Wednesday, Apr 19

The Multinomial Logit Model

Recall that a logistic regression model can be written as

$$\log\left[\frac{P(Y_i = 1)}{1 - P(Y_i = 1)}\right] = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik}.$$

This can also be written as

$$\log(\pi_{i2}/\pi_{i1}) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik},$$

or

$$\pi_{i2}/\pi_{i1} = e^{\beta_0} e^{\beta_1 x_{i1}} \cdots e^{\beta_k x_{ik}},$$

where

$$\begin{aligned} \pi_{i2} &= P(Y_i = 1), \\ \pi_{i1} &= P(Y_i = 0). \end{aligned}$$

Here the ratio of probabilities π_{i2}/π_{i1} is the *odds* that $Y_i = 1$ rather than $Y_i = 0$. Note that odds are basically the probability of one event relative to that of another event.

Let $Y_i = 1, 2, \dots, R$ denote R categories, but not necessarily ordered in any way, and let $\pi_{i1}, \pi_{i2}, \dots, \pi_{iR}$ denote the probability of each category. The *multinomial* logistic regression model can be written as

$$\begin{aligned} \log(\pi_{i2}/\pi_{i1}) &= \beta_0^{(2)} + \beta_1^{(2)} x_{i1} + \cdots + \beta_k^{(2)} x_{ik}, \\ \log(\pi_{i3}/\pi_{i1}) &= \beta_0^{(3)} + \beta_1^{(3)} x_{i1} + \cdots + \beta_k^{(3)} x_{ik}, \\ &\vdots \\ \log(\pi_{iR}/\pi_{i1}) &= \beta_0^{(R)} + \beta_1^{(R)} x_{i1} + \cdots + \beta_k^{(R)} x_{ik}, \end{aligned}$$

for a system of $R - 1$ equations. This can also be written as

$$\begin{aligned} \pi_{i2}/\pi_{i1} &= e^{\beta_0^{(2)}} e^{\beta_1^{(2)} x_{i1}} \cdots e^{\beta_k^{(2)} x_{ik}}, \\ \pi_{i3}/\pi_{i1} &= e^{\beta_0^{(3)}} e^{\beta_1^{(3)} x_{i1}} \cdots e^{\beta_k^{(3)} x_{ik}}, \\ &\vdots \\ \pi_{iR}/\pi_{i1} &= e^{\beta_0^{(4)}} e^{\beta_1^{(4)} x_{i1}} \cdots e^{\beta_k^{(4)} x_{ik}}, \end{aligned}$$

so that the model relates the *odds* of categories 2 through R *relative to* the first category (often called a "baseline" or "reference" category). For example, π_{i3}/π_{i1} is the odds of the third category versus the first category. Applying the exponential function to a parameter or contrast gives an *odds ratio* that concerns the change in this odds.

Some algebra shows that the category probabilities can be written as

$$\begin{aligned}
\pi_{i1} &= 1 - (\pi_{i2} + \pi_{i3} + \cdots + \pi_{iR}), \\
\pi_{i2} &= \frac{e^{\eta_{i2}}}{1 + e^{\eta_{i2}} + e^{\eta_{i3}} + \cdots + e^{\eta_{iR}}} \\
\pi_{i3} &= \frac{e^{\eta_{i3}}}{1 + e^{\eta_{i2}} + e^{\eta_{i3}} + \cdots + e^{\eta_{iR}}} \\
&\vdots \\
\pi_{iR} &= \frac{e^{\eta_{ir}}}{1 + e^{\eta_{i2}} + e^{\eta_{i3}} + \cdots + e^{\eta_{iR}}}
\end{aligned}$$

where

$$\begin{aligned}
\eta_{i2} &= \beta_0^{(2)} + \beta_1^{(2)} x_{i1} + \cdots + \beta_k^{(2)} x_{ik}, \\
\eta_{i3} &= \beta_0^{(3)} + \beta_1^{(3)} x_{i1} + \cdots + \beta_k^{(3)} x_{ik}, \\
&\vdots \\
\eta_{iR} &= \beta_0^{(R)} + \beta_1^{(R)} x_{i1} + \cdots + \beta_k^{(R)} x_{ik}.
\end{aligned}$$

We can write this more compactly as

$$\pi_{ic} = \frac{e^{\eta_{ic}}}{1 + \sum_{t=2}^{K} e^{\eta_{it}}}$$

or

$$\pi_{ic} = \frac{e^{\eta_{ic}}}{\sum_{t=1}^{K} e^{\eta_{it}}},$$

if we let $\eta_{i1} = 0$ since $e^0 = 1$. This is useful for computing and plotting estimated probabilities for each category of the response variable.

