# Neumann series

A **Neumann series** is a mathematical series of the form

$$\sum_{k=0}^{\infty} T^k$$

where $T$ is an operator and $T^k := T^{k-1} \circ T$ its $k$ times repeated application. This generalizes the geometric series.

The series is named after the mathematician Carl Neumann, who used it in 1877 in the context of potential theory. The Neumann series is used in functional analysis. It forms the basis of the Liouville-Neumann series, which is used to solve Fredholm integral equations. It is also important when studying the spectrum of bounded operators.

## Properties

Suppose that $T$ is a bounded linear operator on the normed vector space $X$. If the Neumann series converges in the operator norm, then $\mathrm{Id} - T$ is invertible and its inverse is the series:

$$(\mathrm{Id} - T)^{-1} = \sum_{k=0}^{\infty} T^k,$$

where $\mathrm{Id}$ is the identity operator in $X$. To see why, consider the partial sums

$$S_n := \sum_{k=0}^{n} T^k.$$

Then we have

$$\lim_{n\to\infty} (\mathrm{Id} - T) S_n = \lim_{n\to\infty} \left( \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} \right) = \lim_{n\to\infty} \left( \mathrm{Id} - T^{n+1} \right) = \mathrm{Id}.$$

This result on operators is analogous to geometric series in $\mathbb{R}$, in which we find that:

$$(1-x)\cdot(1+x+x^2+\cdots+x^{n-1}+x^n) = 1-x^{n+1},$$
$$1+x+x^2+\cdots = \frac{1}{1-x}.$$

One case in which convergence is guaranteed is when $X$ is a Banach space and $|T| < 1$ in the operator norm or $\sum |T^n|$ is convergent. However, there are also results which give weaker conditions under which the series converges.

## Example

Let $C \in \mathbb{R}^{3\times 3}$ be given by:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{5}{7} & 0 & \frac{1}{7} \\ \frac{3}{10} & \frac{3}{5} & 0 \end{pmatrix}.$$

We need to show that C is smaller than unity in some norm. Therefore, we calculate:

$$||C||_\infty = \max_i \sum_j |c_{ij}| = \max \left\{ \frac{3}{4}, \frac{6}{7}, \frac{9}{10} \right\} = \frac{9}{10} < 1.$$

Thus, we know from the statement above that $(I - C)^{-1}$ exists.

# Approximate matrix inversion

A truncated Neumann series can be used for approximate matrix inversion. To approximate the inverse of an invertible matrix $\mathbf{A}$, we can assign the linear operator as:

$$T(\mathbf{x}) = (\mathbf{I} - \mathbf{A})\mathbf{x}$$

where $\mathbf{I}$ is the identity matrix. If the norm condition on $T$ is satisfied, then truncating the series at $n$, we get:

$$\mathbf{A}^{-1} \approx \sum_{i=0}^{n} (\mathbf{I} - \mathbf{A})^i$$

# The set of invertible operators is open

A corollary is that the set of invertible operators between two Banach spaces $B$ and $B'$ is open in the topology induced by the operator norm. Indeed, let $S : B \to B'$ be an invertible operator and let $T : B \to B'$ be another operator. If $|S - T| < |S^{-1}|^{-1}$, then $T$ is also invertible. Since $|\mathrm{Id} - S^{-1}T| < 1$, the Neumann series $\sum (\mathrm{Id} - S^{-1}T)^k$ is convergent. Therefore, we have

$$T^{-1}S = (\mathrm{Id} - (\mathrm{Id} - S^{-1}T))^{-1} = \sum_{k=0}^{\infty} (\mathrm{Id} - S^{-1}T)^k$$

Taking the norms, we get

$$|T^{-1}S| \le \frac{1}{1 - |\mathrm{Id} - (S^{-1}T)|}$$

The norm of $T^{-1}$ can be bounded by

$$|T^{-1}| \le \tfrac{1}{1-q} |S^{-1}| \quad \text{where} \quad q = |S - T| \, |S^{-1}|.$$

# Applications

The Neumann series has been used for linear data detection in massive multiuser multiple-input multiple-output (MIMO) wireless systems. Using a truncated Neumann series avoids computation of an explicit matrix inverse, which reduces the complexity of linear data detection from cubic to square.[1]

Another application is the theory of Propagation graphs which takes advantage of Neumann series to derive closed form expression for the transfer function.

# References


1. Wu, M.; Yin, B.; Vosoughi, A.; Studer, C.; Cavallaro, J. R.; Dick, C. (May 2013). "Approximate matrix inversion for high-throughput data detection in the large-scale MIMO uplink" (https://ieeexplore.ieee.org/document/6572301). *2013 IEEE International Symposium on Circuits and Systems (ISCAS2013)*. pp. 2155–2158. doi:10.1109/ISCAS.2013.6572301 (https://doi.org/10.1109%2FISCAS.2013.6572301). hdl:1911/75011 (https://hdl.handle.net/1911%2F75011). ISBN 978-1-4673-5762-3. S2CID 389966 (https://api.semanticscholar.org/CorpusID:389966).

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