

Fundamentals of Avionics Navigation Systems

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Preface

This book is about the fundamentals of navigation systems. A navigation system consists of a set of sensors connected to a computer or a microprocessor with the goal of determining the position and orientation of a user. The navigation algorithm running inside the computer or microprocessor processes the measurements and determines the position, orientation, and sometimes the velocity of the user. A digital cell phone is a good example of a navigation system which is broadly used while driving a vehicle. The sensors inside a cell phone include a Global Positioning System (GPS) receiver, accelerometers, gyroscopes, a microphone, touch and pressure sensors, and light sensors (including a camera). Most cell phones also have magnetometers and proximity sensors. The software program that runs the navigation algorithm and indicates the position of the vehicle in the map is called a phone software application (or simply a phone app). Some examples in this book can be validated by experiments with a cell phone.

The material presented in this book includes over 15 years of experience of the author teaching a class in avionics navigation systems and performing research in guidance, navigation and control of aerospace vehicles. The book grew out of the class notes of the author for a first course on navigation offered to senior undergraduate and first year graduate students taught at Concordia University. This course covered chapters 1 to 5 in 13 weeks, excluding spacecraft trajectories in chapter 3. A sequel course on integration of navigation systems covered the spacecraft trajectories from chapter 3 and all the material in chapters 6 to ???. The first course was offered to both senior undergraduate and graduate students, including industry professionals that were coming back to school for the start of their graduate studies. Given the specifics of this course it was difficult to find a book that would strike the necessary balance between detailed mathematical background and applications. The mathematical background had to be self contained and provided

at an undergraduate level, at the same time that more advanced concepts of navigation had to be addressed including state estimation and application examples. This book was thus designed to meet those constraints. As such, the book provides an alternative approach to most of the books available in the open literature, which target mainly students with an advanced mathematical background in signals and systems theory. The book facilitates self study because it is based on a geometric understanding of navigation systems. This strategy is clear from chapter 1 that starts the discussion with trigonometry and analytical geometry. A review of vector and tensor notation in chapter 2 then allows the treatment of both aircraft and spacecraft trajectories in chapter 3. This is followed by inertial navigation systems (INS) in chapter 4, and the global positioning system (GPS) in chapter 5. The last chapters focus on integration of navigation systems using Kalman filtering. For a graduate class where the background of the students is very strong in mathematics and physics one can perhaps teach the whole material in a single course. However, the more advanced material on Kalman filtering and integration of INS with GPS will perhaps have to be limited in scope. Alternatively, this material can be left to a second sequel course where a final project can be assigned to the students. The latter approach was the one followed by the author at Concordia University where a second course covers integration of navigation systems with a project focused on data collected by a cell phone.

The book follows a pedagogical approach starting from simpler planar geometry of a flat Earth model, and then progressively moving toward more complex spherical and ellipsoidal models of the Earth. The pre-requisites to read this book are basic knowledge of trigonometry and analytical geometry at the high school level, the fundamental concepts of probability and random variables, as well as first year university courses in calculus, linear algebra, and mechanics covering kinematics differential equations and Newton's laws. All this material will be reviewed in the first two chapters. However, the assumption is that the reader has seen these concepts before and needs only a review in a self contained manner. Probability and random vectors will only be used starting on section 5.5 and therefore section 2.9 can be skipped until one arrives at section 5.5. I hope that you enjoy reading the book as much as I enjoyed writing it!

Luis Rodrigues
Montreal, January 2022

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Chapter 1

Introduction

Imagine that you go on a boat trip and you drift away waking up one day surrounded by water with no land in view. How would you be able to determine where you are ? This same question has inspired many novels and movies, including the plot of "All is lost" in 2013, "The life of Pi" in 2012, and "Cast away" in 2000. It is the question "where am I ?" that motivates the material in this book. The answer to this question is the main purpose of a navigation system. Navigation systems are ubiquitous in our lives. Whether you are driving assisted by GPS, or piloting a boat or an aircraft, or even when simply walking or biking using a phone app as a navigation aid, you are using navigation systems. In this chapter we will briefly review the most important historical events in navigation systems up to the appearance of avionics navigation, followed by a review of trigonometry, analytical geometry, and differential Calculus pre-requisites that are necessary to understand the material in the book.

