AERO483/ENGR6471 Integration of Avionics Systems

Homework 3

Problem 1: The bivariate Gaussian distribution is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{\frac{\left(\frac{x-\bar{x}}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\bar{x}}{\sigma_X}\right)\left(\frac{y-\bar{y}}{\sigma_Y}\right) + \left(\frac{y-\bar{y}}{\sigma_Y}\right)^2}{-2\left(1-\rho^2\right)}\right\}$$

where $\rho = \sigma_{XY}\sigma_X^{-1}\sigma_Y^{-1}$ is the correlation coefficient. Show that the random variable $X|Y$ has a Gaussian distribution with the same mean and variance of the affine minimum variance estimator of X.

Problem 2: Consider a system with a single state described by

$$x(t+1) = ax(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

where a is not zero, w and v are zero mean uncorrelated white noise processes with variances equal to q and r, respectively. They are also uncorrelated with the initial condition. Answer the following questions:

1. Design a Kalman filter and explicitly write all equations and assumptions
2. Express the variance of the state as a function of the parameters *a, q,* and *r*.
3. What is the value of *a-ak(t)* as *t* goes to infinity, where *k(t)* is the Kalman gain ?

Problem 3: Consider the INS system on flat Earth described by

$$\begin{bmatrix} \delta x(t+1) \\ \delta v(t+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta v(t) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} w(t)$$

$$\delta y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta v(t) \end{bmatrix} + v(t)$$

where T is the sampling period, w and v are zero mean uncorrelated scalar white noise processes with variances equal to q and r, respectively. They are also uncorrelated with the initial condition. Answer the following questions:

1. Design a Kalman filter and explicitly write all equations and assumptions
2. Express the covariance matrix of the state as a function of *T, q,* and *r*.
3. Assuming that the initial condition has a mean of zero and the identity covariance matrix, write down the estimate for the state at the sampling time *T*
4. Assume now that the available measurement is the position and comes from a GPS. Design a loosely coupled INS-GPS integration system assuming that the Kalman filter initial condition is zero with an initial error covariance matrix equal to sigma squared times the identity. Determine the state estimate at the sampling time *T*. Please note that the measurement matrix is not anymore $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Problem 4: Consider the INS system on a curved Earth described by

$$\begin{bmatrix} \delta\dot{v} \\ \delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & -g \\ \dfrac{1}{R+h} & 0 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta\phi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -w_a \\ -w_g \end{bmatrix}$$
$$\delta y = \delta v + v$$

where *w* and *v* are zero mean uncorrelated scalar white noise processes with covariance matrices equal to *Q* and (scalar) *r*, respectively. They are also uncorrelated with the initial condition. Assume that *Q* is a diagonal matrix and answer the following questions:

1. Compute the zero-order-hold equivalent matrices *A*, *B*.
2. Assuming that $\Omega = \sqrt{g/R+h}$ is small find an approximation for *A*, *B*.
1. Design a Kalman filter for the approximation in 2.
2. Express the covariance matrix of the state as a function of *T, q,* and *r*.
3. Assuming that the initial condition has a mean of zero and the identity covariance matrix, write down the estimate for the state at the sampling time *T*

Problem 5: Consider four satellites that are 1000Km away from the origin. Satellites 1,2 are on the line with equation $y = ax$ where *a* is the largest digit of your ID number. Satellite 1 has a positive *x* coordinate and satellite 2 has a negative *x* coordinate. Satellites 3,4 are on the line with equation $y = -ax$ where *a* is the largest digit of your ID number. Satellite 3 has a positive *x* coordinate and satellite 4 has a negative *x* coordinate. The pseudoranges are

$$\rho_1 = 1534, \rho_2 = 1120, \rho_3 = 1354, \rho_4 = 1210 \text{ [Km]}$$

Assuming that the user is at the point $x_0 = -\begin{bmatrix} a & b \end{bmatrix}^T$, where *a* is the largest digit and *c* is the smallest digit in your ID number answer the following questions:

1. Compute the least squares solution for the user's position

2. Assuming a measurement covariance of $R = 100I$ compute the affine minimum variance estimate of the position if the position deviation has a-priori mean and covariance given by $\delta\bar{r} = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix}^T$, $P = E\left[\left(\delta r - \delta\bar{r}\right)\left(\delta r - \delta\bar{r}\right)^T\right] = 1000I$
3. Consider only satellites 1, 2, and 3 and obtain the least squares solution for the position. Update this solution using the measurement of satellite 4 and recursive least squares

Problem 6: For the steady climb shown in the figure below, assume that there is a gyro that provides measurements of pitch rate that are accurate in short-term for fast maneuvers. Moreover, assume that there are two accelerometers mounted along the body x and z axis. Design a complementary filter and draw its block diagram to fuse all measurements and estimate the angle γ