AMATH 586 SPRING 2020
MIDTERM EXAM — DUE MAY 15 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2020` repository.

This entire exam concerns system of ODEs called the *Toda lattice*. The system is defined using positions p_j, $j = 0, \pm 1, \pm 2, \ldots$ and momenta q_j, $j = 0, \pm 1, \pm 2, \ldots$:

$$\begin{aligned} q_j'(t) &= p_j(t), \\ p_j'(t) &= e^{-(q_j(t) - q_{j-1}(t))} - e^{-(q_{j+1}(t) - q_j(t))}, \quad j = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{1}$$

This is, as defined, an infinite-dimensional ODE system. We will consider finite-dimensional approximations.

Problem 1: Define new variables

$$\begin{aligned} a_j(t) &= \frac{1}{2} e^{-(q_{j+1}(t) - q_j(t))/2}, \\ b_j(t) &= -\frac{1}{2} p_j(t), \quad j = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{2}$$

Using (2), first show that the Toda lattice (1) can be written as

$$\begin{aligned} a_j'(t) &= a_j(t)(b_{j+1}(t) - b_j(t)), \\ b_j'(t) &= 2(a_j^2(t) - a_{j-1}^2(t)), \quad j = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{3}$$

Now consider the time-dependent tridiagonal matrices:

$$T(t) = \begin{bmatrix} \ddots & \ddots & & & & \\ \ddots & b_{j-1}(t) & a_{j-1}(t) & & & \\ & a_{j-1}(t) & b_j(t) & a_j(t) & & \\ & & a_j(t) & b_{j+1}(t) & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}, \quad S(t) = \begin{bmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & a_{j-1}(t) & & & \\ & -a_{j-1}(t) & 0 & a_j(t) & & \\ & & -a_j(t) & 0 & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}$$

Now, show that (3) is equilvalent to

$$T'(t) = S(t)T(t) - T(t)S(t). \tag{4}$$

Hint: Fix j and consider

$$e_j^T S(t) T(t) e_k \text{ and } e_j^T T(t) S(t) e_k, \quad k = j-2, j-1, j, j+1, j+2,$$

where e_j denotes the standard basis vector that is all zero except for a one in the jth entry. Despite the fact that the matrices are bi-infinite, you only need to track the $(j-1)$th, jth and $(j+1)$th entries of vectors such as $e_j^T S(t)$ and $T(t) e_k$.

Problem 2: One finite-dimensional approximation of (4) is to just take a finite section (a square subblock on the diagonal) of both T, S:

$$T_N(t) = \begin{bmatrix} b_1(t) & a_1(t) & & & \\ a_1(t) & b_2(t) & a_2(t) & & \\ & a_2(t) & b_3(t) & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & a_{N-1}(t) & b_N(t) \end{bmatrix},$$

$$S_N(t) = \begin{bmatrix} 0 & a_1(t) & & & \\ -a_1(t) & 0 & a_2(t) & & \\ & -a_2(t) & 0 & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & -a_{N-1}(t) & 0. \end{bmatrix}$$

The finite section choice can be understood by formally setting $q_0 = -\infty$ and $q_{N+1} = +\infty$ and then performing the change of variables (2).

With initial conditions $b_j(0) = 0$, $a_j(0) = 1/2, j = 1, 2, \ldots, N$ and $N = 6$, use your favorite time-stepping method to solve

$$T_N'(t) = S_N(t)T_N(t) - T_N(t)S_N(t),$$

to $t = 100$ and plot the solution. You should notice something striking about the solution. You might want to look at eigenvalues of $T_N(0)$. Comment on this. Repeat this with $b_j(0) = -2$ and $a_j(0) = 1$, $j = 1, 2, \ldots, N$ and $N = 12$.

Problem 3: The Toda lattice in the finite-section case is a Hamiltonian system with Hamiltonian

$$H(p, q) = \frac{1}{2}p_N^2 + \sum_{j=1}^{N-1} \left[\frac{1}{2}p_j^2 + e^{-(q_{j+1}-q_j)} \right].$$

This means that the equations of motion for $q_j(t)$ and $p_j(t)$ can also be written as

(5)
$$\begin{aligned} p_j'(t) &= -\frac{\partial H}{\partial q_j}(p(t), q(t)), \\ q_j'(t) &= \frac{\partial H}{\partial p_j}(p(t), q(t)), \end{aligned}$$

where $p(t) = (p_j(t))_{j=1}^N$ and $q(t) = (q_j(t))_{j=1}^N$. And, by the chain rule, H is conserved:

$$\frac{d}{dt}H(p(t), q(t)) = 0.$$

Sympletic numerical integrators for Hamiltonian systems are designed to preserve conserved quantities and geometric properties of systems they approximate. We can summarize the system (5) as

$$p'(t) = J(q(t)), \quad q'(t) = K(p(t)).$$

One symplectic method is the so-called Störmer–Verlet method and it is given by

$$\begin{aligned} P^* &= P^n + \frac{k}{2} J(Q^n), \\ Q^{n+1} &= Q^n + kK(P^*), \\ P^{n+1} &= P^* + \frac{k}{2} J(Q^{n+1}). \end{aligned}$$

Convert the initial data $b_j(0) = 0$ and $a_j(0) = 1/2$ for $j = 1, 2, \ldots, N$ to $q_j(0), p_j(0)$ for $j = 1, 2, \ldots, N$ and solve the system with the Störmer-Verlet method. Perform a convergence study at $t = 1$ for time steps $k = 2^{-j}$, $j = 1, 2, 3, 4, 5, 6$ (see `https://github.com/trogdoncourses/amath-586-2020/blob/master/notebooks/Astability.ipynb`) to determine the order of the method.

Some hints:

- Since the a_j,b_j variables depend only on a difference of the q_j. You can set one value, say $q_1(0)$, to be whatever value you wish.
- You also might want to write a function to convert between a_j, b_j and p_j, q_j. Here is a `Julia` implementation:

```
to_a = (p,q) -> .5*exp.(-(q[2:end]-q[1:end-1])/2)
to_b = (p,q) -> -.5*p
to_p = (a,b) -> -2*b
function to_q(a,b) # chooses q[1] = 0
    q = fill(0.,length(b))
    q[2:end] = -2*log.(2*a)
    cumsum(q)
end
```

- Here is a `Julia` implementation of J and K:

```
function J(q)
    out = fill(0.,length(q))
    temp = exp.(q[1:end-1] - q[2:end])
    out[1:end-1] -= temp
    out[2:end] += temp
    out
end

function K(p)
    p
end
```

<u>Just for your information</u>: A second order method will satisfy:

$$\text{error at time } T \sim C_T k^2.$$

And the constant C_T is incredibly important as T increases. Symplectic methods can be used to keep C_T from growing too rapidly and they are very important in, say, planetary dynamics over long time scales.

Problem 4: (Extra credit) Form a finite-dimensional approximation of (1) using the boundary condition $q_0(t) = q_{N+1}(t)$ and performing the change of variables (2). Update the matrices T_N and S_N for the periodic case.