

**AMATH 586 SPRING 2021**  
**HOMEWORK 1 — DUE APRIL 9 ON GRADESCOPE BY 11PM**

Be sure to do a `git pull` to update your local version of the `amath-586-2021` repository.

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**Problem 1:** Using the Taylor series representation of the matrix exponential:

(a) Verify the identities

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$$

for an  $n \times n$  matrix  $A$ .

(b) Verify that  $u(t) = e^{tA} \eta$  is indeed the solution of the IVP

$$\begin{cases} u'(t) = Au(t), \\ u(0) = \eta. \end{cases}$$

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**Problem 2:** Construct a system (i.e., needs to be not scalar valued)

$$\begin{cases} u'(t) = f(u(t)), \end{cases}$$

and two choices of initial data  $u_0 \neq v_0$  so that two solutions

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0, \end{cases} \quad \begin{cases} v'(t) = f(v(t)), \\ v(0) = v_0, \end{cases}$$

satisfy

$$(1) \quad \|u(t) - v(t)\|_2 = \|u(0) - v(0)\|_2 e^{Lt}$$

where  $L$  a Lipschitz constant for  $f(u)$ . Recall that we have shown that for any solution

$$\|u(t) - v(t)\|_2 \leq \|u(0) - v(0)\|_2 e^{Lt}.$$

So, you are tasked with showing that this is sharp. Then show that equality (1) fails to hold for  $u'(t) = -f(u(t))$ ,  $v'(t) = -f(v(t))$  with the same initial conditions.

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**Problem 3:** Consider the IVP

$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1(t) - u_2(t), \end{cases}$$

with initial conditions specified at time  $t = 0$ . Solve this problem in two different ways:

- (a) Solve the first equation, which only involves  $u_1$ , and then insert this function into the second equation to obtain a nonhomogeneous linear equation for  $u_2$ . Solve this using (5.8). Check that your solution satisfies the initial conditions and the ODE.
- (b) Write the system as  $u' = Au$  and compute the matrix exponential using (D.30) to obtain the solution.

**Problem 4:** Consider the IVP

$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1 + 2u_2(t), \end{cases}$$

with initial conditions specified at time  $t = 0$ . Solve this problem.

**Problem 5:** Consider the Lotka–Volterra system<sup>1</sup>

$$\begin{cases} u_1'(t) = \alpha u_1(t) - \beta u_1(t)u_2(t), \\ u_2'(t) = \delta u_1(t)u_2(t) - \gamma u_2(t). \end{cases}$$

For  $\alpha = \delta = \gamma = \beta = 1$  and  $u_1(0) = 5, u_2(0) = 0.8$  use the forward Euler method to approximate the solution with  $k = 0.001$  for  $t = 0, 0.001, \dots, 50$ . Plot your approximate solution as a curve in the  $(u_1, u_2)$ -plane and plot your approximations of  $u_1(t)$  and  $u_2(t)$  on the same axes as a function of  $t$ . Repeat this with backward Euler. What do you notice about the behavior of the numerical solutions? The most obvious feature is most apparent in the  $(u_1, u_2)$ -plane.

**Problem 6:** Determine the coefficients  $\beta_0, \beta_1, \beta_2$  for the third order, 2-step Adams-Moulton method. Do this in two different ways:

- (a) Using the expression for the local truncation error in Section 5.9.1,
- (b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s)) ds.$$

Interpolate a quadratic polynomial  $p(t)$  through the three values  $f(U^n)$ ,  $f(U^{n+1})$  and  $f(U^{n+2})$  and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values  $f(U^{n+j})$ . It's easiest to use the “Newton form” of the interpolating polynomial and consider the three times  $t_n = -k$ ,  $t_{n+1} = 0$ , and  $t_{n+2} = k$  so that  $p(t)$  has the form

$$p(t) = A + B(t + k) + C(t + k)t$$

where  $A$ ,  $B$ , and  $C$  are the appropriate divided differences based on the data. Then integrate from 0 to  $k$ . (The method has the same coefficients at any time, so this is valid.)

<sup>1</sup>This is a famous model of predator-prey dynamics.