Friday, Oct 25

Optimum Sample Sizes for Two-Stage Cluster Sampling

One of the advantages of two-stage cluster sampling over one-stage cluster sampling is that we have more control over the cost and precision (in terms of the variance of an estimator) of the survey. This is because there are *two* stages of sampling, and thus *two* sample size decisions.

1. The number of clusters to sample (n).
2. The number of elements to sample from each cluster ($m_1, m_2, \ldots, m_n$).

Between-Group and Within-Group Mean Squares

Assume a simple case where we have the following.

1. Simple random sampling at both stages.
2. All clusters are the same size (i.e., all M_i are equal).
3. The number of elements sampled from each cluster are the same (i.e., all m_i are equal).

To simplify notation, let $\bar{M} = M/N$ be the number of elements per cluster, and let m denote the number of elements sampled from each cluster. In this case the unbiased and ratio estimators are the same. Without loss of generality we will consider $\hat{\mu}$.

The variance of $\hat{\mu}$ can be written as

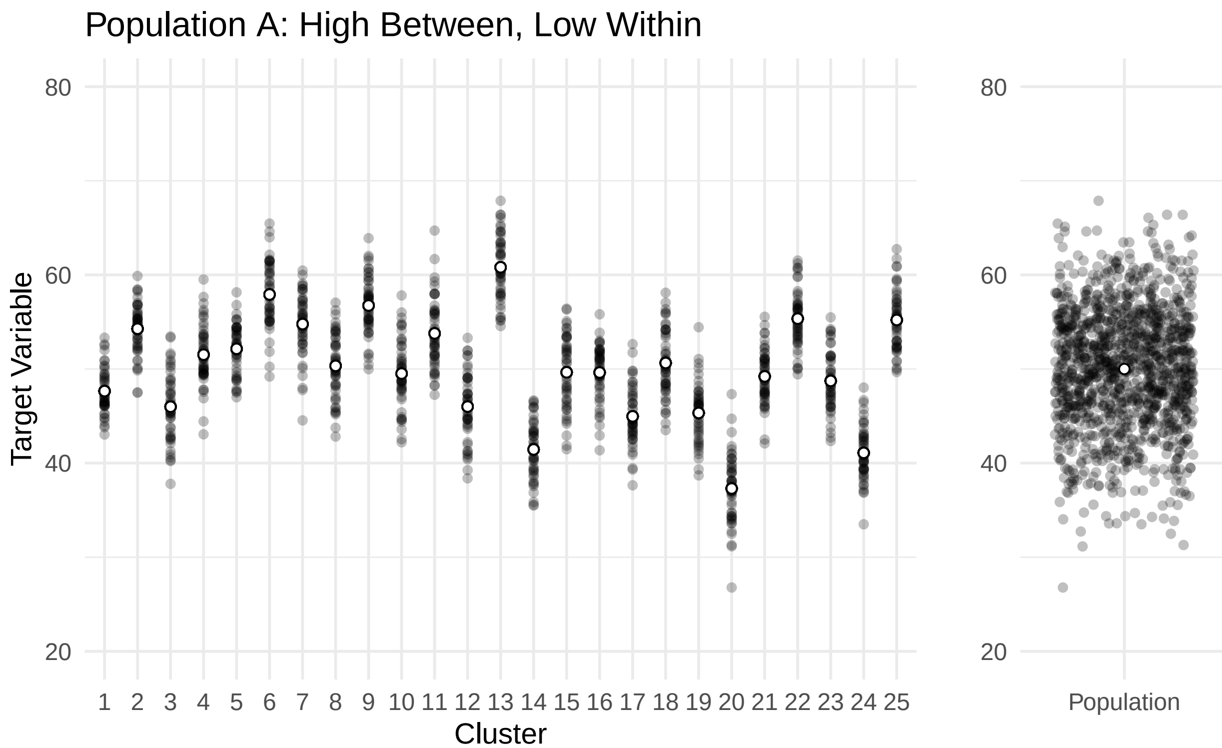
$$V(\hat{\mu}) = \left(1 - \frac{n}{N}\right)\frac{\sigma_b^2}{n\bar{M}} + \left(1 - \frac{m}{\bar{M}}\right)\frac{\sigma_w^2}{nm},$$