Example: Let's consider again the `pneumo` data from the **VGAM** package.

```r
library(VGAM)
m <- vglm(cbind(normal, mild, severe) ~ exposure.time,
  family = multinomial(refLevel = "normal"), data = pneumo)
summary(m)
```

```

Call:
vglm(formula = cbind(normal, mild, severe) ~ exposure.time, family = multinomial(refLevel = "normal"),
    data = pneumo)

Coefficients:
                Estimate Std. Error z value Pr(>|z|)    
(Intercept):1    -4.2917     0.5214   -8.23  < 2e-16 ***
(Intercept):2    -5.0598     0.5964   -8.48  < 2e-16 ***
exposure.time:1   0.0836     0.0153    5.47  4.5e-08 ***
exposure.time:2   0.1093     0.0165    6.64  3.2e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1])

Residual deviance: 13.93 on 12 degrees of freedom

Log-likelihood: -29.54 on 12 degrees of freedom
```

```
Number of Fisher scoring iterations: 5

Warning: Hauck-Donner effect detected in the following estimate(s):
'(Intercept):1', '(Intercept):2'


Reference group is level  1  of the response
```

Note: The categories/levels of the response variable correspond to the order they are specified in `cbind`.

Odds ratios can be obtained in the usual way.

```
exp(cbind(coef(m), confint(m)))
```

```
                             2.5 %  97.5 %
(Intercept):1   0.013682 0.004924 0.03802
(Intercept):2   0.006347 0.001972 0.02043
exposure.time:1 1.087156 1.055079 1.12021
exposure.time:2 1.115481 1.080048 1.15208
```

Here is another nice way to output the parameter estimates.

```
t(coef(m, matrix = TRUE))
```

```
                   (Intercept) exposure.time
log(mu[,2]/mu[,1])      -4.292       0.08357
log(mu[,3]/mu[,1])      -5.060       0.10929
```

Then we can obtain odds ratio as follows.

```
exp(t(coef(m, matrix = TRUE)))
```

```
                   (Intercept) exposure.time
log(mu[,2]/mu[,1])    0.013682         1.087
log(mu[,3]/mu[,1])    0.006347         1.115
```

Plotting the estimated category probabilities can be done as with previous models. First we create a data frame of estimated probabilities by exposure time and category.

```
d <- data.frame(exposure.time = seq(5, 52, length = 100))
d <- cbind(d, predict(m, newdata = d, type = "response"))
head(d)
```

```
  exposure.time normal    mild  severe
1         5.000 0.9692 0.02014 0.01062
2         5.475 0.9679 0.02093 0.01117
3         5.949 0.9665 0.02174 0.01175
4         6.424 0.9651 0.02259 0.01236
5         6.899 0.9635 0.02346 0.01300
6         7.374 0.9620 0.02437 0.01367
```

```
library(tidyr)
d <- d %>% pivot_longer(cols = c(normal, mild, severe),
  names_to = "condition", values_to = "probability")
head(d)
```

```
# A tibble: 6 x 3
  exposure.time condition probability
          <dbl> <chr>           <dbl>
```