1.1 Historical Perspective

Clearly one of the most important events in the history leading to the development of avionics navigation systems was the first powered flight performed by the Wright Brothers. It took place in Kitty Hawk in North Carolina on the 17th of December of 1903. However, navigation systems had already been used for traveling at sea thousands of years before 1903. It is believed that Homo Erectus were the first navigators when they crossed the sea from Euro-Asia to the Indonesian islands 800000 years ago [1]. There is also the

possibility that Homo Sapiens crossed the sea from Indonesia to Australia approximately 60000 years ago [1]. This would have been the first time that land would be out of sight and therefore navigation could not be based on visual cues. There is still no evidence of which means of navigation could have been used in such early sea crossings and "no physical remains suggestive of navigation older than 10500 years have ever been found" [1].

The Minoan, Greek and Phoenician boats were probably the first to use rudimentary means of navigation to carry goods in the Mediterranean around 3500 B.C. Staying close to shore they could navigate by visually inspecting the coast and known landmarks. They are also believed to be the first to navigate at night using bonfires at the top of mountains as navigation aids. It appears that the first navigators to leave shore and venture into the open ocean were native from the Polynesian islands around 1000 B.C. [2]. They used their practical knowledge of currents, winds, and water patterns to navigate the open sea. It is believed that they have reached the islands of Hawaii. The Vikings were the first to sail the North Atlantic to Iceland, Greenland, and Newfoundland (Canada) at the start of the 11th century. They were already using the sun, wind and stars as navigation aids. It is believed that they also released birds at sea to follow them as the birds used their natural navigation skills to find land [3].

In the 12th century an early version of a compass was developed in China based on previous experience with magnetism in a traditional practice called feng-shu [4]. The compass became a navigation aid to determine magnetic north. This instrument could be used even in bad weather when the stars were not visible because of clouds. The compass arrived in Europe at the end of the 13th century. However, to be able to repeatedly travel to the same destination, one important tool was missing: nautical charts or maps indicating navigation routes between waypoints. The first charts to appear were the Portolan charts in the 13th century enabling the later exploration of the Atlantic ocean by competing nations. The first nation to venture into the Atlantic ocean using new navigation tools was Portugal who reached Ceuta (Morocco) in 1415. Vasco da Gama reached India and João Lavrador travelled to Labrador (Canada) in 1498. Brazil was reached by Pedro Álvares Cabral in 1500. To navigate to North America, South America, and India the Portuguese used the astrolabe, which was invented circa 1484. This instrument allowed to measure the elevation of stars. Therefore, latitude could be determined in the Northern hemisphere by measuring the elevation of the North Star Polaris (latitude is positive north and negative south). The

astrolabe has been improved with time and led to the sextant that is still used as a backup navigation instrument.

One year before the Portuguese navigator João Lavrador reached Labrador and southwestern Greenland, the Italian navigator Giovanni Caboto had already reached Canada. The exact location is still disputed between Southern Labrador, Newfoundland, and Cape Breton Island. He was working for King Henry VII of England. Five years before Caboto's travel, Cristoforo Colombo had reached what he called America. The exact place of landing is believed to be one Caribbean island that now belongs to the Bahamas. He thought he would reach Asia traveling west and had initially proposed such a travel to the Portuguese king João II who turned him down in his offer. Colombo's expedition was instead sponsored by the Spanish king Ferdinand and queen Isabella, who also sponsored Cortés in 1519 in a travel to the Yucatán Peninsula. The first circum-navigation travel around the globe in 1520 was headed by a Portuguese navigator called Fernão de Magalhães who was also working for the Spanish king and queen. The travel was finished by a Spanish navigator called Juan Sebastián Elcano after Magalhães died in the Philippines. In this travel Magalhães named the Pacific ocean and the Strait of Magalhães. In 1534 Jacques Cartier reaches Québec (Canada) working for the French king Francis I, the same king that housed Leonardo da Vinci close to the end of his life. Nine years later, in 1543, Portuguese navigators arrive in Japan. Not only Japan but also many other places that these navigators reached in the 15th and 16th centuries were already inhabitted. This prompted the start of commercial trade with goods transported by boats travelling the oceans, such as the trade of spices from India to Europe.

