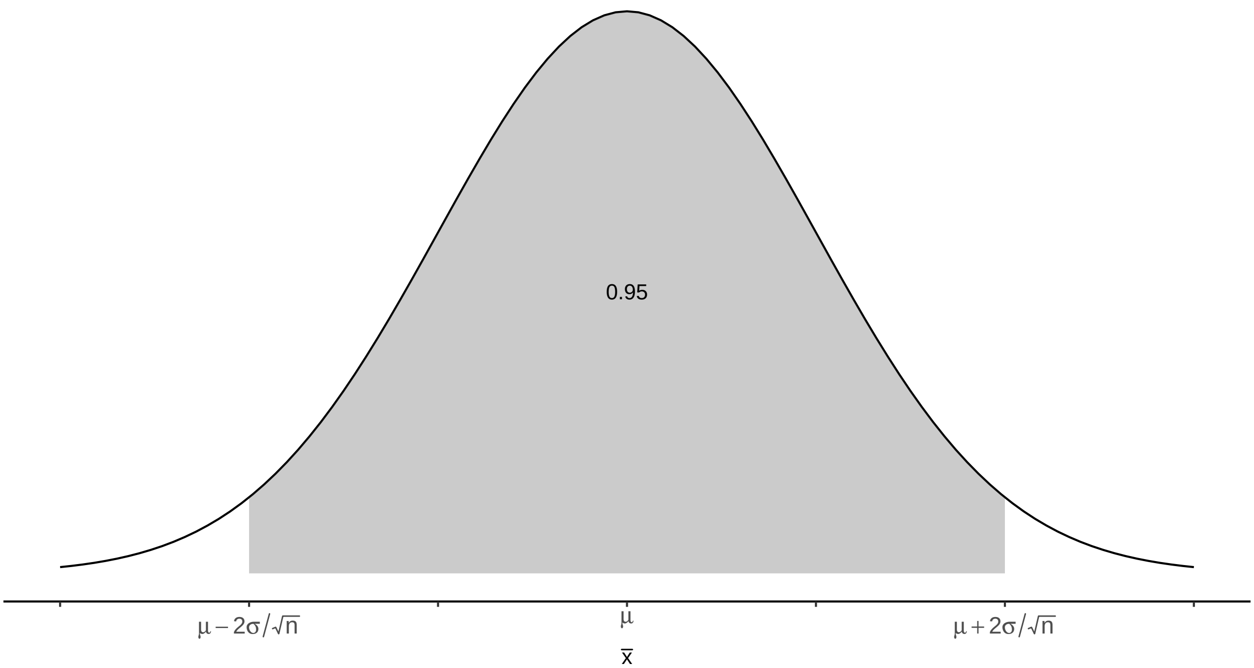
The Sampling Distribution of $\bar{x}$ and Estimation of μ

Note: To keep the notation simple, we will use μ and σ to represent the mean and standard deviation of x, respectively (i.e., we will omit the subscript from μ_x and σ_x).

We know the following about the sampling distribution of $\bar{x}$:

1. The mean of $\bar{x}$ is μ.
2. The standard deviation of $\bar{x}$ is $\sigma/\sqrt{n}$.
3. The shape of the distribution is approximately that of a normal distribution.

This allows us to make statements about the *probability* that $\bar{x}$ will be within a certain distance of μ.



We can say that

$$P\left(\mu - 2\frac{\sigma}{\sqrt{n}} < \bar{x} < \mu + 2\frac{\sigma}{\sqrt{n}}\right) \approx 0.95,$$

which can also be stated as

$$P\left(|\bar{x} - \mu| < 2\frac{\sigma}{\sqrt{n}}\right) \approx 0.95.$$

The probability that the *distance* between μ and $\bar{x}$ will be less than $2\sigma/\sqrt{n}$ is approximately 0.95. We call this distance the **margin of error**.

The choice of a probability of 0.95 is arbitrary, but is a common convention. We can make similar statements for other probabilities. Recall that the **standard error** of $\bar{x}$ is its standard deviation, which is $\sigma/\sqrt{n}$. The probability that the distance between μ and $\bar{x}$ does not exceed the *standard error* is about 0.68. That is

$$P\left(|\bar{x}-\mu| < \frac{\sigma}{\sqrt{n}}\right) \approx 0.68.$$

So the standard error corresponds to a margin of error for a probability of 0.68.

Other Ways to Look at the Error of Estimation

We might call $|\bar{x}-\mu|$ the **error of estimation**. We know some things about the distribution of the error of estimation.

