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The central limit theorem for Markov chains with normal transition operators, started at a point

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Abstract. The central limit theorem and the invariance principle, proved by Kipnis and Varadhan for reversible stationary ergodic Markov chains with respect to the stationary law, are established with respect to the law of the chain started at a fixed point, almost surely, under a slight reinforcing of their spectral assumption. The result is valid also for stationary ergodic chains whose transition operator is normal.

1. Introduction

In 1986 Kipnis and Varadhan [KV] proved the central limit theorem for additive functionals of stationary reversible ergodic Markov chains under a spectral assumption which seems to be almost optimal; a similar result, for stationary ergodic chains with "normal" transition operators, had been previously announced in a note of Gordin and Lifshitz [GL2]. The central limit theorem in both cases was established with respect to the stationary probability law of the chain. For reversible chains, Kipnis and Varadhan proved also the functional central limit theorem (invariance principle), and raised the question of the validity of their results with respect to the law of the Markov chain starting from a point x at time 0 (remark 1.7 of [KV]). The present paper is devoted to this question. Under a slight reinforcing of the spectral assumption made in [KV] we shall show the central limit theorem, and also its functional form, for the Markov chain starting from x, for almost every x with respect to the invariant initial distribution. Moreover, we shall show this result not only for reversible chains, but also for the larger class of chains whose transition operators are "normal".

Before going farther it is necessary to precise that no assumption of "irreducibility" with respect to a measure is made, so the chains are not supposed "Harris recurrent". For "Harris recurrent" chains the question we consider here is solved, and more precise results are known (a recent reference is [C]). Examples of stationary Markov chains with a transition probability which is singular with respect

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to the invariant measure, appear frequently in the study of infinite systems of particles, random walks or processes in random media; in this context the question of the almost sure validity of the limit theorem makes the difference between the "annealed" or "quenched" media. Another simple example is a singular random walk on a compact group.

Our method of proof follows a now classical line. First the additive functional of the Markov chain is decomposed into a sum of a martingale and a cocycle of the shift on the path space of the chain. Then, essentially two arguments are used: a version of the central limit theorem for martingales without stationary increments, and a new pointwise ergodic theorem for "fractional" coboundaries which is proved in our paper [DL2] . Unfortunately, our method does not work under exactly the same spectral assumption as the one made by Kipnis and Varadhan for reversible chains. It requires a slight strengthening, but it yields the desired result for the class of chains generated by transition operators which are normal on the L^2 space of the invariant probability; within this larger class we can show that our result is optimal.

In Section 2 the definitions are given, the problem of the central limit theorem for general Markov chains is described, and some needed known results are recalled. In Section 3 our main result, Theorem 1, is stated and the plan of its proof is explained. Sections 4 and 5 contain the proof of Theorem 1. In Section 6, the functional form of the central limit theorem (the invariance principle), is proved in Theorem 2. In Section 7 a specific question of harmonic analysis is treated, in order to show the limitation of our method and to explain why we need a strengthening of the assumption of [KV]. The special case of random walks on compact groups is studied in Section 8.

2. Definitions, notations, and background

Let $(\mathbb{S}, \mathscr{S}, m)$ be a probability space and $P(x, A)$ a transition probability on it. The Markov operator P is defined by $Pf(x) = \int_{\mathbb{S}} f(y)P(x, dy)$ for every bounded measurable function f on $\mathbb{S}$. The same formula defines $Pf(x)$ for any $f \geq 0$, which need not be finite. Throughout this paper the probability measure m is assumed to be P-invariant, that is, $\int_{\mathbb{S}} Pf dm = \int_{\mathbb{S}} f dm$ for every $f \geq 0$, and ergodic, that is, $Pf = f$ m a.e. for $f \geq 0$ implies that f is constant m a.e. For each $1 \leq p \leq \infty$, the linear operator P induces a linear contraction in the space $L^p_m = L^p(\mathbb{S}, m)$.

We emphasize that we do not assume any kind of m-irreducibility. In the main applications we have in mind, for every x, the measure $P(x, A)$ and all the n-step transitions $P^n(x, A) = P^n 1_A(x)$ are singular with respect to the invariant measure m.

The Markov chain associated to these objects is defined by the canonical construction ([N]). On the infinite product space $\Omega = \mathbb{S}^{\mathbb{N}}$, endowed with the infinite product σ−algebra $\mathscr{S}^{\otimes\mathbb{N}}$, X_n is the n^{th} projection on $\mathbb{S}$. The shift $\theta : \Omega \to \Omega$ is defined by $X_n(\theta\omega) = X_{n+1}(\omega)$, for every $n \geq 0$. For every probability measure ν on the state space $(\mathbb{S}, \mathscr{S})$ the law of the Markov chain $(X_n)_{n\geq 0}$, with transition probability P and initial distribution ν, is the probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathscr{S}^{\otimes\mathbb{N}})$ such that:

$$\mathbb{P}_\nu\left[X_{n+1} \in A \mid X_n = x\right] = P(x, A) \text{ and } \mathbb{P}_\nu\left[X_0 \in A\right] = \nu(A).$$

The expectation with respect to $\mathbb{P}_\nu$ is denoted by $\mathbb{E}_\nu$. For $\nu = \delta_x$, the Dirac measure at x$\in \mathbb{S}$, we write just $\mathbb{P}_x$ or $\mathbb{E}_x$; thus $\mathbb{P}_x$ is the law of the Markov chain starting at the point x. The Markov property reads as follows:

$$\mathbb{E}_\nu\left[f(X_{n+1}) \mid X_n, ..., X_0\right] = Pf(X_n) \quad \mathbb{P}_\nu \text{ a.s.}$$

for any positive or bounded measurable function f on $\mathbb{S}$. The invariance and ergodicity of m with respect to P, yield at once the same properties for the probability measure $\mathbb{P}_m$ on Ω with respect to the shift θ, so $(X_n)_{n\geq 0}$ is stationary under $\mathbb{P}_m$. The mapping $f \to f(X_0)$ defines an isometric embedding of L_m^p into $L^p(\Omega, \mathbb{P}_m)$ for $1\leq p \leq \infty$.

In this setting it is classical to formulate the law of large numbers and the central limit theorem in the following manner. Let f be an m-integrable real function on $\mathbb{S}$, and put $S_n(f) = \sum_{k=0}^{n-1} f(X_k) = \sum_{k=0}^{n-1} f(X_0)\circ\theta^k$. This notation, or S_n when no confusion arises, will be used throughout the paper. The pointwise ergodic theorem of Birkhoff yields the strong law of large numbers: $\lim_n \frac{1}{n} S_n(f) = \int_{\mathbb{S}} f \, dm \;\; \mathbb{P}_m$ $a.s.$ If, moreover, $f \in L_m^2$ and $\int_{\mathbb{S}} f \, dm = 0$, one may ask about the convergence in law of $\frac{1}{\sqrt{n}} S_n(f)$ under $\mathbb{P}_m$, or under $\mathbb{P}_x$ for m a.e. $x \in \mathbb{S}$. When there is convergence in law to a Gaussian distribution one says that the central limit theorem holds for the function f.

When the transition probability P satisfies some good assumptions of absolute continuity, or "irreducibility", or has some "spectral gap" properties, the central limit theorem has been proved for large classes of functions f. This is a story for which we refer to [C] for the study of "irreducible" chains, and to [GH] for the method of the "spectral gap", among many references. In our general setting, without "measure theoretic irreducibility" of P nor any "spectral gap" property, the best known result is the following theorem (note that in our general setting no mixing conditions on P nor on θ are assumed).

Theorem A. *(Gordin, Lifshitz* [GL1]*). If f can be written as $f = g - Pg$ with $g\in L_m^2$, then $\frac{1}{\sqrt{n}} S_n(f)$ converges in law, under the invariant probability measure $\mathbb{P}_m$, to a centered Gaussian distribution with variance* $\int_{\mathbb{S}} (g^2 - (Pg)^2) dm \geq 0$ *(if the variance is 0, the limit is the Dirac measure at 0).*

When $f = g - Pg$, the function f is called an L_m^2 *-coboundary* of P. It is well known [BW] that this condition on f is equivalent to $\sup_n \left\| \sum_{j=0}^{n} P^j f \right\| < \infty$. Here the norm is the L_m^2−norm. In the sequel only L^2−norms shall be considered.