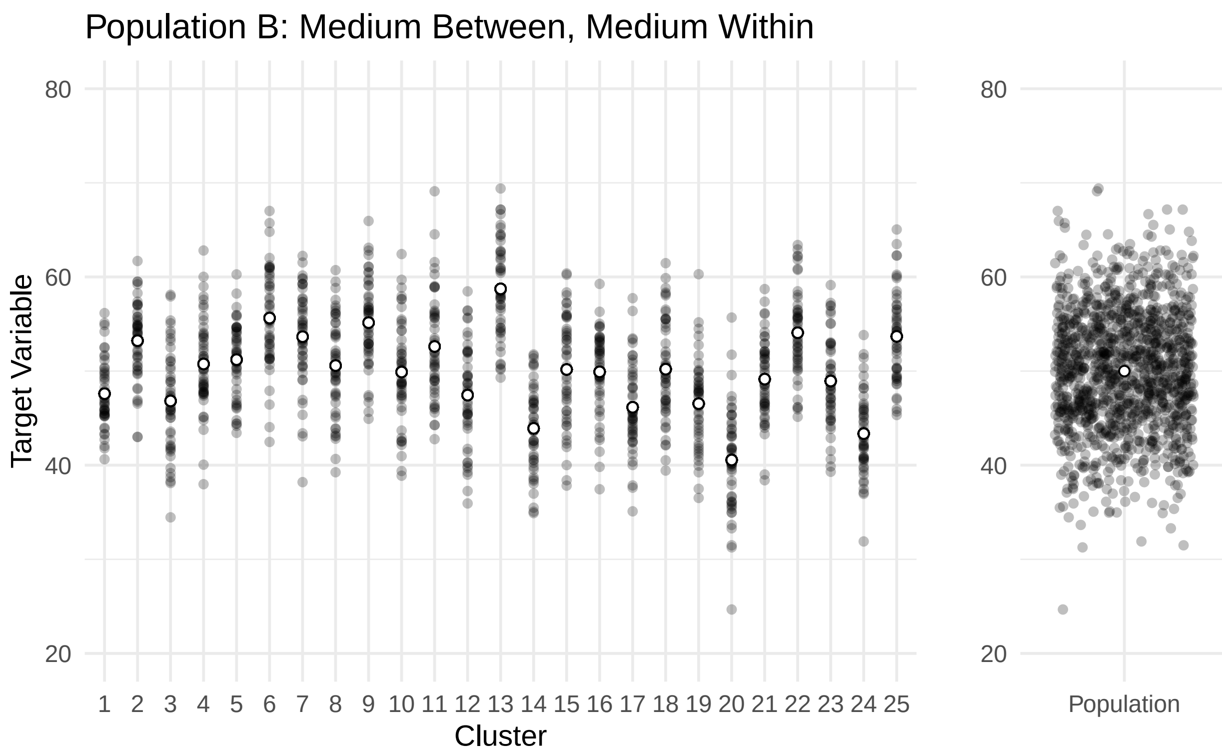
where σ_b^2 and σ_w^2 are the mean squares between-groups and within-groups, respectively, defined as

