

AMATH 586 SPRING 2022
HOMEWORK 3 — DUE MAY 6 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2022` repository.

Problem 1: It is natural to ask if a neighborhood of $z = 0$ can be in the absolute stability region S for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^r \alpha_j U^{n+j} = k \sum_{j=0}^r \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$. Show:

- Consistency implies that $\pi(1; 0) = 0$.
 - Zero-stability and consistency imply that $\rho'(1) \neq 0$.
 - Assuming zero-stability and consistency, suppose $\xi = 1 + \eta(z)$ for z near zero so that $\pi(\xi; z) = \pi(1 + \eta(z); z) = 0$. Compute $\eta'(0)$. Why does this imply that there must be an interval $(0, \epsilon]$ for some small $\epsilon > 0$ that does not lie in the absolute stability region S .
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Problem 2: Consider the system of ODEs

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \\ u_4'(t) \end{bmatrix} = \begin{bmatrix} u_4(t) - \mu u_3(t) + \lambda g(u_1(t) + u_2(t)) \\ \mu u_3(t) - u_4(t) + \lambda g(u_1(t) + u_2(t)) \\ -\sigma u_4(t) \\ \sigma u_3(t) \end{bmatrix}, \quad g(u) = u(1-u)^2, \quad \lambda > 0, \quad \sigma, \mu \in \mathbb{R}.$$

You will show that for some choice of initial conditions the solution is bounded for all t .

- Solve for $u_3(t), u_4(t)$ in terms of $u_3(0) = \eta_3, u_4(0) = \eta_4$.
 - Find an ODE solved by $w(t) := u_1(t) - u_2(t)$. Solve it and show that for any fixed choice of $u_1(0) = \eta_1, u_2(0) = \eta_2, \eta_3, \eta_4$ that $w(t)$ is bounded as $t \rightarrow \infty$.
 - Find an ODE solved by $v(t) := u_1(t) + u_2(t)$. If $0 < \eta_1 + \eta_2 < 1$ argue that $v(t)$ is bounded as $t \rightarrow \infty$.
 - Why does this imply that $u_1(t), u_2(t)$ are bounded as $t \rightarrow \infty$?
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Problem 3: Consider the system of ODEs from the previous problem, using

$$u_1(0) = 0.1, \quad u_2(0) = 0.1, \quad u_3(0) = 0, \quad u_4(0) = 0.1.$$

and $\lambda = 10, \sigma = 200, \mu = 100$, solve this problem with trapezoid with $k = 0.0001$. We will use this as the “ground truth” for the solution. Plot the eigenvalues of the Jacobian as a function of t . Now, solve with forward Euler and leapfrog (starting with one step of forward Euler) with $k = 0.001$. You should find that one method

goes unstable before the other. Use the eigenvalues of the Jacobian to explain this instability. Explain why trapezoid is a good method for this problem.

Note: If you sort the eigenvalues by their imaginary parts at each time step, things might be a bit clearer.

- Problem 4:**
- Plot the absolute stability region for the TR-BDF2 method (8.6).
 - By analyzing $R(z)$, show that the method is both A-stable and L-stable. Hint: To show A-stability, show that $|R(z)| \leq 1$ on the imaginary axis and explain why this is enough.

Problem 5: Let $g(x) = 0$ represent a system of s nonlinear equations in s unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \rightarrow \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$(1) \quad \bar{x} = g(\bar{x}).$$

One way to attempt to compute \bar{x} is with *fixed point iteration*: from some starting guess x^0 , compute

$$(2) \quad x^{j+1} = g(x^j)$$

for $j = 0, 1, \dots$

- Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of \bar{x} , then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).
- Suppose $g(x)$ is differentiable and let $D_x g(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(D_x g(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.
- Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make N correction iterations, i.e., we set

$$\begin{aligned} \hat{U}^0 &= U^n + kf(U^n) \\ \text{for } j &= 0, 1, \dots, N-1 \\ \hat{U}^{j+1} &= U^n + kf(\hat{U}^j) \\ \text{end} \\ U^{n+1} &= \hat{U}^N. \end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region S_N of this method for $N = 2, 5, 10, 20, 50$ and observe that in fact the stability region does not grow much in size.

- (d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues λ of the Jacobian matrix $f'(u)$.
- (e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region S_N of part (c) to converge to as $N \rightarrow \infty$?

Problem 6: Consider the matrix $M_r = I - rT$ where T is the $m \times m$ matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and $r \geq 0$. Find the largest value of c such that M_r is invertible for all $r \in [0, c)$.