The result of Kipnis and Varadhan, which is the starting point of the present study, is an improvement of this statement under the additional assumption that the Markov chain is *reversible.* This condition means that the operator P is *symmetric* on L_m^2 , that is, $P = P^*$ (the dual operator P^* is defined by $\langle Pf, g\rangle = \langle f, P^*g\rangle$ with $\langle f, g\rangle = \int_{\mathbb{S}} f\overline{g} dm$, the scalar product on the space L_m^2). The reversibility is present in several interesting situations. This condition has the probabilistic interpretation that the chain and the reversed chain have the same probability law. Then, for every $f \in L^2_m$ with $\int_{\mathbb{S}} f\,dm = 0$, the spectral measure of f with respect to the symmetric operator P is the positive measure $\mu_f(dt)$ on $[-1,+1]$, without atom at $+1$, which is characterized by $\langle P^n f, f\rangle = \int_{-1}^{+1} t^n \mu_f(dt)$ for every integer n. Then we can state:

Theorem B. *(Kipnis, Varadhan* [KV]*). Let the Markov operator P be reversible, $f \in L^2_m$ with $\int_{\mathbb{S}} f\,dm = 0$ and spectral measure μ_f. If the integral $\displaystyle\int_{-1}^{+1} \frac{1}{1-t}\mu_f(dt)$ is convergent, then $\frac{1}{\sqrt{n}} S_n(f)$ converges in law, under the invariant probability measure $\mathbb{P}_m$, to a centered Gaussian distribution with variance $\displaystyle\int_{-1}^{+1} \frac{1+t}{1-t}\mu_f(dt)$ (if the variance is 0, the limit distribution is the Dirac measure at 0).*

For P reversible this second result is an improvement of the first, since the assumption that $f = g - Pg$, with $g \in L^2_m$, can be formulated in spectral terms as $\displaystyle\int_{-1}^{+1} \frac{1}{(1-t)^2}\mu_f(dt) < +\infty$.

When the Markov operator P is only assumed to be *normal,* that is, when $PP^* = P^*P$ on L^2_m, it turns out that Theorem B is still valid, when its condition on f is given the equivalent form $\displaystyle\int_{\{\lambda\in\mathbb{C}:\,|\lambda|\le 1\}} \frac{1}{|1-\lambda|}\mu_f(d\lambda) < \infty$; this result was already announced in [GL2]; it was proved independently in our note [DL1], by a geometric method without spectral calculus. In [BI, IV.7] , Gordin and Lifshitz prove in fact the following general result, for any P (though their statement is for P *normal*, the normality is used only at one point, to obtain Corollary D below).

Theorem C. *If f satisfies*

i) $\lim_n \| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} P^k f \| = 0$

ii) $\lim_n \sup_{m>0} \left[\| P^n \sum_{k=0}^{m-1} P^k f \|^2 - \| P^{n+1} \sum_{k=0}^{m-1} P^k f \|^2 \right] = 0$

then $\frac{1}{\sqrt{n}} S_n(f)$ converges in law, under $\mathbb{P}_m$, to a centered Gaussian distribution.

Corollary D. *If P is normal and f satisfies $\displaystyle\int \frac{1}{|1-\lambda|}\mu_f(d\lambda) < \infty$, then* i) *and* ii) *hold, and $\frac{1}{\sqrt{n}} S_n(f)$ converges in law, under $\mathbb{P}_m$, to a centered Gaussian distribution with variance $\displaystyle\int \frac{1-|\lambda|^2}{|1-\lambda|^2}\mu_f(d\lambda)$.*

Remark. The integral is taken on the unit disk, which supports the spectral measure μ_f of f for the *normal* operator P; we have $\frac{1-|\lambda|^2}{|1-\lambda|^2} \le \frac{2}{|1-\lambda|}$, and only the finiteness of the smaller integral is used for proving condition ii).

We now show that Theorem C implies Theorem A.

Let $f = (I-P)g$ with $g \in L^2_m$. Since $\sum_{k=0}^{n-1} P^k f = g - P^n g$, condition i) holds; on the other hand we have:

$$\begin{aligned}
&\left\| P^n \sum_{k=0}^{m-1} P^k f \right\|^2 - \left\| P^{n+1} \sum_{k=0}^{m-1} P^k f \right\|^2 \\
&\quad = \left\| P^n (I - P^m) g \right\|^2 - \left\| P^{n+1} (I - P^m) g \right\|^2 \\
&\quad = \left\| P^n g \right\|^2 - 2 \langle P^n g, P^{n+m} g \rangle + \left\| P^{n+m} g \right\|^2 - \left\| P^{n+1} g \right\|^2 \\
&\qquad + 2 \left\langle P^{n+1} g, P^{n+m+1} g \right\rangle - \left\| P^{n+m+1} g \right\|^2 \\
&\quad = \left(\left\| P^n g \right\|^2 - \left\| P^{n+1} g \right\|^2 \right) + \left(\left\| P^{n+m} g \right\|^2 - \left\| P^{n+m+1} g \right\|^2 \right) \\
&\qquad - 2 \left\langle P^{*n} P^n g - P^{*n+1} P^{n+1} g, P^m g \right\rangle .
\end{aligned}$$

Since $(P^{*n} P^n)$ is a decreasing sequence of symmetric nonnegative operators, it converges strongly ([RN]). Hence $\lim_n \left\| P^{*n} P^n g - P^{*n+1} P^{n+1} g \right\| = 0$, and condition ii) holds too.

In [DL1] our method was based on the consideration of the operator $\sqrt{I - P}$. Since we shall use again this operator, we explain now the definition of the operator $(I - P)^\alpha$ for $0 < \alpha < 1$. In the classical power series expansion $(1 - z)^\alpha = 1 - \sum_{n=1}^\infty a_n^{(\alpha)} z^n$ we have $a_n^{(\alpha)} = (-1)^{n-1} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \sim \frac{c}{n^{\alpha+1}}$ as $n \to \infty$; moreover, $a_n^{(\alpha)} > 0$ and $\sum_{n=1}^\infty a_n^{(\alpha)} = 1$ ([Z] vol. 1, p. 77). Thus, for any linear contraction P it is possible to define

$$(I - P)^\alpha = I - \sum_{n=1}^\infty a_n^{(\alpha)} P^n$$

where the convergence is in the operator norm topology. When P is normal this definition coincides with the definition of $(I - P)^\alpha$ based on the spectral calculus. For P symmetric it is not difficult to see that the convergence of the integral $\int_{-1}^{+1} \frac{1}{1-t} \mu_f(dt)$ and the existence of a solution $g \in L_m^2$ to the equation $f = \sqrt{I - P} g$, are equivalent properties for $f \in L_m^2$ with $\int f \, dm = 0$; this fact was implicit in [KV] and explicit in [DFGW]. Similarly, for P normal, $f \in \sqrt{I - P} L_m^2$ if and only if $\int \frac{1}{|1 - \lambda|} \mu_f(d\lambda) < \infty$ [DL2]. For more details we refer to our paper [DL2] , where the ergodic theory of the operators $(I - P)^\alpha$ is developped ; several results of that paper will be used at essential points in the sequel.

To close this section, we discuss, for general P, the existence and the value of the *variance at the limit* $\sigma_f^2 := \lim_n \frac{1}{n} \left\| \sum_{j=0}^{n-1} f(X_j) \right\|^2$ (the norm is the $L^2(\Omega, \mathbb{P}_m)-$norm; in the sequel it will always be clear from the context if the norm or the scalar product are considered in L_m^2 or in $L^2(\Omega, \mathbb{P}_m)$). If σ_f^2 exists and is zero, we have convergence (in probability) to the Dirac measure at 0 and we shall not repeat it in the sequel.

Proposition 1. *Let P be ergodic and let $f \in (I-P)^{\alpha} L^2_m \cap (I-P^*)^{1-\alpha} L^2_m$ with $0 \le \alpha \le 1$. Then* $\lim_n \frac{1}{n} \left\| \sum_{j=0}^{n-1} f(X_j) \right\|^2 = \sigma_f^2 \ge 0$ *exists and* $\sigma_f^2 = \langle (I+P)g, h \rangle$, *where* $f = (I-P)^{\alpha} g = (I-P^*)^{1-\alpha} h$, *with* $\int g dm = \int h dm = 0$ *(for* $\alpha = 0$, $g = f$; *for* $\alpha = 1$, $h = f$*).*

$$\textit{If } P \textit{ is normal, then } f \in (I-P)^{\frac{1}{2}} L^2_m, \textit{ and } \sigma_f^2 = \int \frac{1-|\lambda|^2}{|1-\lambda|^2} \mu_f(d\lambda).$$

Remark. The proposition shows that the variance of the limiting Gaussian distribution given in Theorems A and B and Corollary D is always σ_f^2. In each application it is a specific important problem to distinguish between $\sigma_f^2 > 0$ or $\sigma_f^2 = 0$.

Proof. By the Markov property $\int_{\Omega} f(X_k) f(X_j) d\mathbb{P}_m = \int_{\mathbb{S}} f P^{j-k} f dm$; thus $\int (\sum_{j=0}^{n-1} f(X_j))^2 d\mathbb{P}_m = n \|f\|^2 + 2 \sum_{j=1}^{n-1} \sum_{l=1}^{j} \langle P^l f, f \rangle$. Since $\langle P^l f, f \rangle = \langle P^l (I-P)^{\alpha} g, (I-P^*)^{1-\alpha} h \rangle = \langle P^l (I-P) g, h \rangle$ and $(I-P) \sum_{l=1}^{j} P^l = P - P^{j+1}$, we find that the preceding expression is equal to $n \|f\|^2 + 2 \sum_{j=1}^{n-1} \langle Pg - P^{j+1} g, h \rangle$. According to the ergodic theorem $\lim_n \frac{1}{n} \sum_{j=1}^{n} P^j g = 0$ in L^2_m−norm, and the first assertion follows.

