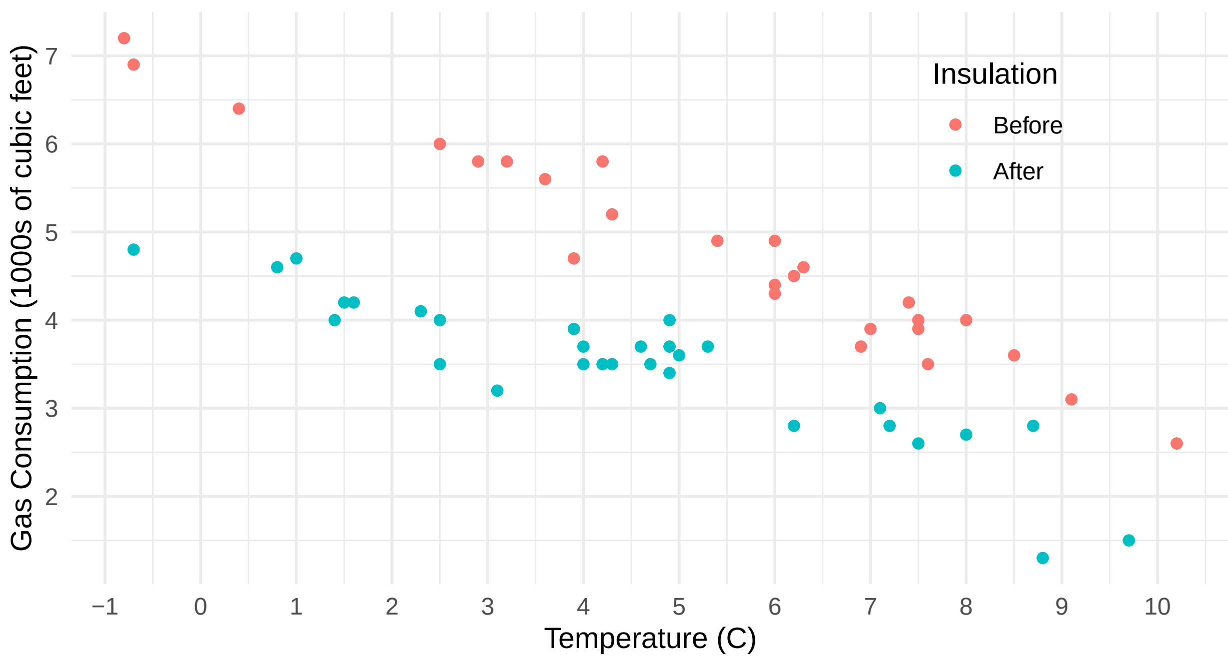
Friday, Jan 19

Quantitative and Categorical Explanatory Variables

```r
library(MASS) # for the whiteside data frame
head(whiteside)
```

```
   Insul Temp Gas
1 Before -0.8 7.2
2 Before -0.7 6.9
3 Before  0.4 6.4
4 Before  2.5 6.0
5 Before  2.9 5.8
6 Before  3.2 5.8
```

Here is a plot of the data (we will consider how to create such plots later).



Consider the following model specified using the `lm` function.

```r
m <- lm(Gas ~ Insul + Temp, data = whiteside)
summary(m)$coefficients
```

```
             Estimate Std. Error   t value     Pr(>|t|)
(Intercept)  6.551329 0.11808641  55.47911 1.256860e-48
InsulAfter  -1.565205 0.09705338 -16.12726 4.310904e-22
Temp        -0.336697 0.01776311 -18.95484 2.864348e-25
```

