

Friday, Jan 26

## Confidence Intervals and Significance Tests

A significance test can be used to derive a confidence interval, and a confidence interval can be used to conduct a significance test. If we have hypotheses for a two-sided test like

$$H_0: \beta_j = c \text{ and } H_a: \beta_j \neq c,$$

then we reject  $H_0$  if and only if the confidence interval for  $\beta_j$  does not contain  $c$ , with a couple of caveats.

1. The confidence level must be  $(1 - \alpha)100\%$  ( $\alpha$  is the *significance level*).
2. The test is two-sided (but one-sided tests match one-sided confidence intervals).

A confidence interval with confidence level  $(1 - \alpha)100\%$  effectively defines all values of the parameter that *would not be rejected* in a two-sided test with significance level  $\alpha$ .

Note that this also applies to a linear function of model parameters ( $\ell$ ). So if we have the hypotheses

$$H_0: \ell = c \text{ and } H_a: \ell \neq c,$$

then we reject  $H_0$  if and only if the confidence interval for  $\ell$  does not contain  $c$ .

**Example:** Consider again the model for the `anorexia` data, but parameterized to compare the two treatment conditions against the control so that the model is

$$E(Y_i) = \begin{cases} \beta_0, & \text{if the } i\text{-th observation is under the control condition,} \\ \beta_0 + \beta_1, & \text{if } i\text{-th observation under cognitive behavioral therapy,} \\ \beta_0 + \beta_2, & \text{if the } i\text{-th observations is under family therapy.} \end{cases}$$

```
library(MASS) # for anorexia data
anorexia$change <- anorexia$Postwt - anorexia$Prewt
anorexia$Treat <- relevel(anorexia$Treat, ref = "Cont")
m <- lm(change ~ Treat, data = anorexia)
cbind(summary(m)$coefficients, confint(m))
```

	Estimate	Std. Error	t value	Pr(> t )	2.5 %	97.5 %
(Intercept)	-0.450	1.476	-0.3048	0.761447	-3.3954	2.495
TreatCBT	3.457	2.033	1.7001	0.093608	-0.5994	7.513
TreatFT	7.715	2.348	3.2854	0.001602	3.0302	12.399

We can produce the same inferences using `contrast`.

```
library(trtools)
contrast(m,
  a = list(Treat = c("CBT", "FT")),
  b = list(Treat = "Cont"),
  cnames = c("Cognitive vs Control", "Family vs Control"))
```

	estimate	se	lower	upper	tvalue	df	pvalue
Cognitive vs Control	3.457	2.033	-0.5994	7.513	1.700	69	0.093608
Family vs Control	7.715	2.348	3.0302	12.399	3.285	69	0.001602

## Joint Hypotheses

**Example:** Consider the following model and hypotheses for the `anorexia` data.

```
library(MASS) # for anorexia data
anorexia$change <- anorexia$Postwt - anorexia$Prewt
m.anorexia <- lm(change ~ Treat, data = anorexia)
summary(m.anorexia)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	3.007	1.398	2.151	0.03499
TreatCont	-3.457	2.033	-1.700	0.09361
TreatFT	4.258	2.300	1.852	0.06838

The model is therefore

$$E(Y_i) = \begin{cases} \beta_0, & \text{if the } i\text{-th observation is under cognitive behavioral therapy,} \\ \beta_0 + \beta_1, & \text{if } i\text{-th observation is under the control condition,} \\ \beta_0 + \beta_2, & \text{if the } i\text{-th observations is under family therapy.} \end{cases}$$

In some cases we might be testing hypothesis like  $H_0 : \beta_2 = 0$  or  $H_0 : \beta_1 - \beta_2 = 0$ . But in other cases we might be testing what is sometimes called a *joint* hypothesis such as

$$H_0 : \beta_1 = 0 \text{ and } \beta_2 = 0 \text{ versus } H_a : \text{not both } \beta_1 = 0 \text{ and } \beta_2 = 0.$$

What does it imply if both  $\beta_1 = 0$  and  $\beta_2 = 0$ ?

**Example:** Consider the following model for the `whiteside` data.