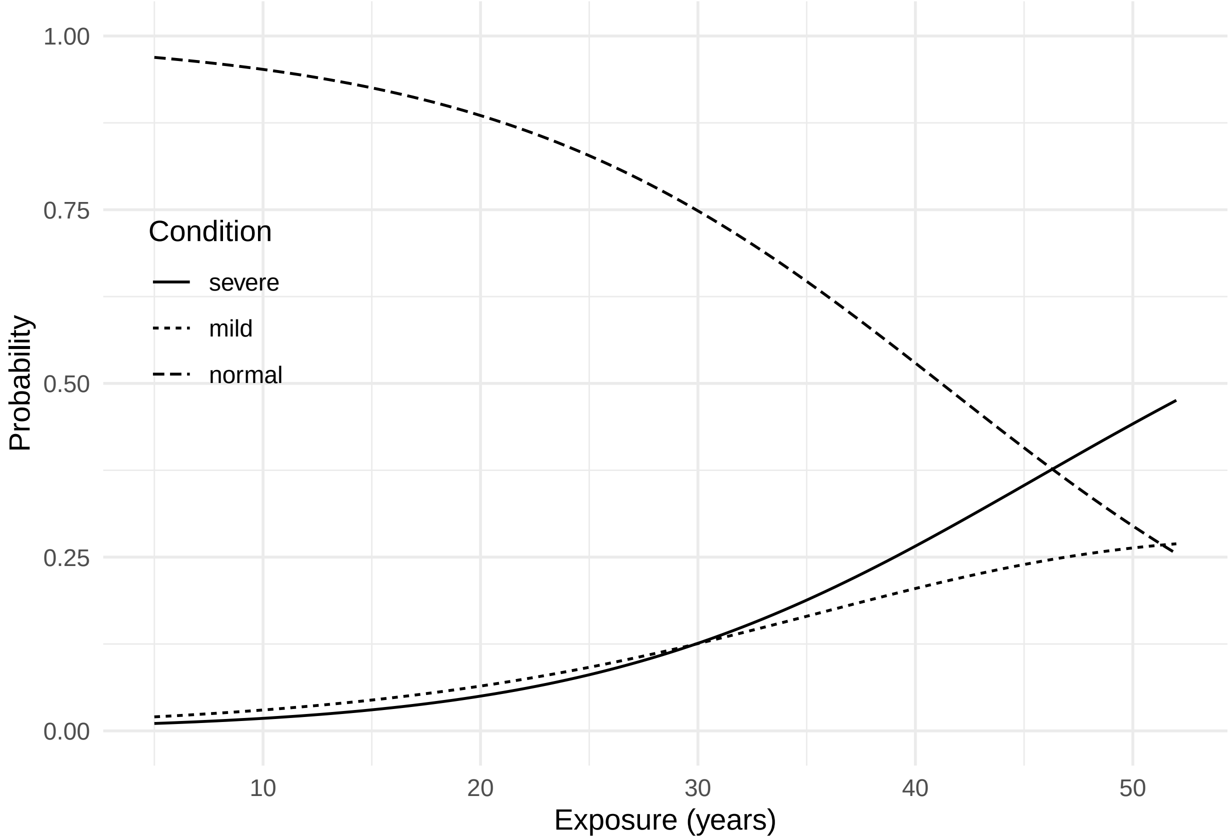
```
1          5    normal         0.969
2          5    mild           0.0201
3          5    severe         0.0106
4          5.47 normal         0.968
5          5.47 mild           0.0209
6          5.47 severe         0.0112
```

Next I reorder the factor levels just for aesthetic purposes.

```
d$condition <- factor(d$condition, levels = c("severe","mild","normal"))
```

And then finally we plot.

```
p <- ggplot(d, aes(x = exposure.time, y = probability)) +
  geom_line(aes(linetype = condition)) +
  ylim(0, 1) + theme_minimal() + theme(legend.position = c(0.1, 0.6)) +
  labs(x = "Exposure (years)", y = "Probability", linetype = "Condition")
plot(p)
```



Example: Consider the data frame `alligator` from the **EffectStars** package.

```
library(EffectStars)
data(alligator)
head(alligator)
```

```
  Food Size Gender    Lake
1 fish <2.3   male Hancock
2 fish <2.3   male Hancock
3 fish <2.3   male Hancock
4 fish <2.3   male Hancock
```

```
5 fish <2.3   male Hancock
6 fish <2.3   male Hancock
```

```
summary(alligator)
```

```
     Food      Size        Gender          Lake   
 bird  :13   <2.3:124   female: 89   George  :63  
 fish  :94   >2.3: 95   male  :130   Hancock :55  
 invert:61                           Oklawaha:48  
 other :32                           Trafford:53  
 rep   :19                                        
```

