Pseudo-Inverse of a Matrix

The pseudo-inverse of a $m \times n$ matrix A is a matrix that generalizes to arbitrary matrices the notion of inverse of a square, invertible matrix. The pseudo-inverse can be expressed from the singular value decomposition (SVD) of A, as follows.

Let the SVD of A be

$$A = U \left(\begin{array}{cc} S & 0 \\ 0 & 0 \end{array} \right) V^T,$$

where U, V are both orthogonal matrices, and S is a diagonal matrix containing the (positive) singular values of A on its diagonal.

Then the pseudo-inverse of A is the $n \times m$ matrix defined as

$$A^\dagger = V \left(\begin{array}{cc} S^{-1} & 0 \\ 0 & 0 \end{array} \right) U^T.$$

Note that $A^\dagger$ has the same dimension as the transpose of A.

This matrix has many useful properties:

- If A is full column rank, meaning $\mathbf{rank}(A) = n \le m$, that is, $A^T A$ is not singular, then A† is a left inverse of A, in the sense that $A^\dagger A = I_n$. We have the closed-form expression

$$A^\dagger = (A^T A)^{-1} A^T.$$

- If A is full row rank, meaning $\mathbf{rank}(A) = m \le n$, that is, AA^T is not singular, then A† is a right inverse of A, in the sense that $AA^\dagger = I_m$. We have the closed-form expression

$$A^\dagger = A^T (AA^T)^{-1}.$$

- If A is square, invertible, then its inverse is $A^\dagger = A^{-1}$.
- The solution to the least-squares problem

$$\min_x \ ||Ax - y||_2$$

with minimum norm is $x^* = A^\dagger y$.

Example: pseudo-inverse of a 4×5 matrix.