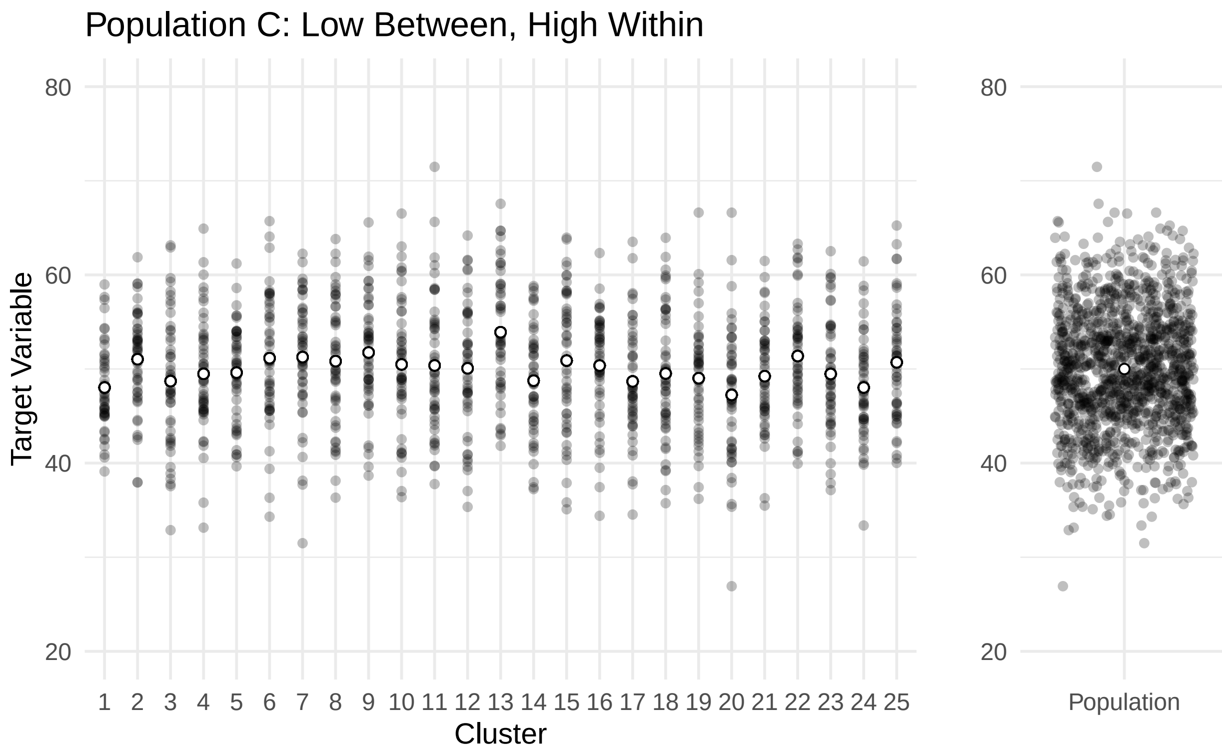
$$\sigma_b^2 = \bar{M}\frac{\sum_{i=1}^N (\mu_i - \mu)^2}{N-1}, \quad \sigma_w^2 = \frac{1}{N}\sum_{i=1}^N \sigma_i^2,$$

where μ_i and σ_i^2 are the mean and variances of the target variable for *all* the elements in the i-th cluster.

Example: Consider populations with $M = 1250$ elements in $N = 25$ clusters, each of size $\bar{M} = 50$.


Population A: High Between, Low Within
Target Variable
80
60
40
20
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
Cluster
Population


Population B: Medium Between, Medium Within
Target Variable
80
60
40
20
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
Cluster
Population



The mean squares for the three populations are given below.

Population	σ_b^2	σ_w^2
A	1551.41	10.39
B	873.23	23.68
C	103.07	38.76

Optimal Sample Sizes

Assume that the total cost of the survey can be computed as

$$C = nc_1 + nmc_2,$$

where c_1 is the *cost-per-cluster* and c_2 is the *cost-per-element*. Minimizing cost for a fixed variance or bound, or minimizing the variance or bound for a fixed cost yields

$$m_{\text{opt}} = \sqrt{\frac{\bar{M}\sigma_w^2}{\sigma_b^2 - \sigma_w^2} \times \frac{c_1}{c_2}}.$$

Note: We must have $1 \le m \le \bar{M}$, and m_{opt} will not necessarily respect this constraint. Also m_{opt} isn't defined if $\sigma_b^2 < \sigma_w^2$.

The sample size for the *number of clusters* (n) to minimize the variance for a fixed cost is

$$n_{\text{opt}} = \frac{C}{c_1 + c_2 m_{\text{opt}}}.$$

Note: Clearly we must have $1 \le n \le N$.