1. The *95th percentile* of the error of estimation is about $2\sigma/\sqrt{n}$ (from above).
2. The *68th percentile* of the error of estimation is about $\sigma/\sqrt{n}$ (from above).
3. The *50th percentile* of the error of estimation is about $0.674\sigma/\sqrt{n}$. This is its *median*.
4. The *mean* of the error of estimation is about $0.798\sigma/\sqrt{n}$.

Example: Let x be the number of dots shown on a fair six-sided die after being rolled. It can be shown that x has a mean of $\mu = 3.5$ and a standard deviation of $\sigma \approx 1.71$. Suppose we roll the die 25 times to produce a sample of $n = 25$ observations. Let $\bar{x}$ be the mean number of dots for this sample. What are the *standard error* and the *margin of error* of $\bar{x}$? What are the median and mean of the error of estimation?

Example: Let x be yield of a chemical reaction under certain circumstances. Assume that x has a mean of $\mu = 10$ g and a standard deviation of $\sigma = 0.5$ g. Let $\bar{x}$ be the mean yield for a sample of $n = 25$ observations of x. What are the *standard error* and the *margin of error* of $\bar{x}$? What are the median and mean of the error of estimation?

Example: Suppose that the mean height of all Hobbits is 100 cm, and the standard deviation of all heights is 10 cm. Let x be the height of one Hobbit, selected at random. Then x has a mean of $\mu = 100$ cm and a standard deviation of $\sigma = 10$ cm. What are the *standard error* and the *margin of error* of $\bar{x}$ computed from a sample of $n = 25$ Hobbits? What are the median and mean of the error of estimation?

Note that calculation of the standard error and margin of error do not require knowing μ. But they do require σ. In practice this parameter would be unknown, so we can *estimate* it using the standard deviation from the sample (i.e., s).

Example: Let x be yield of a chemical reaction under certain circumstances. Let $\bar{x}$ be the mean yield for a sample of $n = 25$ observations of x. Suppose we obtain a sample and find that $s = 0.4$. What are the (estimated) *standard error* and the *margin of error* of $\bar{x}$ What are the (estimated) median and mean of the error of estimation?

Example: Let x be the height of one Hobbit, selected at random. Suppose we obtain a sample of $n = 25$ Hobbits and find that $s = 10.2$. What are the (estimated) *standard error* and the *margin of error* of $\bar{x}$? What are the (estimated) median and mean of the error of estimation?

The Sampling Distribution of $\hat{p}$ and Estimation of p

Note: Recall that $\hat{p}$ is just a special case of $\bar{x}$ that results in some algebraic simplifications.

We know the following about the sampling distribution of $\hat{p}$:

1. The mean of $\hat{p}$ is p.
2. The standard deviation of $\hat{p}$ is $\sqrt{p(1-p)/n}$.
3. The shape of the distribution is approximately that of a normal distribution.

This allows us to make statements about the *probability* that $\hat{p}$ will be within a certain distance of p.

We can say that

$$P\left(p - 2\sqrt{p(1-p)/n} < \hat{p} < p + 2\sqrt{p(1-p)/n}\right) \approx 0.95,$$

which can also be stated as

$$P\left(|\hat{p} - p| < 2\sqrt{p(1-p)/n}\right) \approx 0.95.$$

The probability that the *distance* between p and $\hat{p}$ will be less than $2\sqrt{p(1-p)/n}$ is approximately 0.95. We call this distance the **margin of error**. Note that the *standard error* here is $\sqrt{p(1-p)/n}$.

Example: Let x the side a coin comes up (i.e., "heads" or "tails"). Assume that the coin is fair so that both sides have a probability of 0.5. Suppose we flip the coin 100 times to produce a sample of $n = 100$ observations of x. Let $\hat{p}$ be the proportion of observations on which the coin came up heads. What are the *standard error* and *margin of error* of $\hat{p}$?

Example: Let x be whether or not polymerase chain reaction (PCR) test is successful under certain circumstances. Assume that the probability of success is 0.8. Suppose we conduct 100 tests to produce a sample of $n = 100$ observations of x. Let $\hat{p}$ be the proportion of these on which the test was successful. What are the *standard error* and *margin of error* of $\hat{p}$?

Example: Assume that 20% of all adult Hobbits have foot lice. Suppose we were to obtain a sample of 100 observations of Hobbits and compute the proportion of Hobbits in the sample that have foot lice. What are the *standard error* and *margin of error* of $\hat{p}$?

Note that computing the standard error and the margin of error of $\hat{p}$ require p, which we would not typically know in practice. But it can be estimated from a sample using $\hat{p}$.

