AMATH 586 SPRING 2022
HOMEWORK 3 — DUE MAY 6 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2022` repository.

Problem 1: It is natural to ask if a neighborhood of $z = 0$ can be in the absolute stability region S for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$. Show:

- Consistency implies that $\pi(1; 0) = 0$.
- Zero-stability implies that $\rho'(1) \neq 0$.
- Suppose $\xi = 1 + \eta(z)$ for z near zero so that $\pi(\xi; z) = \pi(1 + \eta(z); z) = 0$. Compute $\eta'(0)$. Why does this imply that there must be an interval $(0, \epsilon]$ for some small $\epsilon > 0$ that does not lie in the absolute stability region S.

Problem 2: Consider the system of ODEs

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \\ u_4'(t) \end{bmatrix} = \begin{bmatrix} u_4(t) - \mu u_3(t) + \lambda g(u_1(t) + u_2(t)) \\ \mu u_3(t) - u_4(t) + \lambda g(u_1(t) + u_2(t) \\ -\sigma u_4(t) \\ \sigma u_3(t) \end{bmatrix}, \quad g(u) = u(1-u)^2, \quad \lambda > 0,\ \sigma, \mu \in \mathbb{R}.$$

You will show that for some choice of initial conditions the solution is bounded for all t.

- Solve for $u_3(t), u_4(t)$ in terms of $u_3(0) = \eta_3, u_4(0) = \eta_4$.
- Find an ODE solved by $w(t) := u_1(t) - u_2(t)$. Solve it and show that for any fixed choice of $u_1(0) = \eta_1, u_2(0) = \eta_2, \eta_3, \eta_4$ that $w(t)$ is bounded as $t \to \infty$.
- Find an ODE solved by $v(t) := u_1(t) + u_2(t)$. If $0 < \eta_1 + \eta_2 < 1$ argue that $v(t)$ is bounded as $t \to \infty$.
- Why does this imply that $u_1(t), u_2(t)$ are bounded as $t \to \infty$?

Problem 3: Consider the system of ODEs from the previous problem, using

$$u_1(0) = 0.1,\ u_2(0) = 0.1,\ u_3(0) = 0, u_4(0) = 0.1.$$

and $\lambda = 10$, $\sigma = 200$, $\mu = 100$, solve this problem with trapezoid with $k = 0.0001$. We will use this as the "ground truth" for the solution. Plot the eigenvalues of the Jacobian as a function of t. Now, solve with forward Euler and leapfrog (starting with one step of forward Euler) with $k = 0.001$. You should find that one method goes unstable before the other. Use the eigenvalues of the Jacobian to explain this instability. Explain why trapezoid is a good method for this problem.

Note: If you sort the eigenvalues by their imaginary parts at each time step, things might be a bit clearer.

Problem 4:
- Plot the absolute stability region for the TR-BDF2 method (8.6).
- By analyzing $R(z)$, show that the method is both A-stable and L-stable. Hint: To show A-stability, show that $|R(z)| \le 1$ on the imaginary axis and explain why this is enough.

Problem 5: Let $g(x) = 0$ represent a system of s nonlinear equations in s unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{1}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess x^0, compute

$$x^{j+1} = g(x^j) \tag{2}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).

(b) Suppose $g(x)$ is differentiable and let $D_x g(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(D_x g(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n)\\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1\\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j)\\
&\qquad \text{end}\\
&U^{n+1} = \hat{U}^N.
\end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region S_N of this method for $N = 2,\ 5,\ 10,\ 20,\ 50$ and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues λ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (c) to converge to as $N \to \infty$?

Problem 6: Consider the matrix $M_r = I - rT$ where T is the $m \times m$ matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and $r \geq 0$. Find the largest value of c such that M_r is invertible for all $r \in [0, c)$.