The model specified above can be written as

$$E(Y_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2},$$

or

$$E(G_i) = \beta_0 + \beta_1 d_i + \beta_2 t_i,$$

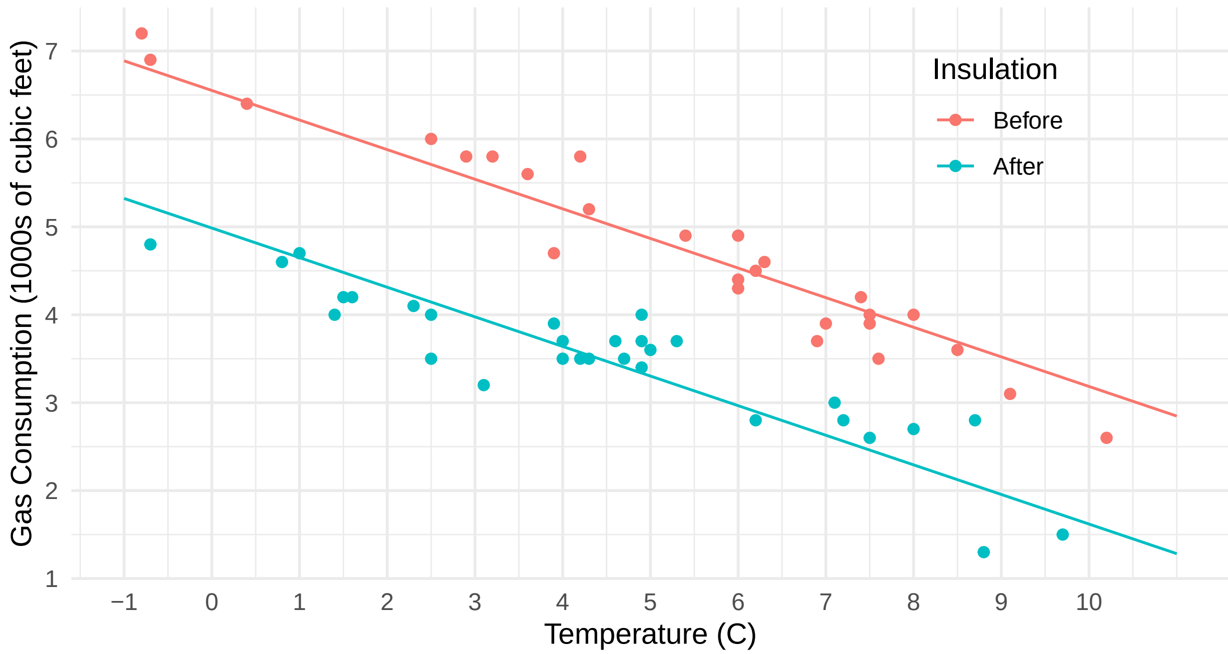
where G_i is gas consumption, t_i is temperature, and d_i is defined as

$$d_i = \begin{cases} 1, & \text{if the } i\text{-th observation is after insulation,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus we can also write the model case-wise as

$$E(G_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if the } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + \beta_2 t_i, & \text{if the } i\text{-th observation is after insulation.} \end{cases}$$

If we plot expected gas consumption versus temperature, we get a line with a slope of β_2 and an intercept of β_0 *before* insulation and $\beta_0 + \beta_1$ *after* insulation.



For the model

$$E(G_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if the } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + \beta_2 t_i, & \text{if the } i\text{-th observation is after insulation,} \end{cases}$$

how can we write the following as functions of β_0, β_1, and/or β_2?

1. The change in the expected gas consumption per unit increase in temperature *before* insulation.
2. The change in the expected gas consumption per unit increase in temperature *after* insulation.
3. The *difference* in the two rates of change described above.
4. The expected gas consumption before and after insulation at 0C, and at 5C.
5. The change in expected gas consumption from before to after insulation at 0C, and at 5C.

Now consider the following model specified using the `lm` function.

```
m <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
# Note: A shortcut for the model formula a + b + a:b is a*b,
# so we could also write the model formula as Gas ~ Insul*Temp.
summary(m)$coefficients
```

```
                 Estimate Std. Error    t value     Pr(>|t|)
(Intercept)     6.8538277 0.13596397  50.409146 7.997414e-46
InsulAfter     -2.1299780 0.18009172 -11.827185 2.315921e-16
Temp           -0.3932388 0.02248703 -17.487358 1.976009e-23
InsulAfter:Temp 0.1153039 0.03211212   3.590665 7.306852e-04
```