For illustration we will just consider just size and gender as explanatory variables.

```
m <- vglm(Food ~ Gender + Size, data = alligator,
  family = multinomial(refLevel = "bird"))
summary(m)
```

```

Call:
vglm(formula = Food ~ Gender + Size, family = multinomial(refLevel = "bird"),
    data = alligator)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)    
(Intercept):1   2.0324     0.5204    3.91  9.4e-05 ***
(Intercept):2   1.9897     0.5265    3.78  0.00016 ***
(Intercept):3   1.1748     0.5640    2.08  0.03724 *  
(Intercept):4  -0.0526     0.6829   -0.08  0.93859    
Gendermale:1    0.6149     0.6338    0.97  0.33197    
Gendermale:2    0.5247     0.6589    0.80  0.42585    
Gendermale:3    0.4185     0.7030    0.60  0.55162    
Gendermale:4    0.5833     0.7841    0.74  0.45691    
Size>2.3:1     -0.7535     0.6439   -1.17  0.24193    
Size>2.3:2     -1.6746     0.6788   -2.47  0.01362 *  
Size>2.3:3     -0.9865     0.7143   -1.38  0.16723    
Size>2.3:4      0.1145     0.7962    0.14  0.88565    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1]), log(mu[,4]/mu[,1]),
log(mu[,5]/mu[,1])

Residual deviance: 588.2 on 864 degrees of freedom

Log-likelihood: -294.1 on 864 degrees of freedom

Number of Fisher scoring iterations: 5

No Hauck-Donner effect found in any of the estimates


Reference group is level  1  of the response
```

To help interpret the output let's check the level order.

```
levels(alligator$Food)
```

```
[1] "bird"   "fish"   "invert" "other"  "rep"
```

Extract parameter estimates and confidence intervals.

```
cbind(coef(m), confint(m))
```

```
                            2.5 %  97.5 %
(Intercept):1  2.03238  1.01239  3.0524
(Intercept):2  1.98965  0.95773  3.0216
(Intercept):3  1.17478  0.06944  2.2801
(Intercept):4 -0.05261 -1.39099  1.2858
Gendermale:1   0.61487 -0.62732  1.8571
Gendermale:2   0.52467 -0.76669  1.8160
Gendermale:3   0.41850 -0.95931  1.7963
Gendermale:4   0.58333 -0.95348  2.1201
Size>2.3:1    -0.75351 -2.01561  0.5086
Size>2.3:2    -1.67459 -3.00492 -0.3443
Size>2.3:3    -0.98650 -2.38642  0.4134
Size>2.3:4     0.11450 -1.44597  1.6750
```

```
t(coef(m, matrix = TRUE))
```

```
                   (Intercept) Gendermale Size>2.3
log(mu[,2]/mu[,1])     2.03238     0.6149  -0.7535
log(mu[,3]/mu[,1])     1.98965     0.5247  -1.6746
log(mu[,4]/mu[,1])     1.17478     0.4185  -0.9865
log(mu[,5]/mu[,1])    -0.05261     0.5833   0.1145
```

Compute odds ratios.

```
exp(t(coef(m, matrix = TRUE)))
```

```
                   (Intercept) Gendermale Size>2.3
log(mu[,2]/mu[,1])      7.6322      1.849   0.4707
log(mu[,3]/mu[,1])      7.3130      1.690   0.1874
log(mu[,4]/mu[,1])      3.2374      1.520   0.3729
log(mu[,5]/mu[,1])      0.9487      1.792   1.1213
```

Note that we can change the reference/baseline category. This changes the model parameterization but does not change the estimated probabilities.

Joint tests of the parameters for each explanatory variable can be conducted (via a likelihood ratio test) using anova.

```
anova(m)
```

```
Analysis of Deviance Table (Type II tests)

Model: 'multinomial', 'VGAMcategorical'

Link: 'multilogitlink'

Response: Food

       Df Deviance Resid. Df Resid. Dev Pr(>Chi)
Gender  4     1.03       868        589   0.9052
Size    4    14.08       868        602   0.0071 **
```

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Note that for other models we should use `anova` by specifying a null model, but here the `anova` function does that automatically.

Here are the estimated probabilities.

