

**AMATH 586 SPRING 2021**  
**HOMEWORK 3 — DUE MAY 7 ON GITHUB BY 11PM**

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**Problem 1:** It is natural to ask if a neighborhood of  $z = 0$  can be in the absolute stability region  $S$  for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^r \alpha_j U^{n+j} = k \sum_{j=0}^r \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial  $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$ . Show:

- Consistency implies that  $\pi(1; 0) = 0$ .
  - Stability implies that  $\rho'(1) \neq 0$ .
  - Suppose  $\xi = 1 + \eta(z)$  for  $z$  near zero so that  $\pi(\xi; z) = \pi(1 + \eta(z); z) = 0$ . Compute  $\eta'(0)$ . Why does this imply that there must be an interval  $(0, \epsilon]$  for some small  $\epsilon > 0$  that does not lie in the absolute stability region  $S$ .
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**Problem 2:** Recall the test problem

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3}v'(t) = 0,$$

where  $\beta_1 < \beta_2 < \beta_3$ . It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\text{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where  $\text{cn}(x, k)$  is the Jacobi cosine function and  $k$  is the elliptic modulus. Some notations use  $\text{cn}(x, m)$  where  $m = k^2$ . The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Write the equation as a system and compute the Jacobian as a function of  $t$ . For  $\beta_1 = -1, \beta_2 = 1, \beta_3 = 2$ , based on an analysis of the Jacobian, suggest methods to solve the problem. Repeat for  $\beta_1 = 0, \beta_2 = 1, \beta_3 = 10$ .

Note: It will help to plot each eigenvalue of the Jacobian as a function of  $t$ . If you sort the eigenvalues by their imaginary parts, things are bit clearer.

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**Problem 3:** • Plot the absolute stability region for the TR-BDF2 method (8.6).

- By analyzing  $R(z)$ , show that the method is both A-stable and L-stable. Hint: To show A-stability, show that  $|R(z)| \leq 1$  on the imaginary axis and explain why this is enough.

**Problem 4:** Let  $g(x) = 0$  represent a system of  $s$  nonlinear equations in  $s$  unknowns, so  $x \in \mathbb{R}^s$  and  $g : \mathbb{R}^s \rightarrow \mathbb{R}^s$ . A vector  $\bar{x} \in \mathbb{R}^s$  is a *fixed point* of  $g(x)$  if

$$(1) \quad \bar{x} = g(\bar{x}).$$

One way to attempt to compute  $\bar{x}$  is with *fixed point iteration*: from some starting guess  $x^0$ , compute

$$(2) \quad x^{j+1} = g(x^j)$$

for  $j = 0, 1, \dots$

- Show that if there exists a norm  $\|\cdot\|$  such that  $g(x)$  is Lipschitz continuous with constant  $L < 1$  in a neighborhood of  $\bar{x}$ , then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).
- Suppose  $g(x)$  is differentiable and let  $D_x g(x)$  be the  $s \times s$  Jacobian matrix. Show that if the condition of part (a) holds then  $\rho(D_x g(\bar{x})) < 1$ , where  $\rho(A)$  denotes the spectral radius of a matrix.
- Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make  $N$  correction iterations, i.e., we set

$$\begin{aligned} \hat{U}^0 &= U^n + kf(U^n) \\ \text{for } j &= 0, 1, \dots, N-1 \\ \hat{U}^{j+1} &= U^n + kf(\hat{U}^j) \\ \text{end} \\ U^{n+1} &= \hat{U}^N. \end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region  $S_N$  of this method for  $N = 2, 5, 10, 20, 50$  and observe that in fact the stability region does not grow much in size.

- Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if  $|k\lambda| < 1$  for all eigenvalues  $\lambda$  of the Jacobian matrix  $f'(u)$ .
- Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region  $S_N$  of part (c) to converge to as  $N \rightarrow \infty$ ?

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**Problem 5:** Consider the matrix  $M_r = I - rT$  where  $T$  is the  $m \times m$  matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and  $r \geq 0$ . Find the largest value of  $c$  such that  $M_r$  is invertible for all  $r \in [0, c)$ .