We can encounter various "limiting cases" when solving for m_{opt} and n_{opt}.

1. If $m_{\text{opt}} < 1$ then set $m_{\text{opt}} = 1$ (i.e., sample just one element per cluster).

2. If $m_{\text{opt}} > \bar{M}$, then set $m_{\text{opt}} = \bar{M}$ (i.e., use one-stage cluster sampling).
3. If $\sigma_b^2 < \sigma_w^2$ then set $m_{\text{opt}} = \bar{M}$ (i.e., use one-stage cluster sampling).
4. If $n_{\text{opt}} \geq N$ then set $n_{\text{opt}} = N$ (i.e., use stratified random sampling).

Example: What would be the optimal sample sizes for the three populations if we have a total budget of $C = 100$, the cost per cluster is $c_1 = 10$, and the cost per element is $c_2 = 1$?

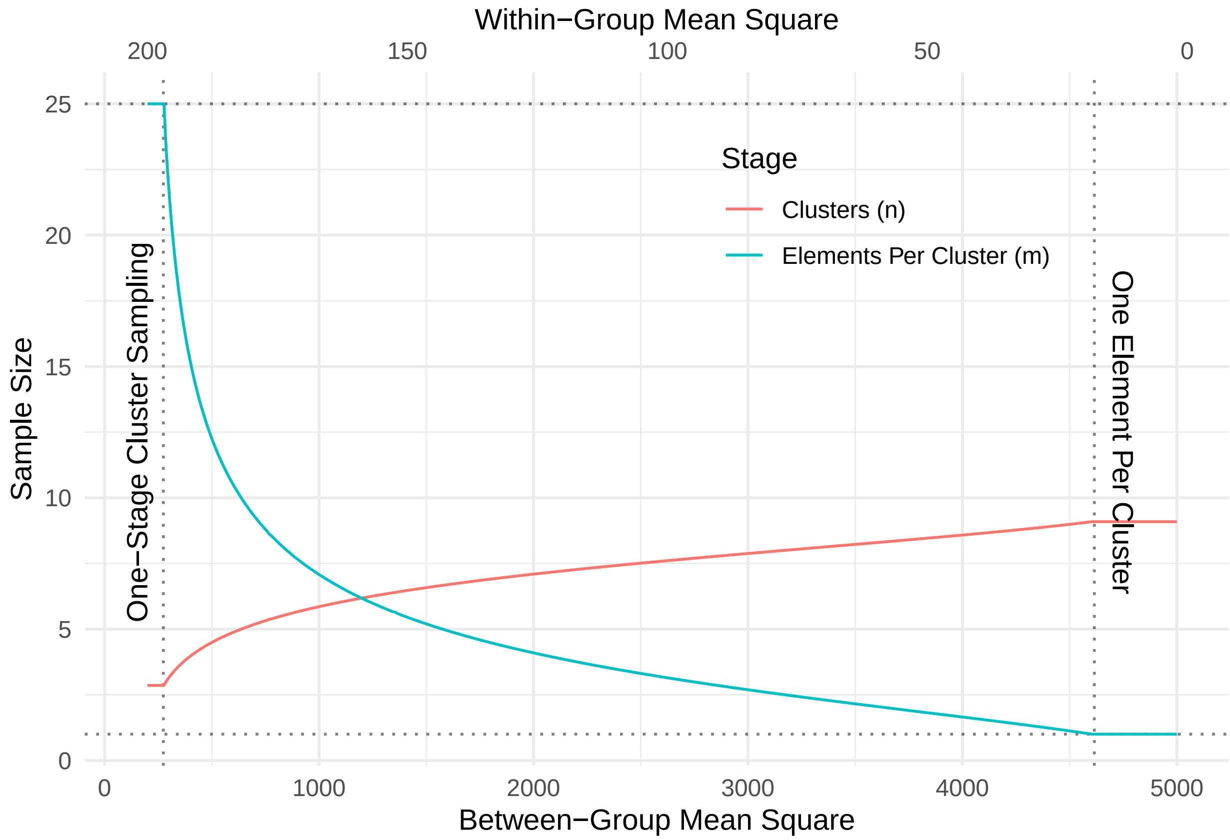
Population	σ_b^2	σ_w^2	m_{opt}	n_{opt}
A	1551.41	10.39	1.836067	8.448752
B	873.23	23.68	3.733201	7.281624
C	103.07	38.76	17.359517	3.655035

