AMATH 586 SPRING 2022
HOMEWORK 1 — DUE APRIL 8 ON GRADESCOPE BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2022` repository.

Problem 1: Using the Taylor series representation of the matrix exponential:

(a) Verify the identities

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathrm{e}^{tA} = A\,\mathrm{e}^{tA} = \mathrm{e}^{tA}\,A$$

for an $n \times n$ matrix A.

(b) Verify that $u(t) = \mathrm{e}^{tA}\,\eta$ is indeed the solution of the IVP

$$\begin{cases} u'(t) = Au(t), \\ u(0) = \eta. \end{cases}$$

Problem 2: Construct a system (i.e., needs to be not scalar valued)

$$\begin{cases} u'(t) = f(u(t)), \end{cases}$$

and two choices of initial data $u_0 \neq v_0$ so that two solutions

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0, \end{cases} \qquad \begin{cases} v'(t) = f(v(t)), \\ v(0) = v_0, \end{cases}$$

satisfy

$$\|u(t) - v(t)\|_2 = \|u(0) - v(0)\|_2\,\mathrm{e}^{Lt} \tag{1}$$

where L a Lipschitz constant for $f(u)$. Recall that we have shown that for any solution

$$\|u(t) - v(t)\|_2 \leq \|u(0) - v(0)\|_2\,\mathrm{e}^{Lt}\,.$$

So, you are tasked with showing that this is sharp. Then show that equality (1) fails to hold for $u'(t) = -f(u(t)), v'(t) = -f(v(t))$ with the same intial conditions.

Problem 3: Consider the IVP

$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1(t) - u_2(t), \end{cases}$$

with initial conditions specified at time $t = 0$. Solve this problem in two different ways:

(a) Solve the first equation, which only involves u_1, and then insert this function into the second equation to obtain a nonhomogeneous linear equation for u_2. Solve this using (5.8). Check that your solution satisfies the initial conditions and the ODE.
(b) Write the system as $u' = Au$ and compute the matrix exponential using (D.30) to obtain the solution.

Problem 4: Consider the IVP

$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1 + 2u_2(t), \end{cases}$$

with initial conditions specified at time $t = 0$. Solve this problem.

Problem 5: Consider the Lotka–Volterra system[1]

$$\begin{cases} u_1'(t) = \alpha u_1(t) - \beta u_1(t)u_2(t), \\ u_2'(t) = \delta u_1(t)u_2(t) - \gamma u_2(t). \end{cases}$$

For $\alpha = \delta = \gamma = \beta = 1$ and $u_1(0) = 5, u_2(0) = 0.8$ use the forward Euler method to approximate the solution with $k = 0.001$ for $t = 0, 0.001, \ldots, 50$. Plot your approximate solution as a curve in the (u_1, u_2)-plane and plot your approximations of $u_1(t)$ and $u_2(t)$ on the same axes as a function of t. Repeat this with backward Euler. What do you notice about the behavior of the numerical solutions? The most obvious feature is most apparent in the (u_1, u_2)-plane.

Problem 6: Determine the coefficients β_0, β_1, β_2 for the third order, 2-step Adams-Moulton method. Do this in two different ways:
(a) Using the expression for the local truncation error in Section 5.9.1,
(b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\, ds.$$

Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n)$, $f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t + k) + C(t + k)t$$

where A, B, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)

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[1] This is a famous model of predator-prey dynamics.