**AMATH 586 SPRING 2023**
**HOMEWORK 5 — DUE MAY 26 BY 11PM**

Be sure to do a `git pull` to update your local version of the `amath-586-2023` repository.
Homeworks must be typeset and uploaded to `Gradescope` for submission.
The submitted homework must include plots and descriptions of your code.
Code should be uploaded to `GitHub`.
You must include your name and `GitHub` username on your assignment.

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**Problem 1:** Consider solving

$$\begin{cases} u_t + u_{xxx} = 0, \quad -1 < x < 1 \\ u(x,0) = \eta(x), \\ u(-1,t) = u(1,t), \\ u_x(-1,t) = u_x(1,t), \\ u_{xx}(-1,t) = u_{xx}(1,t). \end{cases}$$

This is the linear KdV (Airy) equation with periodic boundary conditions.

- Use a second-order accurate centered difference and the trapezoid method as a time-stepper. Can you see dispersive quantization? Use $\eta(x) = 1$ if $-1/2 < x < 1/2$ and $\eta(x) = 0$ otherwise.
- Prove that the method is Lax-Richtmyer stable. Discuss whether or not it is convergent with this initial condition.

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**Problem 2:** Consider solving

$$\begin{cases} u_t + 3(u^2)_x + u_{xxx} = 0, \quad -L < x < L, \\ u(x,0) = \eta(x), \\ u(-L,t) = u(L,t), \\ u_x(-L,t) = u_x(L,t), \\ u_{xx}(-L,t) = u_{xx}(L,t). \end{cases}$$

This is the KdV equation with periodic boundary conditions.

- Use a second-order accurate centered difference and the trapezoid method as a time-stepper to solve this problem with

$$\eta(x) = 4\text{sech}(x)^2.$$

Note that you will need to implement Newton's method for this and use it at each time step.
- You need not prove this, but give an argument that the method Lax-Richtmyer stable. This is easier if you use $(u^2)_x$ in the PDE as opposed to $2uu_x$!

**Problem 3:** Consider solving the small-dispersion (semi-classical) NLS equation

$$\begin{cases} \mathrm{i}\epsilon u_t + \frac{\epsilon^2}{2} u_{xx} + |u|^2 u = 0, \quad -\infty < x < \infty, \\ u(x,0) = A(x) e^{\mathrm{i}S(x)/\epsilon}, \\ A(x) = -\operatorname{sech}(x), \\ S(x) = -\mu \log\cosh(x), \quad \mu = 0.1. \end{cases}$$

Use the Fourier exponential integrator with Runge–Kutta 4 to solve this on $[-L, L]$ for sufficiently large $L$. Use $\epsilon = 0.1$ and $\epsilon = 0.05$ and produce a contour plot of the squared modulus of the solution for $t \in [0, 4]$ for both values of $\epsilon$. Argue that you have chosen $L$ sufficiently large and have chosen a sufficiently large number of Fourier modes.

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**Problem 4:** Consider solving

$$\begin{cases} u_t + 3(u^2)_x + u_{xxx} = 0, \quad -L < x < L, \\ u(x,0) = \eta(x), \\ u(-L,t) = u(L,t), \\ u_x(-L,t) = u_x(L,t), \\ u_{xx}(-L,t) = u_{xx}(L,t). \end{cases}$$

This is the KdV equation with periodic boundary conditions.

- Use a second-order accurate centered difference and the trapezoid method as a time-stepper to solve this problem with

  $$\eta(x) = 4\operatorname{sech}(x)^2, \quad L = 10.$$

  Note that you will need to implement Newton's method for this and use it at each time step.
- You need not prove this, give some rationale as to why you might hope this method is Lax-Richtmyer stable.

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**Problem 5:** Consider solving the small-dispersion (semi-classical) NLS equation

$$\begin{cases} \mathrm{i}\epsilon u_t + \frac{\epsilon^2}{2} u_{xx} + |u|^2 u = 0, \quad -\infty < x < \infty, \\ u(x,0) = A(x) e^{\mathrm{i}S(x)/\epsilon}, \\ A(x) = -\operatorname{sech}(x), \\ S(x) = -\mu \log\cosh(x), \quad \mu = 0.1. \end{cases}$$

Use the Fourier exponential integrator with Runge–Kutta 4 to solve this on $[-L, L]$ for sufficiently large $L$. Use $\epsilon = 0.1$ and $\epsilon = 0.05$ and produce a contour plot of the squared modulus of the solution for $t \in [0, 4]$ for both values of $\epsilon$. Argue that you have chosen $L$ sufficiently large and have chosen a sufficiently large number of Fourier modes.

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**Problem 6:** Consider solving the Kuramoto–Sivashinsky equation with periodic boundary conditions

$$\begin{cases} u_t + uu_x + u_{xx} + u_{xxxx} = 0, \quad -L < x < L, \\ u(x,0) = \eta(x), \\ u(-L,t) = u(L,t), \\ u_x(-L,t) = u_x(L,t), \\ u_{xx}(-L,t) = u_{xx}(L,t), \\ u_{xxx}(-L,t) = u_{xxx}(L,t). \end{cases}$$

Use the ETDRK4 method using Fourier series to solve this problem with $L = 16\pi$, $\eta(x) = \cos(x/16)(1+\sin(x/16))$. Create a contour plot of the solution for $t \in [0, 150]$.