Example: Let x be whether or not polymerase chain reaction (PCR) test is successful under certain circumstances. Suppose we conduct 100 tests to produce a sample of 100 observations of x and observe that the PCR test was successful on 90 of those 100 observations. What are the (estimated) *standard error* and *margin of error* of $\hat{p}$?

Example: Suppose we obtained a sample of 100 of Hobbits and found that 15 had foot lice. What are the (estimated) *standard error* and *margin of error* of $\hat{p}$?

Confidence Intervals

Two kinds of estimation:

1. **Point estimation** is estimation of the value of a parameter with the value of a statistic (e.g., estimation of μ with $\bar{x}$, or estimation of p with $\hat{p}$).
2. **Interval estimation** is the estimation of the value of a parameter with an *interval* of values. The device we will be using for interval estimation is a *confidence interval*.

Confidence Interval for μ

Some algebra shows that if

$$P\left(\mu - 2\frac{\sigma}{\sqrt{n}} < \bar{x} < \mu + 2\frac{\sigma}{\sqrt{n}}\right) \approx 0.95,$$

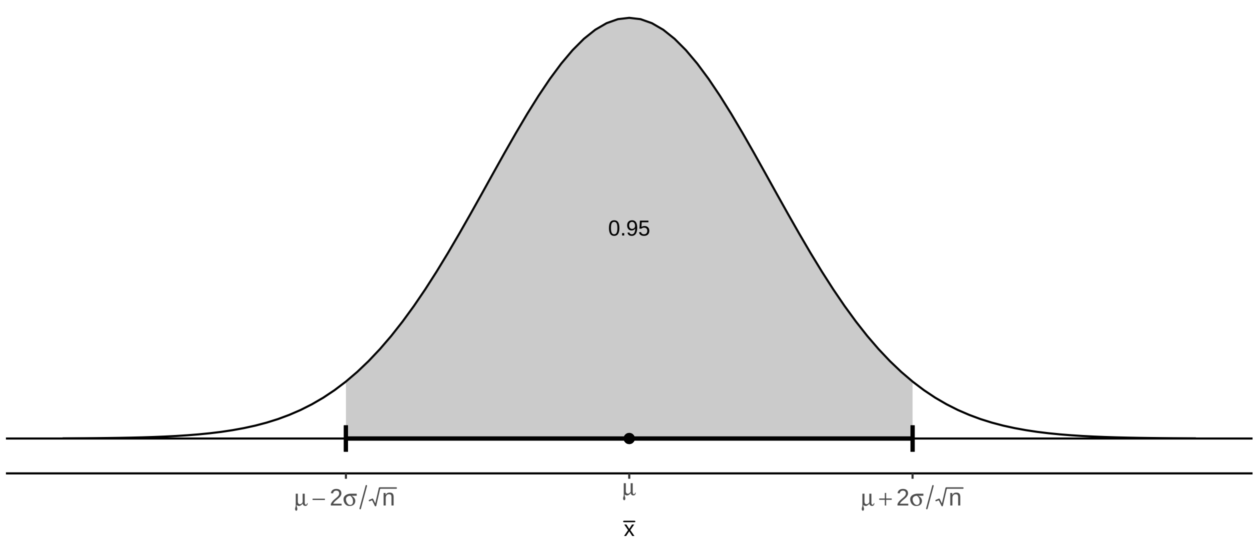
then

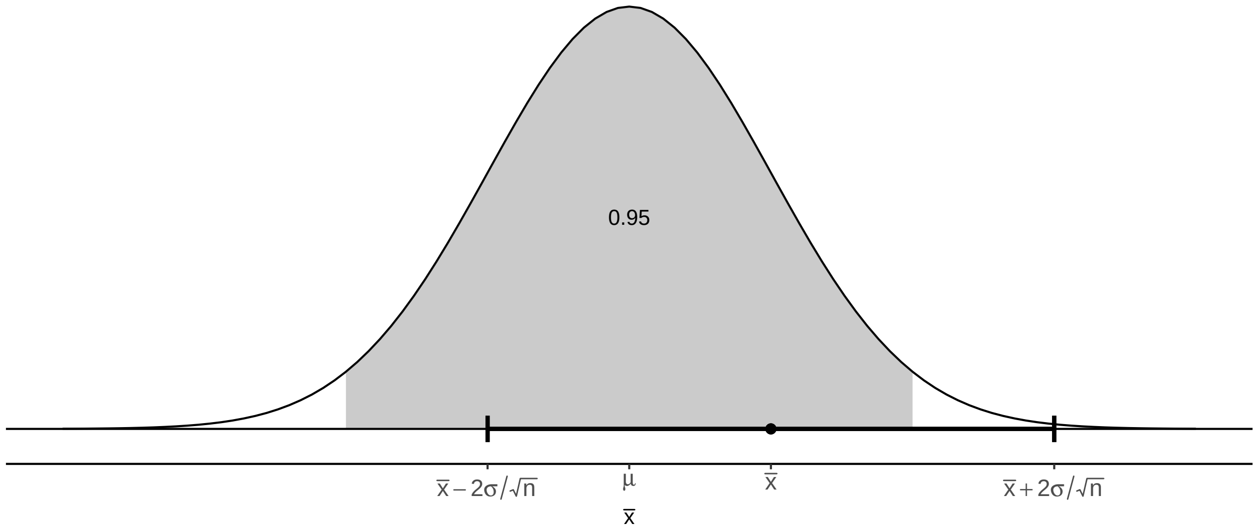
$$P\left(\bar{x} - 2\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2\frac{\sigma}{\sqrt{n}}\right) \approx 0.95,$$