For P normal $(I-P^*)^{1-\alpha} L^2_m = (I-P)^{1-\alpha} L^2_m$ [DL2], so $f \in (I-P)^{\frac{1}{2}} L^2_m$; the formula for σ_f^2 is deduced from $\langle (I+P)g, h \rangle$ by elementary spectral calculus.

Remarks.

1, The above proof shows that for any $f \in L^2_m$, the variance at the limit exists if $\sum_{n=1}^{\infty} \langle P^n f, f \rangle$ converges (and then $\sigma_f^2 = \|f\|^2 + 2 \sum_{n=1}^{\infty} \langle P^n f, f \rangle$). The convergence of the series implies that $\langle P^n f, f \rangle \to 0$, so $P^n f$ converges weakly to 0 [F].

2. If, in addition to the assumptions of Proposition 1, $P^n f$ converges weakly to 0, then $\sum_{n=0}^{\infty} \langle P^n f, f \rangle$ converges to $\langle g, h \rangle$, since g is in the closed linear manifold generated by $\{P^n f\}$ (which yields $P^n g \to 0$ weakly).

3. For P ergodic symmetric, σ_f^2 exists if and only if $f \in \sqrt{I-P} L^2_m$. For P normal, this equivalence is no longer true ([DL1]).

4. For P ergodic symmetric, and $f \in L^2_m$ with $\int f \, dm = 0$, we have $P^n f \to 0$ weakly if and only if $\mu_f(\{-1\}) = 0$ (and then $\|P^n f\| \to 0$). If these conditions hold (e.g., -1 is not in the spectrum of P), then $\sum_{n=0}^{\infty} \langle P^n f, f \rangle = \int_{-1}^{1} \frac{1}{(1-t)} \mu_f(dt)$, so the existence of σ_f^2 is equivalent to $f \in \sqrt{I-P} L^2_m$, or to $\sum_{n=0}^{\infty} \|P^n f\|^2 < \infty$. For P normal, $\sum_{n=0}^{\infty} \langle P^n f, f \rangle$ may converge, even absolutely, for $f \notin \sqrt{I-P L^2_m}$ (see Section 8).

5. For P normal and $f \in \sqrt{I-P} L^2_m$, $\sigma_f^2 = 0$ if and only if $P^* P f = f$. For P symmetric this yields $\sigma_f^2 = 0$ if and only if $Pf = -f$.

3. The main result and the plan of its proof

In the three Theorems A, B, C recalled above, the central limit theorem for $S_n(f)$ is established with respect to the invariant probability measure $\mathbb{P}_m$. The validity of the central limit theorem with respect to the probability measures $\mathbb{P}_x$ for m-almost every $x \in \mathbb{S}$ is a natural question. In other words, the problem is to decide if the central limit theorem holds when the chain $(X_n)_{n\geq 0}$ starts at a fixed point x. It is the main subject of the present paper.

Under the assumption of Theorem A the central limit theorem is valid also under $\mathbb{P}_x$ for m-almost every $x \in \mathbb{S}$ – this is proved by Gordin and Lifshitz in [BI] . In an anterior work [K] , Brown's central limit theorem, for martingales without stationary increments which satisfy Lindeberg's condition, was used to prove a similar result for a random walk in random environment. We too shall use Brown's theorem to establish our main result, which we state now.

Theorem 1. *Let the Markov operator P be normal on L^2_m. If there exist $\alpha > 1/2$ and $g \in L^2_m$ with $f = (I-P)^\alpha g$ (or, equivalently, if $\sup_n \|\frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} P^k f\| < \infty$ for some $\beta > 1/2$), then for m-almost every starting point $x \in \mathbb{S}$, the sequence $\frac{1}{\sqrt{n}} S_n(f)$ converges in law, under the probability measure $\mathbb{P}_x$, to the Gaussian distribution $\mathcal{N}\left(0, \sigma_f^2\right)$.*

Remarks and comments.

1. The condition $f \in (I-P)^\alpha L^2_m$ for some $\alpha > 1/2$ is equivalent to $\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} P^k f \right\| < \infty$ for some $\beta > 1/2$, as it is shown by Proposition 2.16 and Theorem 2.17 of [DL2] . For the convenience of the reader we recall here the following results from [DL2] which help to understand the situation: for $0 < \alpha \leq 1$, there is equivalence between $f \in (I-P)^\alpha L^2_m$ and $\sup_n \left\| \sum_{j=1}^{n} \frac{P^j f}{j^{1-\alpha}} \right\| < \infty$; furthermore this condition implies $\sup_n \left\| \frac{1}{n^{1-\alpha}} \sum_{k=0}^{n-1} P^k f \right\| < \infty$, and for every $\beta > \alpha$, it is implied by $\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} P^k f \right\| < \infty$ ([DL2] Section 2).

2. By Proposition 1, the condition on f implies the existence of σ_f^2. For any (not necessarily normal) Markov operator, the existence of σ_f^2 implies, by Theorem 2.17 of [DL2], that $f(X_0) \in (I-\theta)^{\frac{1}{2}-\epsilon} L^2(\Omega, \mathbb{P}_m)$ for $0 < \epsilon < 1/2$, and Theorem 3.2(iii) of [DL2] yields that $\frac{1}{n^{\frac{1}{2}+\epsilon}} S_n(f) \to 0$ $\mathbb{P}_m$ a.s., and therefore also $\mathbb{P}_x$ a.s. for m a.e. x.

3. The equation $f = (I-P)^\alpha g$ can be written as $f = g - P_\alpha g$ where $P_\alpha = \sum_{j=1}^{\infty} a_j^{(\alpha)} P^j$. A function $f \in (I-P)^\alpha L^2_m$ is called a *fractional coboundary of degree α for P*, and is just an ordinary coboundary for the operator P_α which is a convex combination of the powers of P.

4. When P is normal, the assumption $f = (I-P)^\alpha g$ with $g \in L^2_m$ is equivalent to the convergence of the integral $\displaystyle\int \frac{1}{|1-\lambda|^{2\alpha}} \mu_f(d\lambda)$ with $\mu_f(d\lambda)$ denoting, as above, the spectral measure of f for the normal operator P, which is supported on the unit disk.

5. The subspaces $(I-P)^{\alpha}L_m^2$ are decreasing with α ([DL2]), therefore the assumption $f=(I-P)^{\alpha}g$ with $\alpha>1/2$ is stronger than the assumption of Corollary D.

6. For m-almost every x the limit is the same Gaussian distribution $\mathcal{N}\left(0,\sigma_f^2\right)$ obtained in Corollary D.

We outline now the scheme of the proof of Theorem 1, which will be developed in the next two sections.

The starting idea is to look for a decomposition of the sums $\mathrm{S}_n(f)$ of the form

$$S_n = M_n + \sum_{k=0}^{n-1} W{\circ}\theta^k$$

where M_n is a martingale and W a fractional coboundary of θ, that is, an element of $L^2(\Omega,\mathbb{P}_m)$ which belongs to a subspace $(I-\theta)^{\beta}L^2(\Omega,\mathbb{P}_m)$. When $f=(I-P)g$ the decomposition is

$$S_n = \sum_{k=0}^{n-1}(g(X_{k+1})-Pg(X_k))+g(X_0)-g(X_n)$$

The sum is a martingale and W reduces to $g(X_0)-g(X_1)$; in other words, for f a coboundary of P, the random variable W is a coboundary of θ. This is the argument of the proof of Theorem A ([GL1]) . When $f=\sqrt{I-P}g$, that is, f is a fractional coboundary of degree $1/2$ for P, it is natural to expect that a decomposition of the preceding type holds with W a fractional coboundary of degree $1/2$ for θ, that is $W=\sqrt{I-\theta}Z$ with $Z\in L^2(\Omega,\mathbb{P}_m)$. It turns out that this is not exactly true, even when P is symmetric. All we can get, using the normality of P, is the existence of $\beta>1/2$ such that W$\in (I-\theta)^{\beta}L^2(\Omega,\mathbb{P}_m)$ when $f\in(I-P)^{\alpha}L_m^2$ for some $\alpha>1/2$. This will be shown in Section 4. To complete the proof, in Section 5, it will remain to check that Brown's theorem can be applied to the martingale appearing in the decomposition, and that $\sum_{k=0}^{n-1}W{\circ}\theta^k$ is $o(\sqrt{n})$ $\mathbb{P}_m$ a.s. using $W\in(I-\theta)^{\beta}L^2(\Omega,\mathbb{P}_m)$ with $\beta>1/2$. This last fact is of ergodic nature and was proved in [DL2], Theorem 3.2.

