AMATH 586 SPRING 2022
HOMEWORK 2 — DUE APRIL 22 ON GITHUB BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2021` repository.

Problem 1: Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\mathrm{cn}(x, k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\mathrm{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 6 & 7 — produce a plot and and a table.

Problem 2: Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+3} = U^{n+1} + 2kf(U^n)$,
(b) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$,
(c) $U^{n+1} = U^n$,
(d) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$,
(e) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

Problem 3: The Fibonacci numbers

(a) Determine the general solution to the linear difference equation $U^{n+2} = U^{n+1} + U^n$.
(b) Determine the solution to this difference equation with the starting values $U^0 = 1$, $U^1 = 1$. Use this to determine U^{30}. (Note, these are the *Fibonacci numbers*, which of course should all be integers.)
(c) Show that for large n the ratio of successive Fibonacci numbers U^n/U^{n-1} approaches the "golden ratio" $\phi \approx 1.618034$.

Problem 4: Explicit solution of leapfrog: Consider the difference equation

$$U^{n+1} + U^{n-1} = 2xU^n, \quad n \geq 0,$$
$$U^0 = 1, \quad U^{-1} = 0.$$

Provided that $-1 \leq x \leq 1$ perform the following steps:

(a) Argue that this can be replaced with

$$U^{n+1} + U^{n-1} = (e^{i\theta} + e^{-i\theta})U^n, \quad n \geq 0,$$
$$U^0 = 1, \quad U^{-1} = 0, \quad \theta \in \mathbb{R}.$$

(b) Define $V^n = U^n - e^{i\theta}U^{n-1}$ and find a simpler recurrence relation for V^n. Solve it.

(c) With V^n known, $V^n = U^n - e^{i\theta}U^{n-1}$ gives an inhomogeneous recurrence relation for U^n. Find a formula for U^n. For which value(s) of $x \in [-1, 1]$ is U^n largest?

Note: In this problem you can make the ansatz $U^n = \lambda^n$ and find a quadratic equation to determine two possible values for λ, say, λ_1, λ_2. Then the general solution is a sum of these, $U^n = c_1\lambda_1^n + c_2\lambda_2^n$. This approach will give another representation of the same solution. You are encouraged to do so and derive this remarkable identity!

Problem 5: Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form

$$U^{n+1} = R(z)U^n$$

where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r. For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying

$$R(z) = e^z + \mathrm{O}(z^{p+1}) \quad \text{as } z \to 0.$$

This indicates that the one-step error is $\mathrm{O}(z^{p+1})$ on this problem, as expected for a pth order accurate method.

The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to $u'(t) = \lambda u(t)$. Show that in fact when applied to this problem the method becomes $U^{n+1} = R(z)U^n$ where $R(z)$ is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of e^z through $O(z^4)$ terms.

We will see that this function $R(z)$ is also important in the study of absolute stability of a one-step method.

Problem 6: Determine the function $R(z)$ described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for $u'(t) = \lambda u(t)$. (You might want to convince yourself these are the same method).

Confirm that $R(z)$ agrees with e^z to the expected order.

Note that for this implicit method $R(z)$ will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1-\epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots.$$

Problem 7: Consider the time-dependent matrices

$$T_N(t) = \begin{bmatrix} b_1(t) & a_1(t) & & & \\ a_1(t) & b_2(t) & a_2(t) & & \\ & a_2(t) & b_3(t) & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & a_{N-1}(t) & b_N(t) \end{bmatrix},$$

$$S_N(t) = \begin{bmatrix} 0 & a_1(t) & & & \\ -a_1(t) & 0 & a_2(t) & & \\ & -a_2(t) & 0 & \ddots & \\ & & \ddots & \ddots & a_{N-1}(t) \\ & & & -a_{N-1}(t) & 0. \end{bmatrix}$$

With initial conditions $b_j(0) = 0$, $j = 1, 2, \ldots, N$, $a_j(0) = 1/2, j = 1, 2, \ldots, N-1$ and $N = 6$, use your favorite time-stepping method to solve

$$T_N'(t) = S_N(t)T_N(t) - T_N(t)S_N(t),$$

to $t = 100$ and plot the solution. You should notice something striking about the solution. You might want to look at eigenvalues of $T_N(0)$. Comment on this. Repeat this with $b_j(0) = -2$ and $a_j(0) = 1$, $j = 1, 2, \ldots, N$ and $N = 12$.