```
m.insulation <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
summary(m.insulation)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	6.8538	0.13596	50.409	7.997e-46
InsulAfter	-2.1300	0.18009	-11.827	2.316e-16
Temp	-0.3932	0.02249	-17.487	1.976e-23
InsulAfter:Temp	0.1153	0.03211	3.591	7.307e-04

The model is therefore

$$E(Y_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + (\beta_2 + \beta_3) t_i, & \text{if } i\text{-th observation is after insulation.} \end{cases}$$

We might test a single null hypothesis that the rate of change in expected gas consumption with respect to temperature is the same before and after insulation — i.e.,  $H_0 : \beta_3 = 0$ . But consider the joint hypothesis

$$H_0 : \beta_1 = 0 \text{ and } \beta_3 = 0 \text{ versus } H_a : \text{not both } \beta_1 = 0 \text{ and } \beta_3 = 0.$$

What does it imply if both  $\beta_1 = 0$  and  $\beta_3 = 0$ ?

## The “Analysis of Variance” Calculations

Calculations for inference for linear models is often based on the sums of squares decomposition

$$\underbrace{\sum_{i=1}^n (y_i - \bar{y})^2}_{\text{total}} = \underbrace{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}_{\text{model/regression}} + \underbrace{\sum_{i=1}^n (y_i - \hat{y}_i)^2}_{\text{error/residual}},$$

where  $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \dots + \hat{\beta}_k x_{ik}$ , and the degrees of freedom decomposition

$$\underbrace{n-1}_{\text{total}} = \underbrace{p-1}_{\text{model/regression}} + \underbrace{n-p}_{\text{error/residual}},$$

where  $p$  is the number of  $\beta_j$  parameters, and  $p = k + 1$  if the model includes a  $\beta_0$ . (Note: If the  $\beta_0$  parameter is omitted from the model, the total degrees of freedom becomes  $n$  and the model/regression degrees of freedom becomes  $p$ .)

A *mean square* is a variance-like quantity that is a sum of squares divided by its corresponding degrees of freedom.

Tests can be conducted using the  $F$  test statistic which can be written as

$$F = \frac{(\text{RSS}_{\text{null}} - \text{RSS}_{\text{full}})/(\text{RDF}_{\text{null}} - \text{RDF}_{\text{full}})}{\text{RSS}_{\text{full}}/\text{RDF}_{\text{full}}},$$

where RSS and RDF refer to the *residual* sum of squares and degrees of freedom, respectively. The degrees of freedom for the  $F$  distribution are  $\text{RDF}_{\text{null}} - \text{RDF}_{\text{full}}$  (numerator) and  $\text{RSS}_{\text{full}}$  (denominator). The *full* model is the model we are using, and the *null* (aka “reduced”) model is what the full model reduces to *if the null hypothesis is true*. The  $F$  test statistic can be used for tests of individual and joint hypotheses in linear models.

## Using the anova Function

The `anova` function is particularly useful for testing joint hypothesis, although it can also be used to test a hypothesis about a single parameter.

Applying `anova` to a single model will produce the RSS and RDF in the **Residuals** row.

```
anova(m.anorexia)
```

Analysis of Variance Table

Response: change

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Treat	2	615	307.3	5.42	0.0065 **
Residuals	69	3911	56.7		

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

To conduct a test, the recommended approach is to apply `anova` to a null model and the full model.

```
m.full <- lm(change ~ Treat, data = anorexia)
m.null <- lm(change ~ 1, data = anorexia) # use ~ 1 if no explanatory variables
anova(m.null, m.full)
```

Analysis of Variance Table

Model 1: change ~ 1

Model 2: change ~ Treat

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	71	4525				
2	69	3911	2	615	5.42	0.0065 **

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

```
m.full <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
m.null <- lm(Gas ~ Temp, data = whiteside)
anova(m.null, m.full)
```

## Analysis of Variance Table

Model 1: Gas ~ Temp

Model 2: Gas ~ Insul + Temp + Insul:Temp

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	54	40.0				
2	52	5.4	2	34.6	166	<2e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

The `anova` function can also do a test concerning a single parameter. Here are two approaches to testing the null hypothesis that  $\beta_3 = 0$  in the model

$$E(Y_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + (\beta_2 + \beta_3) t_i, & \text{if } i\text{-th observation is after insulation.} \end{cases}$$

```
m.full <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
m.null <- lm(Gas ~ Insul + Temp, data = whiteside)
anova(m.null, m.full)
```