To close this section, we point out that, in general, Theorem 1 is false for $\alpha=1/2$. For instance, let us consider the Markov operator P defined by an ergodic invertible measure preserving transformation τ. This operator is unitary, hence normal. For $f\in(I-\tau)^{1/2}L_m^2$ we have $\lim_n \frac{1}{\sqrt{n}}\left\|\sum_{k=0}^{n-1} f{\circ}\tau^k\right\|=0$ (Corollary 2.15 of [DL2]). Since the Markov operator P is deterministic, its shift θ on $(\Omega,\mathbb{P}_m)$ is isomorphic to τ on $(\mathbb{S},m)$. The convergence in law asserted by Theorem 1, with respect to $m-$a.e. starting point, reduces in this case to the $m-$a.e. convergence to 0 of $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f{\circ}\tau^k$. By Proposition 3.8 of [DL2] we know that this a.e. convergence does not always hold.

4. The decomposition into a martingale and a fractional θ– coboundary

In this section the Markov operator P is always assumed to be normal on L^2_m. To abbreviate we put $a_n = a_n^{(1/2)}$, so $a_n = (-1)^{n-1}\frac{(1/2)(-1/2)(-3/2)\cdots(3/2-n)}{n!}$ for $n > 1$, that is, $(1-z)^{1/2} = 1 - \sum_{n=1}^{\infty} a_n z^n$. We begin with the following essential lemma.

Lemma 1. *For $g \in L^2_m$ with $\int g dm = 0$ the two series*
$\sum_{j=1}^{\infty} a_j \left(\sum_{l=0}^{j-1} P^l g(X_0) - \sum_{l=0}^{j-1} P^l g(X_1)\right)$ *and*
$\sum_{l=0}^{\infty} 2(l+1)a_{l+1}(P^l g(X_0) - P^l g(X_1))$
converge in $L^2(\Omega, \mathbb{P}_m)$, and are equal.

Remark. Since $na_n \sim c/\sqrt{n}$ there is no separate convergence if the differences are split.

Proof. By commutation of the sums and using $\sum_{j=k}^{\infty} a_k = 2ka_k$ ([DL2], Lemma 2.5), we can write:

$$\begin{aligned}
&\sum_{j=1}^{N} a_j \left[\sum_{l=0}^{j-1}(P^l g(X_0) - P^l g(X_1))\right] \\
&\quad = \sum_{l=0}^{N-1} 2(l+1)a_{l+1}(P^l g(X_0) - P^l g(X_1)) \\
&\qquad -2(N+1)a_{N+1}\sum_{l=0}^{N-1}(P^l g(X_0) - P^l g(X_1)).
\end{aligned}$$

We shall check the Cauchy criterion for the first sum of this expression. Using the Markov property: $\int_\Omega \varphi(X_0)\psi(X_1)d\mathbb{P}_m = \int_{\mathbb{S}} \varphi P(\psi) dm$, and the P-invariance of m we get:

$$\begin{aligned}
&\left\|\sum_{l=M}^{N} 2(l+1)a_{l+1}(P^l g(X_0) - P^l g(X_1))\right\|^2 \\
&= 4\sum_{M\le l,n\le N}(l+1)a_{l+1}(n+1)a_{n+1}\int\Big[P^l g(P^n g - P^{n+1}g) \\
&\quad -P^n g P^{l+1} g + P(P^l g P^n g)\Big] dm \\
&= 8\left\langle \sum_{l=M}^{N}(l+1)a_{l+1}P^l(I-P)g, \sum_{n=M}^{N}(n+1)a_{n+1}P^n g\right\rangle.
\end{aligned}$$

Since P is assumed to be normal, $\sqrt{I-P}L^2_m = \sqrt{I-P^*}L^2_m$, ([DL1], Lemma 2). Hence there exists $h \in L^2_m$ with $\int h dm = 0$ such that $\sqrt{I-P}g = \sqrt{I-P^*}h$. We can write $(I-P)g = \sqrt{I-P}(\sqrt{I-P}g) = \sqrt{I-P}(\sqrt{I-P^*}h)$ in the preceding expression, which is therefore equal to:

$$8\left\langle \sum_{l=M}^{N}(l+1)a_{l+1}P^l\sqrt{I-P}h, \sum_{n=M}^{N}(n+1)a_{n+1}P^n\sqrt{I-P}g\right\rangle$$

by the commutation $PP^* = P^*P$. Theorem 2.7 of [DL2] yields $g = \sum_{l=0}^{\infty} 2(l+1)a_{l+1}P^l\sqrt{I-P}g$, with convergence of the series in L^2_m. Thus Cauchy's criterion is satisfied by the series $\sum_{l=0}^{\infty}(l+1)a_{l+1}(P^l g(X_0) - P^l g(X_1))$. By the classical Kronecker lemma the convergence of this series implies

$\lim_N (N+1)a_{N+1}\sum_{l=0}^{N-1}(P^l g(X_0) - P^l g(X_1)) = 0$ in $L^2(\Omega, \mathbb{P}_m)$,
and the lemma is proved.

Lemma 2. *For $f = \sqrt{I-P}g$ with $g \in L^2_m$ and $\int g dm = 0$ we have*

$$f = \sum_{j=1}^{\infty} a_j\left(\sum_{l=0}^{j-1} P^l(I-P)g\right) = \sum_{l=0}^{\infty} 2(l+1)a_{l+1}(P^l g - P^{l+1}g)$$

with convergence in L^2_m.

Proof. The first equality is from the definition of the coefficients a_n :

$$\sum_{j=1}^{\infty} a_j(\sum_{k=0}^{j-1} P^k(I-P)g) = \sum_{j=1}^{\infty} a_j(I-P^j)g = g - \sum_{j=1}^{\infty} a_j P^j g = f.$$

The second is proved as in Lemma 1. It is also useful to observe that Lemma 2 is a direct consequence of Lemma 1, by an application to the series of the conditional expectation operator knowing X_0, $\mathbb{E}_m(\ |X_0)$, which is an L^2-projector.

Now we are ready to set the decomposition of $S_n(f)$ we are looking for.

Proposition 2. *Let $f = \sqrt{I-P}g = g - \sum_{n=1}^{\infty} a_n P^n g$ with $g \in L^2_m$ and $\int g dm = 0$. We have the decomposition:*

$$S_n(f) = \sum_{k=0}^{n-1} M\circ\theta^k + \sum_{k=0}^{n-1} W\circ\theta^k \quad \mathbb{P}_m \ \ a.s.$$

where the random variables M and W are defined by:

$M = \sum_{j=1}^{\infty} a_j(\sum_{l=0}^{j-1}(P^l g(X_1) - P^{l+1}g(X_0))$
$W = \sum_{j=1}^{\infty} a_j(\sum_{l=0}^{j-1}(P^l g(X_0) - P^l g(X_1)).$

The series are $L^2(\Omega, \mathbb{P}_m)$ convergent. The sum $\sum_{k=0}^{n-1} M\circ\theta^k$ is a martingale with stationary increments under $\mathbb{P}_m$ and

$$\sum_{k=0}^{n-1} W\circ\theta^k = \sum_{l=0}^{\infty} 2(l+1)a_{l+1}(P^l g(X_0) - P^l g(X_n)).$$

Proof. By Lemma 2, $f(X_0) = \sum_{j=1}^{\infty} a_j\left[\sum_{k=0}^{j-1} P^k g(X_0) - \sum_{k=0}^{j-1} P^{k+1}g(X_0)\right]$; by Lemma 1, $W = \sum_{j=1}^{\infty} a_j\left[\sum_{k=0}^{j-1} P^k g(X_0) - \sum_{k=0}^{j-1} P^k g(X_1)\right]$ with convergence in $L^2(\Omega, \mathbb{P}_m)$ in both cases. So $M = f(X_0) - W = \sum_{j=1}^{\infty} a_j\left[\sum_{k=0}^{j-1} P^k g(X_1) - \sum_{k=0}^{j-1} P^{k+1}g(X_0)\right]$ with convergence in $L^2(\Omega, \mathbb{P}_m)$. The decomposition of $S_n(f) = \sum_{k=0}^{n-1} f(X_k)$ follows at once. The martingale property is a consequence of the Markov property since:

$$\mathbb{E}_m\left[M \mid X_0\right] = \sum_{j=1}^{\infty} a_j \left(\mathbb{E}_m \left[\sum_{l=0}^{j-1} P^l g(X_1) \mid X_0 \right] - P \sum_{l=0}^{j-1} P^l g(X_0) \right) = 0$$

where the $L^2(\Omega, \mathbb{P}_m)$ convergence of the series M is used.

Remark. For $f = (I - P)g'$ we compute W with $g = \sqrt{I - P}g'$ and find $W = g'(X_0) - g'(X_1)$, so the present decomposition coincides with the decomposition of Gordin and Lifshitz. In fact the formal equation to solve is the same in all cases: $W = (I - \theta)((I - P)^{-1} f)$. The operator $(I - P)^{-1}$ is unbounded; its domain contains $(I - P)L^2_m$. The series W appearing in the preceding proposition is a "rigorous" solution when f does not belong to the domain of $(I - P)^{-1}$ but can be written as $f = \sqrt{I - P}g$.