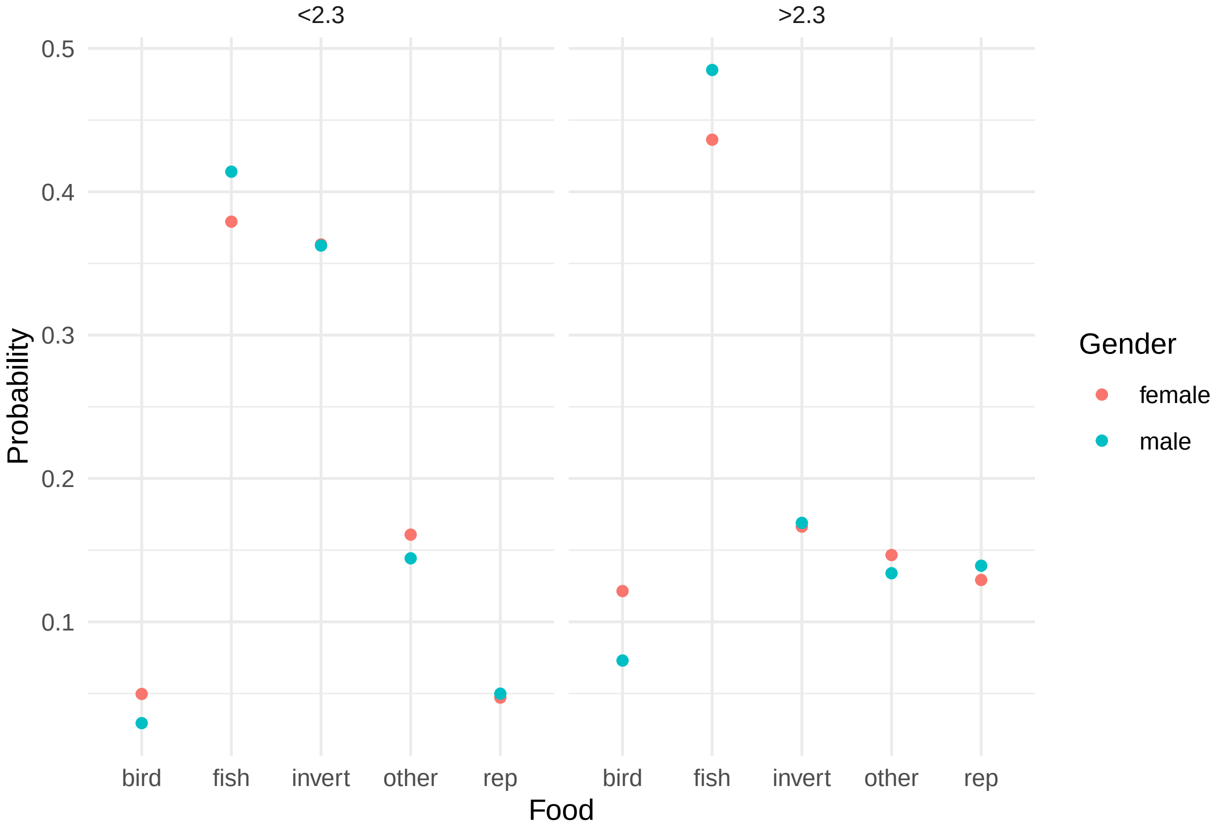
```
d <- expand.grid(Gender = c("female","male"), Size = c("<2.3",">2.3"))
d <- cbind(d, predict(m, newdata = d, type = "response"))
head(d)
```

```
  Gender Size    bird   fish invert  other     rep
1 female <2.3 0.04967 0.3791 0.3633 0.1608 0.04713
2   male <2.3 0.02933 0.4140 0.3625 0.1443 0.04987
3 female >2.3 0.12145 0.4363 0.1664 0.1466 0.12920
4   male >2.3 0.07299 0.4849 0.1690 0.1339 0.13914
```

```
library(tidyr)
d <- d %>% pivot_longer(cols = c(bird, fish, invert, other, rep),
  names_to = "food", values_to = "probability")
head(d)
```

```
# A tibble: 6 x 4
  Gender Size  food   probability
  <fct>  <fct> <chr>        <dbl>
1 female <2.3  bird        0.0497
2 female <2.3  fish        0.379
3 female <2.3  invert      0.363
4 female <2.3  other       0.161
5 female <2.3  rep         0.0471
6 male   <2.3  bird        0.0293
```

```
p <- ggplot(d, aes(x = food, y = probability)) + theme_minimal() +
  geom_point(aes(color = Gender)) + facet_wrap(~ Size) +
  labs(x = "Food", y = "Probability", color = "Gender")
plot(p)
```