Notice what happens as m_{opt} and n_{opt} as σ_b^2 increases, and notice what happens to m_{opt} and n_{opt} as σ_w^2 increases.

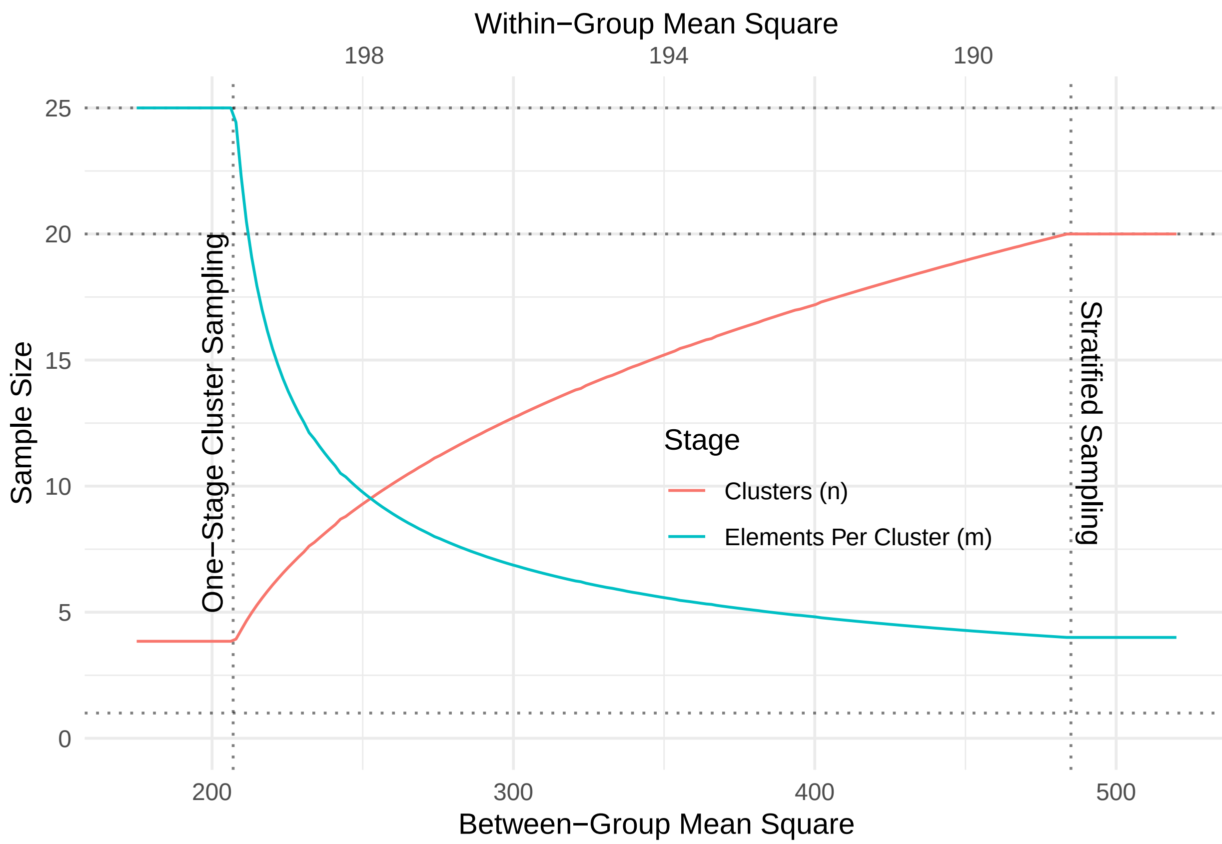
Clearly the solution is approximate — we would need to round the sample sizes.