Proposition 3. *With the notations of the preceding proposition, we have that if $f \in (I - P)^\alpha L^2_m$ with $\alpha \geq 1/2$, then* $\lim_n \frac{1}{n^{1-\alpha}} \left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\| = 0$, *and $W \in (I - \theta)^\beta L^2(\Omega, P_m)$ for every $\beta < \alpha$. In particular,* $\lim_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\| = 0$.

Proof. Since the subspaces $(I - P)^\alpha L^2_m$ decrease with α, we have $f \in \sqrt{I - P}L^2_m$. With the same computation as in the proof of Lemma 1 we get:

$$\left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\|^2 = \left\| \sum_{l=0}^{\infty} 2(l + 1)a_{l+1}(P^l g(X_0) - P^l g(X_n) \right\|^2$$
$$= 8 \left\langle \sum_{l=0}^{\infty} (l + 1)a_{l+1} P^l (I - P^n)g, \sum_{l=0}^{\infty} (l + 1)a_{l+1} P^l g \right\rangle.$$

Using the normality of P we write $f = \sqrt{I - P}g = \sqrt{I - P^*}h$, and then $(I - P^n)g = \sqrt{I - P^*} \sum_{j=0}^{n-1} P^j \sqrt{I - P}h$. By Theorem 2.7 of [DL2] we get:

$$\left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\|^2 = 2 \left\langle \sum_{j=0}^{n-1} P^j h, g \right\rangle \leq 2 \left\| \sum_{j=0}^{n-1} P^j h \right\| \|g\| .$$

Since h may be chosen with $\int h dm = 0$, the ergodicity of P yields $\lim_n \frac{1}{n} \left\| \sum_{j=0}^{n-1} P^j h \right\| = 0$, and the last assertion is proved.

Since $(I - P)^\gamma L^2_m = (I - P^*)^\gamma L^2_m$ for every $0 < \gamma < 1$ by Proposition 4.1 of [DL2], when $f \in (I - P)^\alpha L^2_m$ with $\alpha > 1/2$ we have $h \in (I - P)^{\alpha - 1/2} L^2_m$ and $g \in (I - P^*)^{\alpha - 1/2} L^2_m$. Hence:

$$\left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\|^2 = 2 \left\langle \sum_{j=0}^{n-1} P^j h, g \right\rangle$$
$$= 2 \left\langle \sum_{j=0}^{n-1} P^j (I - P)^{\alpha - 1/2} h', (I - P^*)^{\alpha - 1/2} g' \right\rangle$$

$$= 2\left\langle \sum_{j=0}^{n-1} P^j (I-P)^{2\alpha-1} h', g' \right\rangle$$
$$\leq 2 \left\| \sum_{j=0}^{n-1} P^j (I-P)^{2\alpha-1} h' \right\| \|g'\| .$$

Then Corollary 2.15 of [DL2] yields $\lim_n \frac{1}{n^{2-2\alpha}} \left\| \sum_{j=0}^{n-1} P^j (I-P)^{2\alpha-1} h' \right\| = 0$. Therefore $\lim_n \frac{1}{n^{1-\alpha}} \left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\| = 0$.

Now $W \in (I-\theta)^\beta L^2_m$ for every $0 < \beta < \alpha$ by Theorem 2.17 of [DL2] .

Remarks.

1. We shall come back to the problem of the exact degree of the fractional $\theta-$coboundary W , especially for $\alpha = 1/2$, in Section 6.

2. The preceding decomposition of S_n is unique in the following sense: if $f = M' + W'$ where $\sum_{k=0}^{n-1} M' \circ \theta^k$ is a martingale and $\lim_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} W' \circ \theta^k \right\| = 0$ then $M' = M$ and $W' = W$. Indeed, necessarily $\lim_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} (M - M') \circ \theta^k \right\| = 0$ and also $\left\| \sum_{k=0}^{n-1} (M - M') \circ \theta^k \right\|^2 = n \left\| M - M' \right\|^2$ by the martingale property and the $\theta-$invariance of $\mathbb{P}_m$, hence $M = M'$ $\mathbb{P}_m$ a.s.

5. End of the proof of Theorem 1

In this section we still assume that the Markov operator P is normal. Let $f \in \sqrt{I-P} L^2_m$ be given. In the decomposition

$$S_n(f) = \sum_{k=0}^{n-1} M \circ \theta^k + \sum_{k=0}^{n-1} W \circ \theta^k \quad \mathbb{P}_m \;\; a.s.$$

the first sum is a martingale with stationary increments for the probability measure $\mathbb{P}_m$, by Proposition 2, and $\lim_n \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} W \circ \theta^k \right\| = 0$, by Proposition 3. Then the central limit theorem for martingales with stationary increments, due to Billingsley and Ibragimov ([Bi] , [I]), yields the convergence in law of $\frac{1}{\sqrt{n}} S_n(f)$, under $\mathbb{P}_m$. The limit distribution is the centered Gaussian distribution of variance $\int_\Omega M^2 d\mathbb{P}_m$. By the martingale property $\left\| \sum_{k=0}^{n-1} M \circ \theta^k \right\|^2 = n \|M\|^2$; using the Cauchy-Schwarz inequality, and Propositions 1 and 3, $\lim_n \frac{1}{n} \int_\Omega S_n(f) \left(\sum_{k=0}^{n-1} W \circ \theta^k \right) d\mathbb{P}_m = 0$. Hence $\lim_n \frac{1}{n} \|S_n(f)\|^2 = \sigma_f^2 = \|M\|^2$. If $\sigma_f^2 = 0$ then $M = 0$ $a.s.$ and the limit distribution is the Dirac measure at 0. This is a new approach to the result of [DL1] .

We now come to the consideration of the probability measures $\mathbb{P}_x$ on Ω. We begin with recalling Brown's theorem [Br] , which is an extension of the theorem of Billingsley and Ibragimov.

Theorem E. *(Brown) Let $(Y_k)_{k\geq 1}$ be a sequence of real random variables, adapted to the increasing sequence of sub-$\sigma-$algebras $(\mathscr{F}_k)_{k\geq 1}$, with $E(Y_k^2) < \infty$ for every k. Let us put $S_n = \sum_{k=1}^n Y_k$, $V_n^2 = \sum_{k=1}^n E(Y_k^2 \mid \mathscr{F}_{k-1})$ and $\sigma_n^2 = E(V_n^2)$. The sequence $\sigma_n^{-1} S_n$ converges in law to the standard Gaussian distribution $\mathscr{N}(0,1)$ when the three following conditions hold:*

a) $E(Y_k \mid \mathscr{F}_{k-1}) = 0$ for every $k \geq 1$ (i.e. S_n is a martigale).

b) $\lim_n \sigma_n^{-2} V_n^2 = 1$ a.s.

c) $\lim_n \sigma_n^{-2} \sum_{k=1}^n E(Y_k^2 1_{\{|Y_k|>\delta\sigma_n\}}) = 0$ for every $\delta > 0$ (similar to Lindeberg's condition).

We shall check the three conditions of this theorem for $Y_k = M \circ \theta^{k-1}$ and $\mathscr{F}_k = \sigma(X_0, .., X_k)$, with respect to $\mathbb{P}_x$, for $m-a.e.\ x \in \mathbb{S}$, in the only significant case $\sigma_f^2 > 0$.

By the Fubini theorem we deduce from Proposition 2 that $\mathbb{E}_x(M \mid X_0) = 0$ $\mathbb{P}_x$ $a.s.$ for m $a.e.$ x; by stationarity we have also $\mathbb{E}_x(M \circ \theta^k \mid X_k) = 0$ $\mathbb{P}_x$ $a.s.$ for m $a.e.$ x.

To check condition $b)$ we put $\varphi(x) = \mathbb{E}_x(M^2)$; it is a nonnegative element of L_m^1. Then $\mathbb{E}_x(M^2 \circ \theta^k \mid X_k, .., X_0) = \mathbb{E}_{X_k}(M^2 \circ \theta^k) = \varphi(X_k)$ $\mathbb{P}_m$ $a.s.$ and $V_n^2 = \sum_{k=0}^{n-1} \varphi(X_0) \circ \theta^k$. By the pointwise ergodic theorem for the shift θ we get $\lim_n \frac{1}{n} V_n^2 = \int_\Omega M^2 d\mathbb{P}_m$, $\mathbb{P}_m$ $a.s.$, so $\mathbb{P}_x$ $a.s.$ for $m-a.\ e.\ x \in \mathbb{S}$ by the Fubini theorem. On the other hand, $\sigma_n^2(x) = \mathbb{E}_x(V_n^2) = \sum_{k=0}^{n-1} P^k \varphi(x)$ $ma.e.$ By the pointwise ergodic theorem for the Dunford-Schwartz operator P, we get $\lim_n \frac{1}{n}\sigma_n^2(x) = \int_{\mathbb{S}} \varphi dm = \int_\Omega M^2 d\mathbb{P}_m = \sigma_f^2$ m $a.e.$ Since σ_f^2 is assumed positive, condition $b)$ follows at once for $\mathbb{P}_x$, for m $a.e.$ x.