Category-Specific Explanatory Variables

The multinomial logit model can be extended when explanatory variables vary by *response category*. For example, consider the data frame `TravelMode` from the **AER** package.

```
library(AER)
data(TravelMode)
head(TravelMode, 8)
```

```
  individual  mode choice wait vcost travel gcost income size
1          1   air     no   69    59    100    70     35    1
2          1 train     no   34    31    372    71     35    1
3          1   bus     no   35    25    417    70     35    1
4          1   car    yes    0    10    180    30     35    1
5          2   air     no   64    58     68    68     30    2
6          2 train     no   44    31    354    84     30    2
7          2   bus     no   53    25    399    85     30    2
8          2   car    yes    0    11    255    50     30    2
```

Here waiting time (`wait`), vehicle cost (`vcost`), and travel time (`travel`) vary by travel mode, but household income (`income`) varies only by the respondent. For simplicity let's only consider waiting time and income as explanatory variables. A multinomial logit model can then be written as

$$\begin{aligned}
\log(\pi_{ia}/\pi_{ic}) &= \beta_0^{(a)} + \beta_1(\text{wait}_i^{(a)} - \text{wait}_i^{(c)}) + \beta_2^{(a)}\text{income}_i, \\
\log(\pi_{it}/\pi_{ic}) &= \beta_0^{(t)} + \beta_1(\text{wait}_i^{(t)} - \text{wait}_i^{(c)}) + \beta_2^{(t)}\text{income}_i, \\
\log(\pi_{ib}/\pi_{ic}) &= \beta_0^{(b)} + \beta_1(\text{wait}_i^{(b)} - \text{wait}_i^{(c)}) + \beta_2^{(b)}\text{income}_i.
\end{aligned}$$

If we define

$$\begin{aligned}
\eta_i^{(a)} &= \beta_0^{(a)} + \beta_1(\text{wait}_i^{(a)} - \text{wait}_i^{(c)}) + \beta_2^{(a)}\text{income}_i, \\
\eta_i^{(t)} &= \beta_0^{(t)} + \beta_1(\text{wait}_i^{(t)} - \text{wait}_i^{(c)}) + \beta_2^{(t)}\text{income}_i, \\
\eta_i^{(b)} &= \beta_0^{(b)} + \beta_1(\text{wait}_i^{(b)} - \text{wait}_i^{(c)}) + \beta_2^{(b)}\text{income}_i,
\end{aligned}$$

and $\eta_i^{(c)} = 0$, then we can write the category probabilities as

$$\begin{aligned}
\pi_{ia} &= \frac{e^{\eta_i^{(a)}}}{e^{\eta_i^{(a)}} + e^{\eta_i^{(t)}} + e^{\eta_i^{(b)}} + e^{\eta_i^{(c)}}}, \\
\pi_{it} &= \frac{e^{\eta_i^{(t)}}}{e^{\eta_i^{(a)}} + e^{\eta_i^{(t)}} + e^{\eta_i^{(b)}} + e^{\eta_i^{(c)}}}, \\
\pi_{ib} &= \frac{e^{\eta_i^{(b)}}}{e^{\eta_i^{(a)}} + e^{\eta_i^{(t)}} + e^{\eta_i^{(b)}} + e^{\eta_i^{(c)}}}, \\
\pi_{ic} &= \frac{e^{\eta_i^{(c)}}}{e^{\eta_i^{(a)}} + e^{\eta_i^{(t)}} + e^{\eta_i^{(b)}} + e^{\eta_i^{(c)}}}.
\end{aligned}$$

The quantities $e^{\eta_i^{(a)}}$, $e^{\eta_i^{(b)}}$, $e^{\eta_i^{(b)}}$, and $e^{\eta_i^{(c)}}$ could be loosely interpreted as the relative value or "utility" of each response/choice to the respondent/chooser.

Example: The `mlogit` function from the **mlogit** package will estimate a multinomial logistic regression model of this type.[1]

```
library(mlogit)
m <- mlogit(choice ~ wait | income, reflevel = "car",
  alt.var = "mode", chid.var = "individual", data = TravelMode)
summary(m)
```

```

Call:
mlogit(formula = choice ~ wait | income, data = TravelMode, reflevel = "car",
    alt.var = "mode", chid.var = "individual", method = "nr")

Frequencies of alternatives:choice
  car   air train   bus
0.281 0.276 0.300 0.143

nr method
5 iterations, 0h:0m:0s
g'(-H)^-1g = 0.000429
successive function values within tolerance limits

Coefficients :
                  Estimate Std. Error z-value Pr(>|z|)
(Intercept):air    5.98299    0.80797    7.40  1.3e-13 ***
(Intercept):train  5.49392    0.63354    8.67  < 2e-16 ***
(Intercept):bus    4.10653    0.67020    6.13  8.9e-10 ***
wait              -0.09773    0.01053   -9.28  < 2e-16 ***
income:air        -0.00597    0.01151   -0.52    0.604
income:train      -0.06353    0.01367   -4.65  3.4e-06 ***
income:bus        -0.03002    0.01511   -1.99    0.047 *
```

[1] This model can also be estimated using the `vglm` function from the **VGAM** package, although the syntax is very different.

