

AMATH 586 SPRING 2021
HOMEWORK 2 — DUE APRIL 23 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2021` repository.

Problem 1: Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3}v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\operatorname{cn}(x, k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\operatorname{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 6 & 7 — produce a plot and a table.

Problem 2: Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

- (a) $U^{n+3} = U^{n+1} + 2kf(U^n)$,
 - (b) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$,
 - (c) $U^{n+1} = U^n$,
 - (d) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$,
 - (e) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.
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Problem 3: For the one-step method (6.17), with Ψ given in (6.18), show that the Lipschitz constant is $L' = L + \frac{k}{2}L^2$ where L is the Lipschitz constant for f .

Problem 4: The Fibonacci numbers

- (a) Determine the general solution to the linear difference equation $U^{n+2} = U^{n+1} + U^n$.
- (b) Determine the solution to this difference equation with the starting values $U^0 = 1, U^1 = 1$. Use this to determine U^{30} . (Note, these are the *Fibonacci numbers*, which of course should all be integers.)

- (c) Show that for large n the ratio of successive Fibonacci numbers U^n/U^{n-1} approaches the “golden ratio” $\phi \approx 1.618034$.

Problem 5: Explicit solution of leapfrog: Consider the difference equation

$$\begin{aligned} U^{n+1} + U^{n-1} &= 2\lambda U^n, \quad n \geq 0, \\ U^0 &= 1, \quad U^{-1} = 0. \end{aligned}$$

Provided that $-1 \leq \lambda \leq 1$ perform the following steps:

- (a) Argue that this can be replaced with

$$\begin{aligned} U^{n+1} + U^{n-1} &= (e^{i\theta} + e^{-i\theta})U^n, \quad n \geq 0, \\ U^0 &= 1, \quad U^{-1} = 0, \quad \theta \in \mathbb{R}. \end{aligned}$$

- (b) Define $V^n = U^n - e^{i\theta}U^{n-1}$ and find a simpler recurrence relation for V^n . Solve it.
- (c) With V^n known, $V^n = U^n - e^{i\theta}U^{n-1}$ gives an inhomogeneous recurrence relation for U^n . Find a formula for U^n . For which value(s) of $\lambda \in [-1, 1]$ is U^n largest?

Problem 6: Any r -stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r . For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a p th order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + O(z^{p+1}) \quad \text{as } z \rightarrow 0.$$

This indicates that the one-step error is $O(z^{p+1})$ on this problem, as expected for a p th order accurate method.

The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to $u'(t) = \lambda u(t)$. Show that in fact when applied to this problem the method becomes $U^{n+1} = R(z)U^n$ where $R(z)$ is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of e^z through $O(z^4)$ terms.

We will see that this function $R(z)$ is also important in the study of absolute stability of a one-step method.

Problem 7: Determine the function $R(z)$ described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for $u'(t) = \lambda u(t)$. (You might want to convince yourself these are the same method).

Confirm that $R(z)$ agrees with e^z to the expected order.

Note that for this implicit method $R(z)$ will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1 - \epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots .$$