The verification of condition $c)$ uses again the pointwise ergodic theorem for P. Given a constant $C > 0$ put $\psi_C(x) = \mathbb{E}_x(M^2 1_{\{|M|>C\}})$; it is a nonnegative element of L_m^1. Then $\mathbb{E}_x(M^2 \circ \theta^{k-1} 1_{\{|M| \circ \theta^{k-1} > C\}}) = P^k \psi_C(x)$, and

$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} P^k \psi_C(x) = \int_{\mathbb{S}} \psi_C dm$ m $a.\ e.$ Since $\sigma_n^2(x) \to \infty$ m $a.e.$
$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_x(M^2_\circ \theta^{k-1} 1_{\{|M| \circ \theta^{k-1} > \sigma_n(x)\}}) \leq \int_{\mathbb{S}} \psi_C dm$ m $a.e.$
Since $\lim_{C\to\infty} \int_{\mathbb{S}} \psi_C dm$ =0, condition $c)$ holds under $\mathbb{P}_x$, for m $a.e.$ x.

Thus the sequence $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} M \circ \theta^k$ converges in law to the Gaussian distribution $\mathscr{N}(0, \sigma_f^2)$ under the probability measure $\mathbb{P}_x$, for m $a.e.$ x. The constance of the variance is due to the ergodicity of P.

To complete the proof of the convergence in law of $\frac{1}{\sqrt{n}} S_n(f)$, we need Theorem 3.2(i) of [DL2] . It is an ergodic theorem which says that if φ is a fractional L^2-coboundary of degree $\alpha > 1/2$, for an ergodic, probability preserving transformation τ, then $\lim_n \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ \tau^k = 0$ $a.\ e.$ By Proposition 3, this theorem can be applied to W with respect to θ on $(\Omega, \mathbb{P}_m)$ when $f \in (I-P)^\alpha L_m^2$ with $\alpha > 1/2$, thus $\lim_n \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W \circ \theta^k = 0$ $\mathbb{P}_x$ $a.s.$ for $m-a.e.\ x \in \mathbb{S}$. The proof of Theorem 1 is then complete.

Remark. According to Theorem 2.3 of $[C]$, for a Markov chain which is "irreducible" and "ergodic", that is, Harris recurrent with a unique recurrence class, aperiodic and with finite invariant measure, the central limit theorem holds for every starting point as soon as $S_n/\sqrt{n}$ is bounded in probability. In the present general situation such a strong result is impossible as shown, for instance, by the example at the end of Section 3 (see also the examples of Section 8).

6. The functional form of the central limit theorem

It is natural to ask if the functional form of the central limit theorem that we just proved, is valid too. The functional central limit theorem is also called the invariance principle. For P symmetric and $f = \sqrt{I-P}g$ with $g \in L^2_m$, Kipnis and Varadhan proved the invariance principle under the stationary law of the chain $\mathbb{P}_m$. Their beautiful argument uses strongly the reversibility. Here we shall show that for P normal and $f \in (I-P)^\alpha L^2_m$ with $\alpha > 1/2$, that is, under the same assumptions as in Theorem 1, also the functional form of the central limit theorem holds under $\mathbb{P}_x$, for $m-$almost every $x \in \mathbb{S}$.

Let us precise the meaning of the functional central limit theorem in our situation. For $f \in L^2_m$ we consider, for every n, the continuous function $\xi_n(t, f)$ on $0 \le t \le 1$ which is composed of straight line segments joining the points $(\frac{k}{n}, \frac{1}{\sigma_f\sqrt{n}} S_k(f))$, $k = 0, 1, 2, ..., n$. If the sequence of probability measures on the space $\mathscr{C}[0,1]$ of the continuous functions on the interval $[0,1]$, determined by the distributions of $\xi_n(t, f)$ under $\mathbb{P}_m$ or $\mathbb{P}_x$, converges weakly to the normalized Wiener measure, we say that the functional central limit theorem holds for f under $\mathbb{P}_m$ or $\mathbb{P}_x$ (the considered probability measure). Then we can state our second theorem.

Theorem 2. *Let P be normal. If $f \in L^2_m$ satisfies the assumptions of Theorem 1, the functional central limit theorem holds for f under $\mathbb{P}_x$, for $m-$almost every starting point $x \in \mathbb{S}$.*

Proof. We use again the decomposition $S_n(f) = \sum_{k=0}^{n-1} M \circ \theta^k + \sum_{k=0}^{n-1} W \circ \theta^k$, $\mathbb{P}_m$ $a.s.$, proved in Proposition 2. We consider, for every n, the continuous function $\zeta_n(t, f)$ on $0 \le t \le 1$ which is composed of straight line segments joining the points $(\frac{k}{n}, \frac{1}{\sigma_f\sqrt{n}} \sum_{j=0}^{k-1} M \circ \theta^j)$, $k = 0, 1, 2, ..., n$. Since the functional central limit theorem holds for a martingale satisfying the assumptions of Brown's theorem (Theorem 3 of [Br], the distribution of the continuous process $\zeta_n(t, f)$ under $\mathbb{P}_x$ converges weakly to the normalized Wiener measure, for $m-$almost every starting point $x \in \mathbb{S}$. Thus, following a well known argument, it is enough to prove that $\lim_n \sup_{0 \le t \le 1} |\xi_n(t, f) - \zeta_n(t, f)| = 0$ in probability with respect to $\mathbb{P}_x$, for $m-$almost every x. However, $\sup_{0 \le t \le 1} |\xi_n(t, f) - \zeta_n(t, f)| \le \frac{1}{\sigma_f\sqrt{n}} \sup_{0 \le k \le n} \left| \sum_{j=0}^{k-1} W \circ \theta^j \right|$. By Theorem 3.2 of [DL2], which was already used above, $\lim_n \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} W \circ \theta^j = 0$, $\mathbb{P}_m$ $a.s.$ Using Fubini's theorem and an elementary property of real convergent sequences, we get $\lim_n \frac{1}{\sqrt{n}} \sup_{0 \le k \le n} \left| \sum_{j=0}^{k-1} W \circ \theta^j \right| = 0$, $\mathbb{P}_x$ $a.s.$ for $m-$almost every starting point $x \in \mathbb{S}$, and the proof is complete.

7. Analysis of the case $\alpha = 1/2$ for reversible chains

The original problem of Kipnis and Varadhan was to prove the central limit theorem for P symmetric and $f \in \sqrt{I-P}L^2_m$, when the chain starts at a fixed point. In our proof of Theorem 1, given in Sections 4 and 5, Proposition 3 is an essential point. However, for $f \in \sqrt{I-P}L^2_m$, this proposition yields only that the function W constructed in Proposition 2 is in $(I-\theta)^\beta L^2(\Omega, \mathbb{P}_m)$ for $\beta < 1/2$, which is not sufficient for the a.e. convergence of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W\circ\theta^k$. It turns out that for reversible chains, that is, for $P = P^*$, a better result than Proposition 3 can be proved by using a deeper harmonic analysis. In this section, we shall present this result, in order to make precise what is missing to solve the case $\alpha = 1/2$.

Recall that θ is the one-sided shift on the space $\Omega = \mathbb{S}^{\mathbb{N}}$. In order to characterize fractional coboundaries for θ, we will use the well known two-sided natural extension of θ (no symmetry or normality of P is assumed in this discussion). This two-sided shift, denoted by $\tilde\theta$, is an invertible measure preserving transformation on the two-sided infinite product $\tilde\Omega := \mathbb{S}^{\mathbb{Z}}$ endowed with the σ-algebra $\mathscr{F} := \mathscr{S}^{\otimes\mathbb{Z}}$ and the two-sided Markovian probability measure $\tilde{\mathbb{P}}_m$, whose projection on $\mathbb{S}^{\mathbb{N}}$ is $\mathbb{P}_m$. In $\tilde\Omega$ we have the coordinate projections $(X_n)_{-\infty<n<\infty}$, and we define $\mathscr{F}^+$ as the σ-algebra generated by $(X_n)_{n\ge 0}$. Then $(\Omega, \mathscr{S}^{\otimes\mathbb{N}}, \mathbb{P}_m, \theta)$ is isomorphic to $(\tilde\Omega, \mathscr{F}^+, \tilde{\mathbb{P}}_{m\,|\mathscr{F}^+}, \tilde\theta)$, and $L^2(\Omega, \mathscr{S}^{\otimes\mathbb{N}}, \mathbb{P}_m)$ is identified with $L^2(\tilde\Omega, \mathscr{F}^+, \tilde{\mathbb{P}}_m)$, which is invariant under the operator $\tilde T\psi := \psi\circ\tilde\theta$. Under this identification, which maps $\phi \in L^2(\Omega, \mathbb{P}_m)$ to $\tilde\phi(\tilde\omega) = \phi(X_0(\tilde\omega), X_1(\tilde\omega), \ldots)$, the restriction of $\tilde T$ represents the isometry induced by θ in $L^2(\Omega, \mathbb{P}_m)$. It follows from Theorem 2.11 of [DL2] that $\phi \in (I-\theta)^\alpha L^2(\Omega, \mathbb{P}_m)$ if and only if $\tilde\phi \in (I-\tilde T)^\alpha L^2(\tilde\Omega, \tilde{\mathbb{P}}_m)$. Since $\langle\phi\circ\theta^k, \phi\rangle = \langle\tilde\phi\circ\tilde\theta^k, \tilde\phi\rangle$ for $k \ge 0$, the Fourier coefficients of the spectral measure μ of $\tilde\phi$ with respect to the unitary operator $\tilde T$ are $b_k = \langle\phi\circ\theta^k, \phi\rangle$ for $k \ge 0$; for ϕ real we have $b_{-k} = b_k$. We can call μ the spectral measure of ϕ with respect to θ, and its Fourier coefficients are defined using only ϕ and θ. From the above discussion and the condition for normal operators, we obtain that $\phi \in (I-\theta)^\alpha L^2(\Omega, \mathbb{P}_m)$ if and only if $\int_0^{2\pi} \frac{1}{\left|1-e^{it}\right|^{2\alpha}} \mu(dt) < \infty$.