In the 16th century there were two very important inventions for navigation: the Mercator projection and the chip log. Gerardus Mercator was a Flemish geographer and cartographer who invented a conformal mapping technique to represent the spherical globe on a planar map while preserving local directions and shapes. This is only possible at the cost of an inflation in the size of objects as one moves from the equator to the poles. This phenomenon can be clearly seen by the inflated size of Antarctica in a world map. The chip log was a rope line knotted at regular intervals. This rope was tossed overboard a ship and the number of knots that would pass by the ship in a given time interval would be counted. This allowed the navigators to determine speed in knots (a unit still used today) and then determine position by dead-reckoning (integration of the velocity over time).

In the 17th century the first measurement of a nautical mile was made by

Richard Norwood, who was English. It was also an Englishman called John Harrison who invented the first chronometer to be accurate to one tenth of a second per day, which allowed its use in the determination of longitude [5]. Harrison won a British prize in 1764, which had been created many years before. This prize was offered to motivate the competition for the first design of a chronometer accurate enough to be used in navigation. Captain James Cook used Harrison's chronometer in his travels to circumvent the Earth, which was the second circum-navigation of the globe. When James Cook returned to England in 1779 his calculations of longitude were correct to within 8 nautical miles. In 1884 the meridian of Greenwich in England was adopted by international agreement as the Prime Meridian of zero degrees, which is still used today as a reference meridian to determine longitude. Longitude is positive in the direction east and negative in the direction west of the Greenwich meridian.

During the 19th century the first radios started to be used on ships at sea. This technology ended up replacing Harrison's chronometer in the 20th century. In 1907, only four years after the flight of the Wright brothers in Kitty Hawk, Elmer Sperry introduced the gyroscopic compass, which points to true north, instead of magnetic north (as does the magnetic compass). Sperry's invention opened the doors to the fields of avionics navigation and automatic flight control. The 20th century saw also the development of several navigation sensors linked to radio navigation, such as Very High Frequency Omni Range (VOR), Distance Measuring Equipment (DME), Radio Detection and Ranging (Radar), Long Range Navigation (Loran), and Global Navigation Satellite Systems (GNSS). Another technology developed during the 20th century was Inertial Navigation Systems (INS), namely gimbaled INS and strapdown INS systems. This book will focus on the avionics navigation systems developed during the 20th and 21st centuries.

1.2 Euclidean Geometry

In this section we review the fundamental geometric concepts for navigation in a space with planar or non-curved geometry. These spaces are often called Euclidean after the Greek philosopher Euclides. The notion of planar distance is called Euclidean distance. Euclidean spaces rely on the principles of Euclidean geometry, which is based on the notions of points, lines, line segments, angles between line segments, and circles. Euclidean geometry is

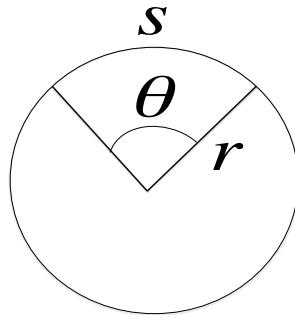


Figure 1.1: Circular arc

based on the following axioms¹:

1. A line may be drawn between any two points.
2. Any line segment may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles (ninety degrees between perpendicular lines) are equal.
5. For any point not on a given line, there is exactly one line through the point not meeting the given line.

We will denote points by a capital letter, for example A . Line segments between two points will be denoted by two capital letters indicating the points, starting with the point of origin. For example AB is the line segment between point A and point B . An angle will be denoted by three capital letters with the letter in the middle indicating the vertex of the angle. For example ABC is the angle formed by the line segments AB and BC and the vertex of the angle is at point B . Sometimes we will use the short notation B to denote the angle ABC . Angles can be measured in radians by dividing the arc length corresponding to the angular displacement by the radius of the circular arc (see figure 1.1). This measure of the angle is always well defined