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -192
McFadden R^2:  0.322
Likelihood ratio test : chisq = 183 (p.value = <2e-16)
```

```
cbind(coef(m), confint(m))
```

```
                               2.5 %     97.5 %
(Intercept):air    5.982989  4.39940  7.5665779
(Intercept):train  5.493920  4.25221  6.7356264
(Intercept):bus    4.106526  2.79295  5.4201038
wait              -0.097731 -0.11838 -0.0770853
income:air        -0.005967 -0.02853  0.0165931
income:train      -0.063531 -0.09033 -0.0367306
income:bus        -0.030019 -0.05964 -0.0003948
```

```
exp(cbind(coef(m), confint(m)))
```

```
                             2.5 %    97.5 %
(Intercept):air   396.6241 81.4020 1932.5157
(Intercept):train 243.2087 70.2608  841.8706
(Intercept):bus    60.7354 16.3291  225.9026
wait                0.9069  0.8884    0.9258
income:air          0.9941  0.9719    1.0167
income:train        0.9384  0.9136    0.9639
income:bus          0.9704  0.9421    0.9996
```

Example: Here the response variable is the choice of one of three types of soda. Note that the **PoEdata** package must be installed using `devtools::install_github("https://github.com/ccolonescu/PoEdata")`.

```
library(dplyr)
library(PoEdata)
data(cola)

mycola <- cola %>% mutate(mode = rep(c("Pepsi","7-Up","Coke"), n()/3)) %>%
   select(id, mode, choice, price, feature, display) %>%
   mutate(feature = factor(feature, levels = 0:1, labels = c("no","yes"))) %>%
   mutate(display = factor(display, levels = 0:1, labels = c("no","yes")))

head(mycola)
```

```
  id  mode choice price feature display
1  1 Pepsi      0  1.79      no      no
2  1  7-Up      0  1.79      no      no
3  1  Coke      1  1.79      no      no
4  2 Pepsi      0  1.79      no      no
5  2  7-Up      0  1.79      no      no
6  2  Coke      1  0.89     yes     yes
```

```
m <- mlogit(choice ~ price + feature + display | 1, data = mycola,
   alt.var = "mode", chid.var = "id")
summary(m)
```

```
Call:
```

```
mlogit(formula = choice ~ price + feature + display | 1, data = mycola,
    alt.var = "mode", chid.var = "id", method = "nr")

Frequencies of alternatives:choice
 7-Up  Coke Pepsi
0.374 0.280 0.346

nr method
4 iterations, 0h:0m:0s
g'(-H)^-1g = 0.00174
successive function values within tolerance limits

Coefficients :
                  Estimate Std. Error z-value Pr(>|z|)
(Intercept):Coke   -0.0907     0.0640   -1.42   0.1564
(Intercept):Pepsi   0.1934     0.0620    3.12   0.0018 **
price              -1.8492     0.1887   -9.80  < 2e-16 ***
featureyes         -0.0409     0.0831   -0.49   0.6229
displayyes          0.4727     0.0935    5.05  4.3e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -1810
McFadden R^2:  0.0891
Likelihood ratio test : chisq = 354 (p.value = <2e-16)
```

```r
exp(cbind(coef(m), confint(m)))
```

```
                         2.5 % 97.5 %
(Intercept):Coke  0.9133 0.8057 1.0353
(Intercept):Pepsi 1.2134 1.0745 1.3702
price             0.1574 0.1087 0.2278
featureyes        0.9600 0.8157 1.1297
displayyes        1.6043 1.3355 1.9271
```

Example: Consider the following data on choices of two options of traveling by train.

