**AMATH 586 SPRING 2020**
**HOMEWORK 3 — DUE MAY 8 ON GITHUB BY 11PM**

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**Problem 1:** It is natural to ask if a neighborhood of $z = 0$ can be in the absolute stability region $S$ for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$. Show:

- Consistency implies that $\pi(1; 0) = 0$.
- Stability implies that $\rho'(1) \neq 0$.
- Suppose $\xi = 1 + \eta(z)$ for $z$ near zero so that $\pi(\xi; z) = \pi(1 + \eta(z); z) = 0$. Compute $\eta'(0)$. Why does this imply that there must be an interval $(0, \epsilon]$ for some small $\epsilon > 0$ that does not lie in the absolute stability region $S$.

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**Problem 2:** Recall the test problem

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\mathrm{cn}(x, k)$ is the Jacobi cosine function and $k$ is the elliptic modulus. Some notations use $\mathrm{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Write the equation as a system and compute the Jacobian. For $\beta_1 = 0, \beta_2 = 1, \beta_3 = 10$, based on an analysis of the Jacobian, suggest methods to solve the problem.

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**Problem 3:**

- Plot the absolute stability region for the TR-BDF2 method (8.6).
- By analyzing $R(z)$, show that the method is both A-stable and L-stable. Hint: To show A-stability, show that $|R(z)| \leq 1$ on the imaginary axis and explain why this is enough.

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**Problem 4:** Let $g(x) = 0$ represent a system of $s$ nonlinear equations in $s$ unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{1}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess $x^0$, compute

$$x^{j+1} = g(x^j) \tag{2}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).

(b) Suppose $g(x)$ is differentiable and let $D_x g(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(D_x(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make $N$ correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n) \\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1 \\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j) \\
&\qquad \text{end} \\
&U^{n+1} = \hat{U}^N.
\end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region $S_N$ of this method for $N = 2,\ 5,\ 10,\ 20,\ 50$ and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues $\lambda$ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region $S_N$ of part (c) to converge to as $N \to \infty$?

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**Problem 5:** Consider the matrix $M_r = I - rT$ where $T$ is the $m \times m$ matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and $r \geq 0$. Find the largest value of $c$ such that $M_r$ is invertible for all $r \in [0, c)$.