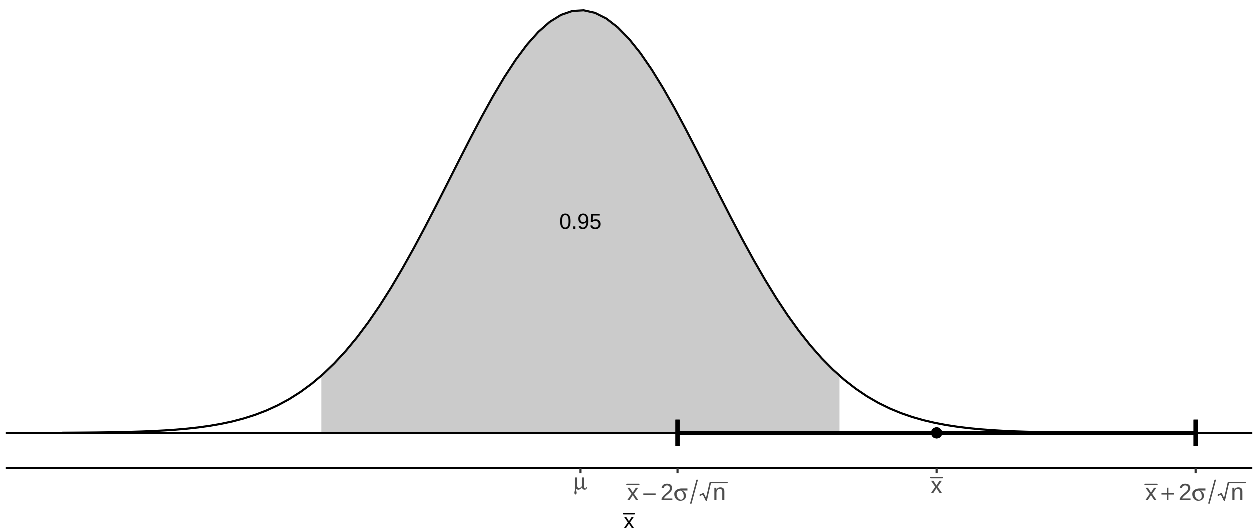
The *confidence interval*

$$\bar{x} \pm 2\frac{\sigma}{\sqrt{n}} \Leftrightarrow \left(\bar{x} - 2\frac{\sigma}{\sqrt{n}},\ \bar{x} + 2\frac{\sigma}{\sqrt{n}}\right).$$

has a probability of approximately 0.95 of containing μ.







Note: In *practice*, we need to replace σ with s since σ will be unknown.

Example: Let μ be the mean height of *all* Hobbits. A recent survey found that in a random sample of 64 Hobbits the mean height was $\bar{x} = 95$ cm and the standard deviation was $s = 16$ cm. What is the *confidence interval* for estimating μ?

Confidence Interval for p

We can similarly derive a confidence interval for p as

$$\hat{p} \pm 2\sqrt{p(1-p)/n} \Leftrightarrow \left(\hat{p} - 2\sqrt{p(1-p)/n},\ \hat{p} + 2\sqrt{p(1-p)/n}\right).$$

Note: In *practice*, we need to replace p with $\hat{p}$ since p will be unknown.

Example: Let x be whether or not polymerase chain reaction (PCR) test is successful under certain circumstances. Suppose we conduct 100 tests to produce a sample of 100 observations of x and observe that the PCR test was successful on 90 of those 100 observations. What is the confidence interval for estimating the probability of a successful PCR test?

Example: Suppose we obtained a sample of 100 of Hobbits and found that 15 had foot lice. What is the confidence interval for estimating the proportion of all Hobbits that have foot lice?