We now return to our problem. To make the exposition shorter we assume, in the remainder of this section, P symmetric of nonnegative type, that is, with spectrum contained in the interval $[0, 1]$. This restriction on the spectrum allows us to ignore the eventual singularity at -1, which is inessential in our discussion.

To begin with, the covariance coefficients of $W = \sum_{j=1}^\infty a_j (\sum_{l=0}^{j-1} (P^l g(X_0) - P^l g(X_1))$, with respect to the unitary operator induced by the two-sided shift $\tilde\theta$, are:

$$b_k = \left\langle W\circ\theta^k, W\right\rangle = -\left\langle g, (I-P)P^{|k|-1}g\right\rangle \text{ for } k \ne 0, \text{ and } b_0 = 2\,\|g\|^2 .$$

This computation is similar to the one used in the proof of Proposition 3.

If $f = \sqrt{I-P}g$ then $\int_0^1 \frac{1}{1-s}\mu_f(ds) < \infty$, and $b_k = -\int_0^1 s^{|k|-1}\mu_f(ds)$ for $k \ne 0$, so we have:

$$-\sum_{k=1}^{\infty} b_k = \int_0^1 \sum_{k=1}^{\infty} s^{k-1} \mu_f(ds) = \int_0^1 \frac{1}{1-s} \mu_f(ds) = \|g\|^2 < \infty;$$

moreover, $\sum_{k \in \mathbb{Z}} b_k = 0$. Therefore the spectral measure of W with respect to θ has a continuous density $\psi(t) = \sum_{k \in \mathbb{Z}} b_k e^{itk}$ on $[0, 2\pi]$. Using $b_0 = -2\sum_{k=1}^{\infty} b_k$ we get

$$\psi(t) = 2\mathscr{R}e \sum_{k=1}^{\infty} (1 - e^{itk}) \int_0^1 s^{k-1} \mu_f(ds) = 2\int_0^1 \frac{1+s}{1-s} \left(\frac{1-\cos t}{\left|1 - se^{it}\right|^2}\right) \mu_f(ds).$$

We know that $W \in (I - \theta)^{\beta} L^2(\Omega, \mathbb{P}_m)$ if and only if $\displaystyle\int_0^{2\pi} \frac{1}{\left|1 - e^{it}\right|^{2\beta}} \psi(t) dt < \infty$.

On the other hand, $\frac{1-\cos t}{|1-e^{it}|^{2\beta}} = 2^{1-2\beta} (\sin \frac{t}{2})^{2-2\beta}$ and $\frac{1-s^2}{|1-se^{it}|^2} = 2K(s,t)$ is twice the Poisson kernel $K(s,t)$ for $0 < s < 1$. This analysis is summarized in the following proposition:

Proposition 4. *Let the Markov operator P be symmetric of nonnegative type. For $f = \sqrt{I - P} g$ the random variable $W = \sum_{j=1}^{\infty} a_j (\sum_{l=0}^{j-1} (P^l g(X_0) - P^l g(X_1))$ satisfies $W \in (I - \theta)^{\beta} L^2(\Omega, \mathbb{P}_m)$ if and only if the integral:*

$$\mathscr{I}_{\beta} := \int_0^1 \frac{1}{(1-s)^2} \int_0^{2\pi} (\sin \frac{t}{2})^{2-2\beta} K(s,t) \, dt \, \mu_f(ds)$$

converges.

We separate now the case $\beta > 1/2$ from $\beta = 1/2$.

Lemma 4. *Let $\beta > 1/2$; if $\displaystyle\int_0^1 \frac{1}{(1-s)^{2\beta}} \mu_f(ds) < \infty$ (i.e., $f \in (I - P)^{\beta} L^2_m$), then $\mathscr{I}_{\beta} < \infty$.*

Proof. By the classical inequality ([Z] vol. 1, p. 96): $K(s,t) \leq \frac{\pi^2}{2} \frac{1-s}{(1-s)^2 + t^2}$ for $(1/2) \leq s < 1$, and also $\left|\sin \frac{t}{2}\right| < \left|\frac{t}{2}\right|$ for $|t| < \pi$, we get:

$$\int_0^{2\pi} (\sin \frac{t}{2})^{2-2\beta} K(s,t) \, dt \leq C \int_0^{\pi} \frac{t^{2-2\beta}(1-s)}{(1-s)^2 + t^2} dt$$
$$= C(1-s)^{2-2\beta} \int_0^{\pi/(1-s)} \frac{v^{2-2\beta}}{1+v^2} dv.$$

For $\beta > 1/2$ the integral $\displaystyle\int_0^{\infty} \frac{v^{2-2\beta}}{1+v^2} dv$ converges, and

$$\mathscr{I}_{\beta} = \int_0^1 \frac{1}{(1-s)^2} \mathscr{O}((1-s)^{2-2\beta}) \mu_f(ds) < \infty.$$

Lemma 5. *The integral $\mathscr{I}_{1/2}$ converges if and only if $\displaystyle\int_0^1 \frac{\ln(1-s)}{1-s} \mu_f(ds)$ converges.*

Proof. The Fourier coefficients of $\sin\frac{t}{2}$ are $\gamma_n = \frac{1}{2\pi}\int_0^{2\pi}(\sin\frac{t}{2})e^{-int}dt = \frac{2}{\pi(1-4n^2)}$. Using the expansion $K(s,t) = \frac{1}{2}\sum_{n\in\mathbb{Z}} s^{|n|}e^{int}$ ([Z] vol. 1, p. 96), we get

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}(\sin\frac{t}{2})K(s,t)\,dt &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\gamma_n s^{|n|} = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2}{4n^2-1}s^n \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2}{4n^2-1}(1-s^n) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2}{4n^2-1}(1-s)(\sum_{l=0}^{n-1}s^l) \\
&= (1-s)\frac{1}{\pi}\sum_{l=0}^{\infty}(\sum_{n=l+1}^{\infty}\frac{2}{4n^2-1})s^l = (1-s)\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{s^l}{2l+1}
\end{aligned}$$

which is asymptotically $\frac{1}{2\pi}(1-s)|\ln(1-s)|$, as $s \to 1^-$. This proves the lemma.

In conclusion, we have proved the following proposition, which makes Proposition 3 precise.

Proposition 5. *Let the Markov operator P be symmetric of nonnegative type. If $f \in (I-P)^\alpha L^2_m$ with $\alpha > 1/2$, then $W \in (I-\theta)^\alpha L^2(\Omega,\mathbb{P}_m)$. Moreover, $W \in (I-\theta)^{1/2}L^2(\Omega,\mathbb{P}_m)$ if and only if f is such that $\int_0^1 \frac{\ln(1-s)}{1-s}\mu_f(ds)$ converges.*

From this study we observe that there are two obstacles to proving Theorem 1 for $\alpha = 1/2$ by our method. First, when f is a fractional coboundary of degree 1/2 for P we don't obtain in general a decomposition of $S_n(f)$ as a sum of a martingale and a cocycle of θ defined by a fractional coboundary of degree 1/2 (see Section 8 for an example). Second, from Theorem 3.2 and Proposition 3.8 of [DL2], we know that $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W\circ\theta^k$ need not converge $\mathbb{P}_m$ $a.s.$ even if W is a fractional L^2-coboundary of degree 1/2 for θ.

8. Application to random walks on compact groups

Let G be a compact group, with $\mathscr{B}$ the $\sigma-$algebra of Borel subsets and m the normalized Haar measure. Let ν be a probability measure on G.

The random walk on G defined by ν is the Markov chain whose transition probability is given by the convolution by ν. The Markov operator is

$$Pf(x) = f * \nu(x) = \int_G f(xy)\nu(dy).$$

The invariance of the Haar measure under P is obvious, and it is well known that the ergodicity of P is equivalent to the property that ν is not supported by a proper closed subgroup of G. In the sequel we always assume this condition, to put ourselves in the setting of the preceding sections.