## Analysis of Variance Table

Model 1: Gas ~ Insul + Temp

Model 2: Gas ~ Insul + Temp + Insul:Temp

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	53	6.77				
2	52	5.43	1	1.34	12.9	0.00073 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

```
summary(m.full)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	6.8538	0.13596	50.409	7.997e-46
InsulAfter	-2.1300	0.18009	-11.827	2.316e-16
Temp	-0.3932	0.02249	-17.487	1.976e-23
InsulAfter:Temp	0.1153	0.03211	3.591	7.307e-04

Comment: When conducting a test concerning one parameter (or a single linear function of the model parameters), the  $F$  and  $t$  test statistics have the relationship  $t^2 = F$  and produce the same p-values.

## Example: Three Approaches to One Test

Consider again the model for the `anorexia` data, but suppose we parameterized the model differently.

```
anorexia$Treat <- relevel(anorexia$Treat, ref = "Cont")
m.anorexia <- lm(change ~ Treat, data = anorexia)
summary(m.anorexia)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-0.450	1.476	-0.3048	0.761447
TreatCBT	3.457	2.033	1.7001	0.093608
TreatFT	7.715	2.348	3.2854	0.001602

The model is therefore

$$E(Y_i) = \begin{cases} \beta_0, & \text{if the } i\text{-th observation is from the control group,} \\ \beta_0 + \beta_1, & \text{if the } i\text{-th observation is from the cognitive-behavioral therapy group,} \\ \beta_0 + \beta_2, & \text{if the } i\text{-th observations is from the family therapy group.} \end{cases}$$

Now consider a test of the null hypothesis that the expected weight change is the same regardless of which of the two therapies (i.e., cognitive-behavioral or family) is used. This is the null hypothesis that  $\beta_1 = \beta_2$  or, equivalently,  $\beta_1 - \beta_2 = 0$ .

1. Using `lincon` we can test this null hypothesis as follows.

```
m <- lm(change ~ Treat, data = anorexia)
trtools::lincon(m, a = c(0, 1, -1))
```

	estimate	se	lower	upper	tvalue	df	pvalue
(0,1,-1),0	-4.258	2.3	-8.845	0.3299	-1.852	69	0.06838

This is because the null hypothesis can be written as

$$\ell = 0 \times \beta_0 + 1 \times \beta_1 + (-1) \times \beta_2 = \beta_1 - \beta_2.$$

2. Using `contrast` we can test this null hypothesis as follows.

```
m <- lm(change ~ Treat, data = anorexia)
trtools::contrast(m, a = list(Treat = "CBT"), b = list(Treat = "FT"))
```

	estimate	se	lower	upper	tvalue	df	pvalue
	-4.258	2.3	-8.845	0.3299	-1.852	69	0.06838

3. Using `anova` we can test this null hypothesis as follows.

```
anorexia$therapy <- ifelse(anorexia$Treat == "Cont", "control", "therapy")
head(anorexia)
```

	Treat	Prewt	Postwt	change	therapy
1	Cont	80.7	80.2	-0.5	control
2	Cont	89.4	80.1	-9.3	control
3	Cont	91.8	86.4	-5.4	control
4	Cont	74.0	86.3	12.3	control
5	Cont	78.1	76.1	-2.0	control
6	Cont	88.3	78.1	-10.2	control

```
tail(anorexia)
```

	Treat	Prewt	Postwt	change	therapy
67	FT	82.1	95.5	13.4	therapy
68	FT	77.6	90.7	13.1	therapy
69	FT	83.5	92.5	9.0	therapy
70	FT	89.9	93.8	3.9	therapy
71	FT	86.0	91.7	5.7	therapy
72	FT	87.3	98.0	10.7	therapy

```
m.full <- lm(change ~ Treat, data = anorexia)
m.null <- lm(change ~ therapy, data = anorexia)
anova(m.null, m.full)
```