The model specified above can be written as

$$E(Y_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3},$$

or

$$E(G_i) = \beta_0 + \beta_1 d_i + \beta_2 t_i + \beta_3 d_i t_i,$$

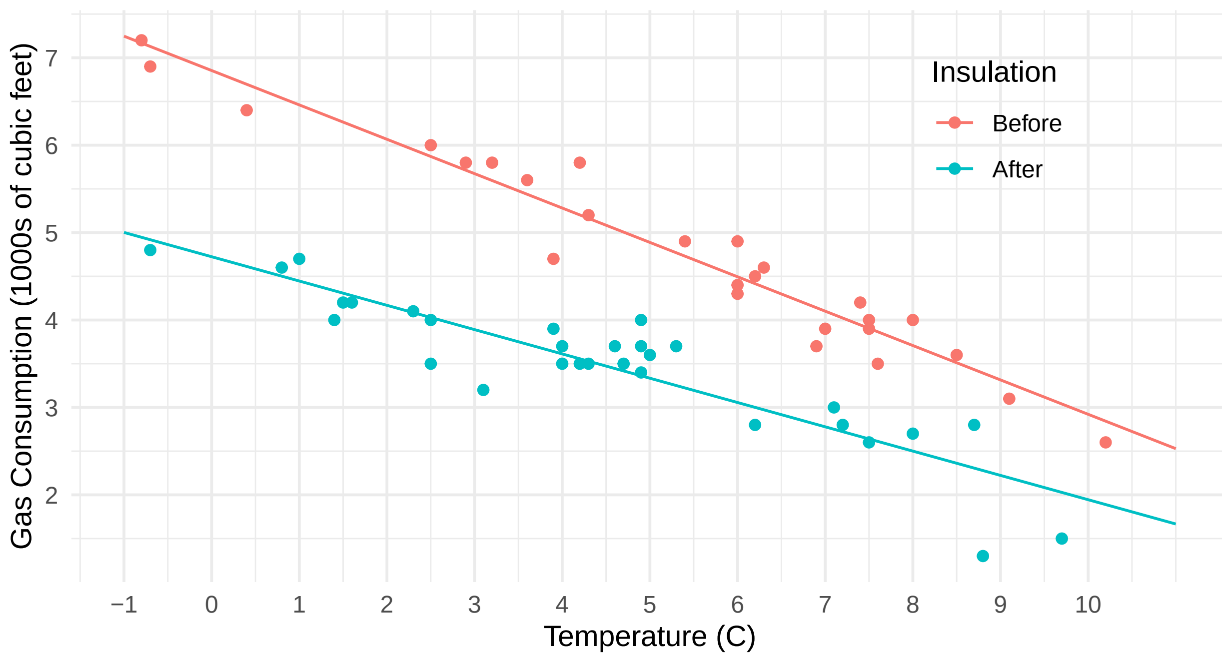
where again G_i is gas consumption, t_i is temperature, and d_i is defined as

$$d_i = \begin{cases} 1, & \text{if the } i\text{-th observation is after insulation,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus we can also write the model case-wise as

$$E(G_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if the } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + (\beta_2 + \beta_3) t_i, & \text{if the } i\text{-th observation is after insulation.} \end{cases}$$

If we plot expected gas consumption versus temperature, we get a line with a slope of β_2 *before* insulation and $\beta_2 + \beta_3$ *after* insulation, and an intercept of β_0 *before* insulation and $\beta_0 + \beta_1$ *after* insulation.



For the model

$$E(G_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if the } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + (\beta_2 + \beta_3) t_i, & \text{if the } i\text{-th observation is after insulation,} \end{cases}$$

how can we write the following as functions of β_0, β_1, β_2 and/or β_3?

1. The rate of change in the expected gas consumption per unit increase in temperature *before* insulation.
2. The rate of change in the expected gas consumption per unit increase in temperature *after* insulation.
3. The *difference* in the two rates of change described above.
4. The expected gas consumption before and after insulation at 0C, and at 5C.
5. The change in expected gas consumption from before to after insulation at 0C, and at 5C.

Identifying Model Terms in Linear Models in R

A linear model can be written as

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$$

that has k *terms*, where the j-th term is $\beta_j x_j$. The row labels returned by `summary` such as

```
summary(m)$coefficients
```

```
                  Estimate Std. Error    t value     Pr(>|t|)
(Intercept)      6.8538277 0.13596397  50.409146 7.997414e-46
InsulAfter      -2.1299780 0.18009172 -11.827185 2.315921e-16
Temp            -0.3932388 0.02248703 -17.487358 1.976009e-23
InsulAfter:Temp  0.1153039 0.03211212   3.590665 7.306852e-04
```

let us identify each term in the model using the following rules.

1. If the label is `(Intercept)` them the first term is β_0. Otherwise β_0 is excluded.
2. If the label is the name of a *quantitative* variable (e.g., `Temp`), then the model includes the term $\beta_j x_j$ where x_j is the value of that variable for a given observation.
3. If the label is the name of a *categorical/factor* variable with a category/level "suffix" (e.g., `InsulAfter`), then the model includes the term $\beta_j x_j$ where x_j is an *indicator variable* for when the observation of the categorical/factor variable equals that category/level.[1]
4. If the label is has a colon (`:`) between two labels (e.g., `InsulAfter:Temp`), then the model includes the term $\beta_j x_j$ where x_j is the *product* of the values of two x variables defined by the two labels (i.e., x defined by `InsulAfter` times the x defined by `Temp`).