Fubini's theorem yields $P^*f(x) = f * \nu^*(x) = \int f(xy^{-1})\nu(dy)$ where ν^* is the image measure of ν by the map $x \to x^{-1}$. Thus P is symmetric on L^2_m if and only if ν is symmetric on G, that is $\nu = \nu^*$. Under this assumption Theorem B and our Theorems 1 and 2 apply for any compact group G.

We shall assume, from now on, that G is abelian and compact; the group operation will be denoted by $+$. Then P is a normal operator on L^2_m for every ν, since:

$$P^*Pf(x) = PP^*f(x) = \int\int f(x+y-z)\nu(dy)\nu(dz).$$

The dual group $\widehat{G}$ of G is discrete. Ergodicity of P is then equivalent to $\widehat{\nu}(\chi) \neq 1$ for every non-identity character $\chi \in \widehat{G}$, where $\widehat{\nu}(\chi) = \int_G \chi d\nu$. A function $f \in L^2_m$ has the Fourier expansion $f = \sum_{\chi\in\widehat{G}} \widehat{f}(\chi)\chi$ with convergence in L^2_m, where $\widehat{f}(\chi) = \int f\overline{\chi}dm$. Using the expansions of f and $g \in L^2_m$, the equation:

$$f = (I-P)^\alpha g = g * (\delta_e - \sum_{n=1}^{\infty} a_n^{(\alpha)}\nu^{*n})$$

can be written as:

$$\widehat{f}(\chi) = (1-\widehat{\nu}(\chi))^\alpha \widehat{g}(\chi) \text{ for every character } \chi \in \widehat{G}.$$

Therefore, by the result of [GL2] or [DL1] , the central limit theorem holds for f with $\int f dm = 0$, the starting point being $m-$distributed, when $\sum_{1\neq\chi\in\widehat{G}} \frac{|\widehat{f}(\chi)|^2}{|(1-\widehat{\nu}(\chi))|}$ $< \infty$. Theorems 1 and 2 have the following corollary.

Corollary 3. *Let ν be ergodic on G. If for $f \in L^2_m$ with $\int f dm = 0$ there exists $\alpha > 1/2$ such that*

$$\sum_{1\neq\chi\in\widehat{G}} \frac{|\widehat{f}(\chi)|^2}{|(1-\widehat{\nu}(\chi))|^{2\alpha}} < \infty,$$

then the functional central limit theorem holds for f and the random walk generated by ν, started at $x \in G$, for $m-$almost every $x \in G$.

A simple and striking example is given by the following singular random walk on the one dimensional torus (which was considered in [P]).

On $G = \mathbb{R}/\mathbb{Z}$ we take $\nu = p\delta_a + q\delta_{-a}$ with $a \in [0,1]$ irrational, and $0 < p < 1$, $q = 1-p$. The random walk defined by ν, started at x, performs on the orbit of x under the ergodic transformation $T : x \to x + a$ (mod 1), a simple random walk with the probability of a step +1 equal to p. The Fourier coefficients of ν are $\widehat{\nu}(n) = pe^{2i\pi na} + qe^{-2i\pi na}$. The quantity $|1-\widehat{\nu}(n)|$ is bounded away from

0 together with the fractional part $\{na\}$. When the fractional part $\{na\}$ tends to 0, $|1-\widehat{\nu}(n)|^{2\alpha} \sim C\,\{na\}^{2\alpha}$ if $p \neq 1/2$, or $|1-\widehat{\nu}(n)|^{2\alpha} \sim C\,\{na\}^{4\alpha}$ if $p = 1/2$. With $\widehat{f}(n)$ denoting the $n^{th}-$Fourier coefficient of f our study yields the following result: when $p \neq 1/2$ (resp. $p = 1/2$), if $\sum_{0\neq n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 / \{na\}^{2\alpha} < \infty$ (resp. $\sum_{0\neq n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 / \{na\}^{4\alpha} < \infty$) for some $\alpha > 1/2$, then the functional central limit theorem holds for f and the random walk started at x, for almost every $x \in \mathbb{R}/\mathbb{Z}$. If the same series converges only with $\alpha = 1/2$ we know that the central limit theorem holds when the starting point of the random walk is distributed according to the Lebesgue measure, but we do not know if it is valid when the starting point is fixed.

In this context we can present an example, which will complete our previous discussion of the problem of obtaining Theorem 1 for $\alpha = 1/2$ by our methods.

We construct a symmetric Markov operator of positive type, obtained by a convolution on $\mathbb{R}/\mathbb{Z}$ (with normalized Haar measure m), and a function $f \in \sqrt{I-P}L^2_m$, such that the decomposition $S_n(f) = M_n + \sum_{k=0}^{n-1} W\circ\theta^k$ of Proposition 2 yields $W \notin \sqrt{I-\theta}L^2_m$ (though W is a fractional θ-coboundary of degree β for any $0 < \beta < 1/2$, by Proposition 3).

For $a \in [0,1]$ irrational, define on $\mathbb{R}/\mathbb{Z}$ the probability measure $\eta = \frac{1}{4}\big(\delta_a + \delta_{-a}\big) + \frac{1}{2}\delta_0$. Since η is symmetric, the convolution operator $Pf := f*\eta$ is symmetric on L^2_m. The Fourier coefficients of η are $\widehat{\eta}(n) = \frac{1}{2}(\cos(2\pi na)+1) = \cos^2(\pi na)$, so P is ergodic. Since $Pe_n = \widehat{\eta}(n)e_n$ for $n \in \mathbb{Z}$, where $e_n(x) = \mathrm{e}^{2\pi inx}$, for $f \in L^2_m$ with Fourier series $f = \sum_{n\in\mathbb{Z}} u_n e_n$ we have $\langle P^k f, f\rangle = \sum_n |u_n|^2 \cos^{2k}(\pi na)$. Hence P is of positive type, and for every $f \in L^2_m$ the spectral measure μ_f is supported in [0,1], with moments $c_k(f) = \int_0^1 t^k d\mu_f(t) = \langle P^k f, f\rangle$. The condition $f \in \sqrt{I-P}L^2_m$ is equivalent to $\int_0^1 \frac{1}{1-t} d\mu_f(t) < \infty$, or when $u_0 = \int f\,dm = 0$, to

$$\sum_{k=0}^{\infty}\langle P^k f, f\rangle = \sum_{0\neq n\in\mathbb{Z}} |u_n|^2 \frac{1}{1-\cos^2(\pi na)} < \infty.$$

By Proposition 5, $W \in \sqrt{I-\theta}L^2(\Omega,\mathbb{P}_m)$ if and only if $\int_0^1 \frac{|\ln(1-t)|}{1-t} d\mu_f(t) < \infty$. Multiplying power series, for $0 \leq t < 1$ we have $|\ln(1-t)|/(1-t) = \sum_{k=1}^{\infty}\big(\sum_{j=1}^{k}\frac{1}{j}\big)t^k$. Using Lebesgue's theorem, we see that, when $u_0 = 0$, the condition for W is equivalent to

$$\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}\frac{1}{j}\Big)c_k(f) = \sum_{n\in\mathbb{Z}} |u_n|^2 \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k}\frac{1}{j}\Big)\cos^{2k}(\pi na)$$

$$= \sum_{0 \neq n \in \mathbb{Z}} |u_n|^2 \frac{|\ln\left(1-\cos^2(\pi n a)\right)|}{1-\cos^2(\pi n a)} < \infty.$$

Since a is irrational, there exists a subsequence $\{n_\ell\}$ of positive integers with $\cos^2(\pi n_\ell a) \to 1$ such that $s_\ell := 1 - \cos^2(\pi n_\ell a)$ satisfies $|\ln s_\ell| \geq \ell$ for every $\ell > 0$. Defining $u_n = 0$ for $n \notin \{n_\ell\}$ and $u_{n_\ell} = \sqrt{s_\ell}/\ell$, we have

$$\sum_{0 \neq n \in \mathbb{Z}} |u_n|^2 \frac{|\ln\left(1-\cos^2(\pi n a)\right)|}{1-\cos^2(\pi n a)} = \sum_{\ell} \frac{s_\ell}{\ell^2} \frac{|\ln s_\ell|}{s_\ell} = \infty$$

while $\displaystyle\sum_{0 \neq n \in \mathbb{Z}} |u_n|^2 \frac{1}{1-\cos^2(\pi n a)} < \infty.$

Remark. The shift θ in the previous example is exact, and the two-sided shift $\tilde{\theta}$ is mixing. Now let T be the unitary operator induced by $\tilde{\theta}$. By the discussion of the preceding section, the function W of the example satisfies $\sum_{k=1}^{\infty} |\langle T^k W, W\rangle| = \sum_{k=1}^{\infty} \langle P^{k-1} f, f\rangle < \infty$, and, by Proposition 3, $\frac{1}{\sqrt{n}} \| \sum_{k=0}^{n-1} T^k W\| \to 0$, although $W \notin \sqrt{I-T} L^2(\tilde{\Omega}, \tilde{\mathbb{P}}_m)$.

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