¹axioms are facts assumed to be self evident that do not require proof

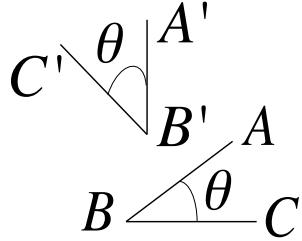


Figure 1.2: Congruent angles

and independent of the radius, provided the radius is positive. If we denote the arc length by s , the radius by r , and the angle by θ , then

$$\theta = \frac{s}{r} \quad (1.1)$$

which is unitless because it is the ratio of the arc length and the radius of the circle. Two angles are called *congruent* if they have the same measure. From the axioms of Euclidean geometry one can prove that if AB is a line segment orthogonal (perpendicular) to $A'B'$ and BC is orthogonal to $B'C'$ then the angles ABC and $A'B'C'$ are congruent (see figure 1.2).

1.3 Planar Trigonometry

As seen in the previous section, Euclidean geometry uses the notions of points, lines and angles between lines. Three line segments that meet each other at the end points form a geometric object called a triangle. It can be proved from the axioms of Euclidean geometry that the sum of all angles in a planar triangle is 180 degrees. If the angles of a triangle are denoted A , B , and C , one can therefore write

$$A + B + C = 180^\circ. \quad (1.2)$$

Trigonometry studies triangles, which are essential geometric objects in navigation systems. It relates the lengths of the sides of a triangle with the angles. A right triangle is a triangle that has a ninety degree angle, which we denote by angle C in figure 1.3. The side that is opposite to the angle C , which has length c , is called the hypotenuse. The other two sides are called

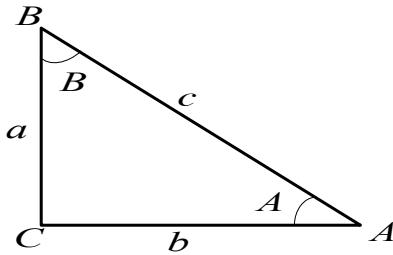


Figure 1.3: Right triangle

the legs of the triangle. For a right triangle as the one in figure 1.3 one can define the following ratios, called sine, cosine, and tangent, respectively:

$$\sin(A) = \frac{a}{c} \quad (1.3)$$

$$\cos(A) = \frac{b}{c} \quad (1.4)$$

$$\tan(A) = \frac{a}{b} \quad (1.5)$$

Additionally $\sin(B) = \cos(A)$, $\cos(B) = \sin(A)$, $\tan(B) = \cot(A)$, where

$$\cot(A) = \frac{1}{\tan(A)} \quad (1.6)$$

is called the cotangent of the angle A . The angles A and B are called complementary angles because their sum is 90° . The fact that the angle A has the same symbol as the vertex point A is a slight abuse of notation that will be used throughout the book to avoid defining too many variables. If a triangle is not a right triangle then it is called an oblique triangle (see figure 1.4). For oblique triangles one can write the law of sines as

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}. \quad (1.7)$$

From this law one can conclude that when two angles have the same sine then the length of the sides that are opposite to these angles is the same.

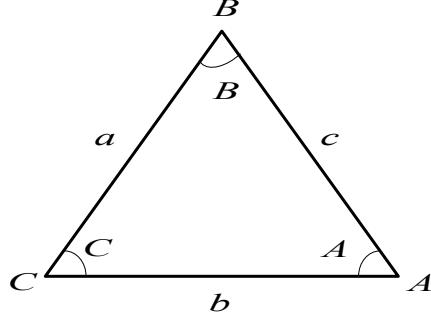


Figure 1.4: Oblique triangle

The law of cosines can be written as

$$c^2 = a^2 + b^2 - 2ab \cos(C). \quad (1.8)$$

When the angle C is a right angle then the oblique triangle becomes a right triangle and the law of cosines yields the theorem of Pythagoras

$$c^2 = a^2 + b^2. \quad (1.9)$$

Example 1.3.1 An aircraft is at point B at the same distance of two airports A and C . The distance between the airports is 100 km and the angle B is equal to 150° . What is the altitude of the aircraft ?