Example: Here's an alternative parameterization of the model used above.

```
m <- lm(Gas ~ -1 + Insul + Insul:Temp, data = whiteside)
summary(m)$coefficients
```

```
                   Estimate Std. Error   t value     Pr(>|t|)
InsulBefore       6.8538277 0.13596397  50.40915 7.997414e-46
InsulAfter        4.7238497 0.11809668  39.99985 9.918382e-41
InsulBefore:Temp -0.3932388 0.02248703 -17.48736 1.976009e-23
InsulAfter:Temp  -0.2779350 0.02292426 -12.12405 8.936039e-17
```

How would we write this model?

[1] There are alternative "systems" to indicator variables when dealing with categorical explanatory variables. One example is where the reference category/level gets a value of -1 instead of 0. These have some applications when using specialized parameterizations of linear models and simplify some calculations, but they can often be avoided. Most software (including R) uses the indicator/dummy variable system by default.

Linear Functions of Parameters

"The default output isn't necessarily useful, and what is useful is not necessarily the default output." — Me

For a regression model like

$$E(Y_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik},$$

a quantity of interest can often be written as a *linear function of the model parameters* (sometimes also called a *linear combination* or *linear contrast* of the parameters). This can be written in general as

$$\ell = a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \cdots + a_k\beta_k + b,$$

where $a_0, a_1, \ldots, a_k$ and b are specified coefficients. There are several ways to do this on R. One way is to use the `lincon` function (for **lin**ear **con**trast) from the **trtools** package.

Example: Consider the following model.

```r
m <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
summary(m)$coefficients
```

```
                 Estimate Std. Error    t value     Pr(>|t|)
(Intercept)     6.8538277 0.13596397  50.409146 7.997414e-46
InsulAfter     -2.1299780 0.18009172 -11.827185 2.315921e-16
Temp           -0.3932388 0.02248703 -17.487358 1.976009e-23
InsulAfter:Temp  0.1153039 0.03211212   3.590665 7.306852e-04
```

```r
library(trtools)    # so we can use lincon below
options(digits = 4) # for display purposes
```

How do we write the following as linear combinations and then make inferences about that linear combination?

1. The rate of change in expected gas consumption before insulation is β_2.

```r
lincon(m, a = c(0,0,1,0), b = 0)
```

```
            estimate      se   lower   upper tvalue df    pvalue
(0,0,1,0),0  -0.3932 0.02249 -0.4384 -0.3481 -17.49 52 1.976e-23
```

2. The rate of change in expected gas consumption after insulation is $\beta_2 + \beta_3$.

```r
lincon(m, a = c(0,0,1,1), b = 0)
```

```
            estimate      se   lower   upper tvalue df    pvalue
(0,0,1,1),0  -0.2779 0.02292 -0.3239 -0.2319 -12.12 52 8.936e-17
```

3. The *difference* in the two rates of change described above (after minus before) is β_3.

```r
lincon(m, a = c(0,0,0,1), b = 0)
```

```
            estimate      se   lower  upper tvalue df    pvalue
(0,0,0,1),0   0.1153 0.03211 0.05087 0.1797  3.591 52 0.0007307
```

4. The expected gas consumption before and after insulation at 5C are $\beta_0 + \beta_2 5$ and $\beta_0 + \beta_1 + (\beta_2 + \beta_3)5$, respectively.

```r
lincon(m, a = c(1,0,5,0), b = 0) # before
```

```
             estimate      se lower upper tvalue df    pvalue
(1,0,5,0),0     4.888 0.06383  4.76 5.016  76.57 52 3.885e-55
```

```r
lincon(m, a = c(1,1,5,5), b = 0) # after
```

```
             estimate      se lower upper tvalue df    pvalue
(1,1,5,5),0     3.334 0.06024 3.213 3.455  55.35 52 6.772e-48
```

5. The change in the expected gas consumption from before to after insulation at 5C is $\beta_1 + \beta_3 5$.

```r
lincon(m, a = c(0,1,0,5), b = 0)
```

```
             estimate      se lower  upper tvalue df    pvalue
(0,1,0,5),0    -1.553 0.08777 -1.73 -1.377  -17.7 52 1.155e-23
```

Note: Because in many cases we have $b = 0$, the `b` argument has this as a default value and can be omitted if $b = 0$. So we can write `lincon(m, a = c(0,1,0,5))` instead of `lincon(m, a = c(0,1,0,5), b = 0)`.