How do we get σ_b^2 and σ_w^2 *in practice*?

Example: The following plot shows m_{opt} and n_{opt} as a function of the between-group and within-group mean squares for a population of $N = 100$ clusters with $\bar{M} = 25$ elements in each cluster. The costs are $c_1 = 10$ and $c_2 = 1$. The fixed total cost is $C = 100$.



Example: The following plot shows m_{opt} and n_{opt} as a function of the between-group and within-group mean squares for a population of now $N = 20$ clusters, with the cost-per-cluster reduced to $c_1 = 1$.


Within−Group Mean Square
198
194
190
25
20
15
10
5
0
Sample Size
One−Stage Cluster Sampling
Stratified Sampling
Stage
Clusters (n)
Elements Per Cluster (m)
200
300
400
500
Between−Group Mean Square

Numerical Solution

We can also solve this problem *numerically*. Here is the numerical solution to the first example above.

```
library(Rsolnp)

# Function to compute the variance of the estimator.
vf <- function(x, msb, msw, N, Mbar, c1, c2) {
  m <- x[1]
  n <- x[2]
  return((1 - n/N) * msb / (n * Mbar) + (1 - m/Mbar) * msw/(n*m))
}

# Function to compute the cost of the survey.
cf <- function(x, msb, msw, N, Mbar, c1, c2) {
  m <- x[1]
  n <- x[2]
  return(n*c1 + n*m*c2)
}

# Find n and m to minimize the variance subject to the
# constraint that the cost must equal 100.
tmp <- solnp(pars = c(5,5), fun = vf, eqfun = cf, eqB = 100,
  N = 25, Mbar = 50, msb = 1551.41, msw = 10.39, c1 = 10, c2 = 1, LB = c(1,1), UB = c(50,25))
```

```
Iter: 1 fn: 3.4968   Pars:  3.71283 7.09572
Iter: 2 fn: 3.1969   Pars:  2.72390 7.80416
Iter: 3 fn: 3.1021   Pars:  2.17386 8.19659
Iter: 4 fn: 3.0797   Pars:  1.90797 8.39335
Iter: 5 fn: 3.0767   Pars:  1.84024 8.44547
Iter: 6 fn: 3.0766   Pars:  1.83610 8.44873
Iter: 7 fn: 3.0766   Pars:  1.83607 8.44875
Iter: 8 fn: 3.0766   Pars:  1.83607 8.44875
solnp--> Completed in 8 iterations
```

```
tmp$pars
```

```
[1] 1.836067 8.448752
```

```
round(tmp$pars)
```

```
[1] 2 8
```

Multi-Stage Cluster Sampling

A multi-stage cluster sampling design is a natural extension of a two-stage cluster sampling design. A k-stage cluster sampling design where $k \geq 2$ can be designed by applying clusters sampling designs *recursively*.

A three-stage cluster sampling design can be described as follows.

1. Partition the M elements in a population into clusters.
2. Select n_1 *primary* sampling units using a probability sampling design.
3. For each of the n_1 sampled primary units, partition the elements into sub-clusters.
4. Select n_2 *secondary* sampling units using a probability sampling design.
5. For each of the n_2 sampled secondary units, sample *tertiary* sampling units (i.e., elements) using a probability sampling design, and observe the target variable for these sampled elements.

Multi-stage cluster sampling designs are useful when elements are formed into groups *hierarchically*. Here are some examples of the three levels of sampling units in three-stage cluster sampling designs.

Sampling Unit			
primary	secondary	tertiary	Target Variable
pallet	box	widget	weight
neighborhood	block	household	income
county	farm	field	acres of wheat
school	classroom	student	test score
day	hour	minute	number of fish
plot	sub-plot	tree	volume

In principle, any probability sampling design can be used at each stage. But often only the first stage uses a design other than SRS (e.g., PPS, stratified random sampling). Tyipcally later stages use SRS.