```r
library(mlogit)
data(Train)
head(Train)
```

```
  id choiceid choice price_A time_A change_A comfort_A price_B time_B change_B comfort_B
1  1        1      A    2400    150        0         1    4000    150        0         1
2  1        2      A    2400    150        0         1    3200    130        0         1
3  1        3      A    2400    115        0         1    4000    115        0         0
4  1        4      B    4000    130        0         1    3200    150        0         0
5  1        5      B    2400    150        0         1    3200    150        0         0
6  1        6      B    4000    115        0         0    2400    130        0         0
```

There are multiple choices for each respondent (`id`), which can induce dependencies among the observations, but we will ignore that here. With only two choices the model reduces to logistic regression where we use the *differences* of the properties of the choices as explanatory variables.

```r
m <- glm(choice == "A" ~ I(price_A - price_B) + I(time_A - time_B),
   family = binomial, data = Train)
summary(m)$coefficients
```

```
                      Estimate Std. Error z value  Pr(>|z|)
(Intercept)           0.018735  3.936e-02   0.476 6.341e-01
I(price_A - price_B) -0.001024  5.939e-05 -17.237 1.400e-66
I(time_A - time_B)   -0.013968  2.288e-03  -6.106 1.021e-09
```

```
exp(cbind(coef(m), confint(m)))
```

```
                             2.5 % 97.5 %
(Intercept)          1.0189 0.9433 1.1007
I(price_A - price_B) 0.9990 0.9989 0.9991
I(time_A - time_B)   0.9861 0.9817 0.9905
```

The price is in cents of guilders and the time is in minutes. For interpretation let's convert the scale of these variables to guilders (equal to 100 cents) and hours (equal to 60 minutes).

```
m <- glm(choice == "A" ~ I((price_A - price_B)/100) + I((time_A - time_B)/60),
   family = binomial, data = Train)
summary(m)$coefficients
```

```
                           Estimate Std. Error z value  Pr(>|z|)
(Intercept)                 0.01874   0.039362   0.476 6.341e-01
I((price_A - price_B)/100) -0.10236   0.005939 -17.237 1.400e-66
I((time_A - time_B)/60)    -0.83808   0.137251  -6.106 1.021e-09
```

```
exp(cbind(coef(m), confint(m)))
```

```
                                   2.5 % 97.5 %
(Intercept)                1.0189 0.9433 1.1007
I((price_A - price_B)/100) 0.9027 0.8921 0.9131
I((time_A - time_B)/60)    0.4325 0.3301 0.5654
```

Here is how we would estimate this model using `mlogit`. The data first need to be reformatted which can be done using the `dfidx` function from the **mlogit** package.

```
mytrain <- dfidx(Train, shape = "wide", choice = "choice",
   varying = 4:11, sep = "_")
head(mytrain)
```

```
~~~~~~~
 first 10 observations out of 5858
~~~~~~~
   id choiceid choice price time change comfort idx
1   1        1   TRUE  2400  150      0       1 1:A
2   1        1  FALSE  4000  150      0       1 1:B
3   1        2   TRUE  2400  150      0       1 2:A
4   1        2  FALSE  3200  130      0       1 2:B
5   1        3   TRUE  2400  115      0       1 3:A
6   1        3  FALSE  4000  115      0       0 3:B
7   1        4  FALSE  4000  130      0       1 4:A
8   1        4   TRUE  3200  150      0       0 4:B
9   1        5  FALSE  2400  150      0       1 5:A
10  1        5   TRUE  3200  150      0       0 5:B

~~~ indexes ~~~~
   id1 id2
1    1   A
2    1   B
3    2   A
```

```
4    2   B
5    3   A
6    3   B
7    4   A
8    4   B
9    5   A
10   5   B
indexes:  1, 2
```

```r
m <- mlogit(choice ~ I(price/100) + I(time/60) | -1, data = mytrain)
summary(m)
```

```

Call:
mlogit(formula = choice ~ I(price/100) + I(time/60) | -1, data = mytrain,
    method = "nr")

Frequencies of alternatives:choice
    A     B
0.503 0.497

nr method
4 iterations, 0h:0m:0s
g'(-H)^-1g = 1.86E-07
gradient close to zero

Coefficients :
             Estimate Std. Error z-value Pr(>|z|)
I(price/100) -0.10235    0.00594   -17.2  < 2e-16 ***
I(time/60)   -0.83684    0.13722    -6.1  1.1e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -1850
```