Analysis of Variance Table

Model 1: change ~ therapy

```

Model 2: change ~ Treat
      Res.Df  RSS Df Sum of Sq    F Pr(>F)
1         70 4105
2         69 3911  1       194 3.43  0.068 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

Note that the null model can be written as

$$E(Y_i) = \begin{cases} \beta_0, & \text{if the } i\text{-th observation is from the control group,} \\ \beta_0 + \beta_1, & \text{if the } i\text{-th observation is from the therapy group,} \end{cases}$$

or

$$E(Y_i) = \begin{cases} \beta_0, & \text{if the } i\text{-th observation is from the control group,} \\ \beta_0 + \beta_1, & \text{if the } i\text{-th observation is from the cognitive-behavioral therapy group,} \\ \beta_0 + \beta_1, & \text{if the } i\text{-th observations is from the family therapy group.} \end{cases}$$

So this model is effectively equivalent to the full model with  $\beta_1 = \beta_2$ .

## The Trouble with ANOVA Tables

I **do not** recommended trying to produce tests by applying `anova` to a *single* model object. While it can produce desired tests *in some cases* and *if used correctly*, it often produces confusing results. For example, the following produces a test of the null hypothesis  $H_0 : \beta_1 = 0, \beta_2 = 0$  for the `anorexia` model.

```

m <- lm(change ~ Treat, data = anorexia)
anova(m)

```

Analysis of Variance Table

```

Response: change
      Df Sum Sq Mean Sq F value Pr(>F)
Treat    2    615    307.3    5.42 0.0065 **
Residuals 69   3911     56.7
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

But the tests shown here are maybe not what you think they are.

```

m <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
anova(m)

```

Analysis of Variance Table

```

Response: Gas
      Df Sum Sq Mean Sq F value    Pr(>F)
Insul    1    22.3     22.3   214.2 < 2e-16 ***
Temp     1    45.9     45.9   439.9 < 2e-16 ***
Insul:Temp 1     1.3      1.3    12.9 0.00073 ***
Residuals 52     5.4      0.1
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

If you know what you are doing, there are alternatives to `anova` that are perhaps better (e.g., the `Anova` function from the `car` package), but there is often a more clear approach using two models in `anova`, using `contrast` or `lincon`, or using the `emmeans` package (which we will discuss later).

**Note:** Another potentially confusing test is one that appears at the bottom of `summary`. It tests the null hypothesis that all  $\beta_j$  (except  $\beta_0$ ) equal zero. For the model for the `anorexia` data it is the same as the test conducted earlier.

```
m <- lm(change ~ Treat, data = anorexia)
summary(m)
```

Call:

```
lm(formula = change ~ Treat, data = anorexia)
```

Residuals:

Min	1Q	Median	3Q	Max
-12.56	-4.54	-1.01	3.85	17.89

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-0.45	1.48	-0.30	0.7614
TreatCBT	3.46	2.03	1.70	0.0936 .
TreatFT	7.71	2.35	3.29	0.0016 **

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.53 on 69 degrees of freedom

Multiple R-squared: 0.136, Adjusted R-squared: 0.111

F-statistic: 5.42 on 2 and 69 DF, p-value: 0.0065

But for the model for the `whiteside` data the utility of this test is questionable.

```
m <- lm(Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
summary(m)
```

Call:

```
lm(formula = Gas ~ Insul + Temp + Insul:Temp, data = whiteside)
```

Residuals:

Min	1Q	Median	3Q	Max
-0.9780	-0.1801	0.0376	0.2093	0.6380

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	6.8538	0.1360	50.41	< 2e-16 ***
InsulAfter	-2.1300	0.1801	-11.83	2.3e-16 ***
Temp	-0.3932	0.0225	-17.49	< 2e-16 ***
InsulAfter:Temp	0.1153	0.0321	3.59	0.00073 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.323 on 52 degrees of freedom

Multiple R-squared: 0.928, Adjusted R-squared: 0.924

F-statistic: 222 on 3 and 52 DF, p-value: <2e-16

Just because R gives you output does not mean it is useful!

**Note:** The `Residual standard error` showns by `summary` is the square root of the residual/error mean square (i.e., the square root of the ratio of the residual sum of squares to the residual degrees of freedom).

The degrees of freedom shown after **Residual standard error** is the residual degrees of freedom.