Solution: If the aircraft is at the same distance of airports A and C then $a = c$ in figure 1.5. From the law of sines (1.7) we can then conclude that the angles A and C are equal. We now connect the point B to the line segment between points A and C in a perpendicular line. This splits the oblique triangle from figure 1.4 into two similar right triangles as seen in figure 1.5. Using the equation (1.2) we can conclude that $A = C = 90^\circ - 0.5B = 15^\circ$. From the definition of tangent in equation (1.5) the altitude is

$$h = \frac{b}{2} \tan(A) = 50 \tan\left(\frac{15}{180}\pi\right) = 13.4 \text{ km.}$$

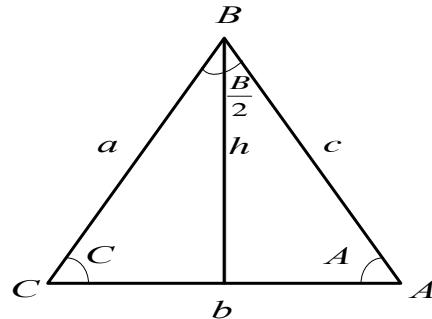


Figure 1.5: Oblique triangle split in two right triangles

1.4 Spherical Trigonometry

Planar trigonometry is only valid for short range local navigation because the Earth has curvature. For long range navigation on a spherical Earth one must consider spherical trigonometry. In spherical trigonometry arcs of great circles take the role of line segments in planar trigonometry. A great circle is a circle at the surface of a sphere whose radius is equal to the radius of the sphere. It is formed by the intersection of the surface of the sphere with a plane passing through the center of the sphere. If we consider Earth as a sphere then examples of great circles are the equator and all circles passing through both the North Pole and the South Pole, which are called meridians.

A spherical triangle is a geometric figure of three sides obtained by joining three arcs of great circles (see figure 1.6). If the sphere has unit radius then the arc lengths of great circles can be measured in radians or degrees because from equation (1.1) they have the same magnitude as the central angle. To determine all trigonometric relations in a spherical triangle one can consider that the sphere has unit radius, without loss of generality. This is possible because the transformation from the triangle on the unit sphere to any sphere with radius greater than one leaves the angles invariant. Transformations that locally preserve angles are called conformal mappings. In this section we will therefore assume a unit radius sphere whenever determining angles. The arc lengths for spherical triangles on the surface of the Earth will then be found as a function of the previously determined angles by equation (1.1), where the radius of the Earth is used as the parameter r .

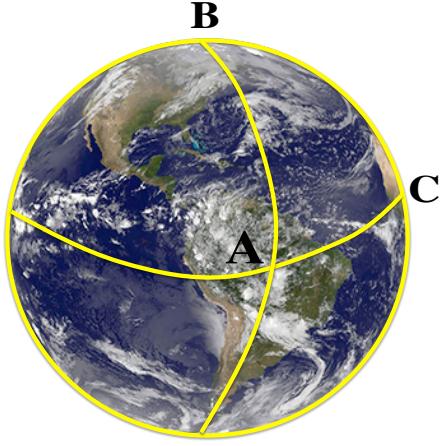


Figure 1.6: Spherical triangle on Earth (globe taken from en.wikipedia.org)

Figure 1.7 shows in more detail the spherical triangle BCA depicted in figure 1.6, including the center of the great circle arcs, which is denoted by point O . We observe that the angle at the vertex C is a right angle. The angle BOC is in the vertical plane whereas the angle AOC is in the horizontal plane. The triangles ODE , DEF , OEF , ODF , and OHG are all right triangles. To obtain trigonometric equations we will relate the spherical triangle AOC with the planar triangles DEF and OFD . Since both line segments DF and FE are orthogonal to OA then the angle DFE is the same as the angle of the spherical triangle at the vertex A . Therefore, assuming a unit sphere,

$$\sin(a) = \sin(BOC) = \frac{DE}{OD} = \frac{DE}{DF} \frac{DF}{OD} = \sin(A) \sin(c), \quad (1.10)$$

$$\cos(c) = \cos(DOF) = \frac{OF}{OD} = \frac{OF}{OE} \frac{OE}{OD} = \cos(b) \cos(a). \quad (1.11)$$

The same procedure can be used to determine all trigonometric relations known as Napier's rules. These rules are divided into two groups. The

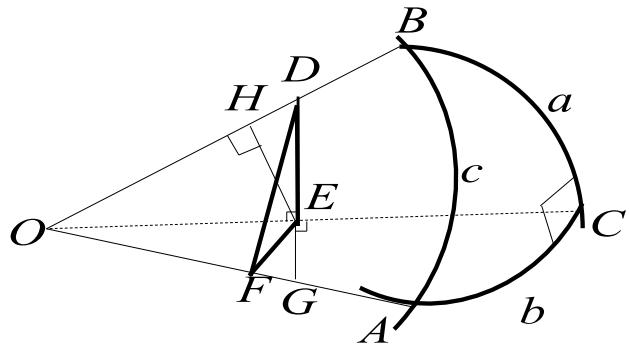


Figure 1.7: Spherical triangle BCA (adapted from [6])

Napier's rules in group 1 are

$$\sin(b) = \tan(a) \cot(A), \quad (1.12)$$

$$\cos(c) = \cot(A) \cot(B), \quad (1.13)$$

$$\sin(a) = \tan(b) \cot(B), \quad (1.14)$$

$$\cos(A) = \tan(b) \cot(c), \quad (1.15)$$

$$\cos(B) = \tan(a) \cot(c), \quad (1.16)$$

and the Napier's rules in group 2 are

$$\sin(a) = \sin(A) \sin(c), \quad (1.17)$$

$$\cos(A) = \sin(B) \cos(a), \quad (1.18)$$

$$\cos(B) = \sin(A) \cos(b), \quad (1.19)$$

$$\sin(b) = \sin(B) \sin(c), \quad (1.20)$$

$$\cos(c) = \cos(a) \cos(b). \quad (1.21)$$

All of these relations can be determined by the Napier's pie depicted in figure 1.8 using the following two rules:

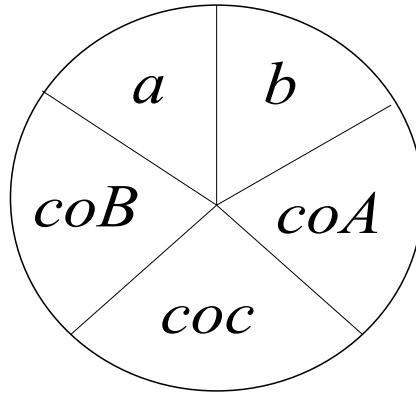


Figure 1.8: Napier's pie

1. The sine of any sector in the pie is the product of the tangents of the two adjacent sectors.
2. The sine of any sector in the pie is the product of the cosines of the two opposite sectors.

The notation co appearing before an angle means that the sine is changed to the cosine (and vice-versa) and the tangent is changed to the cotangent (and vice-versa). For example, the equation (1.20) is obtained by starting at the sector of the pie with the angle b and using rule 2. Notice that the notation co appears before the angles B and c of the two sectors of the pie opposite to the sector with the angle b . Therefore, the product of cosines becomes the product of sines in rule 2 when applied to the sector with the angle b .

The Napier's rules are only valid for right triangles. It can be shown (see [6] for a proof) that for oblique spherical triangles the law of sines is

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)}, \quad (1.22)$$

and the law of cosines is

$$\cos(c) = \cos(b)\cos(a) + \sin(b)\sin(a)\cos(C). \quad (1.23)$$

Note that if the angles a, b, c are small then the spherical triangle can be approximated by a planar triangle. Under this assumption, if we use the

Taylor series approximations

$$\begin{aligned}\sin(a) &\approx a \\ \cos(c) &\approx 1 - 0.5c^2\end{aligned}\quad (1.24)$$

the law of sines (1.22) and law of cosines (1.23) become equations (1.7) and (1.8), respectively. The Napier's rule (1.21) becomes the theorem of Pythagoras using the same approximation. All other Napier's rules can also be simplified to equivalent relations valid for planar trigonometry using the approximations (1.24). There is however one relation that exists for planar triangles that does not have a counterpart for a spherical triangle, which is equation (1.2). This can be seen by forming a triangle that has the North Pole as one vertex and two points in the equator as the two other vertices. The sum of the two angles at the equator vertices is 180° . Therefore the sum of all angles has to be higher than 180° . This is always true for spherical triangles.

Example 1.4.1 [6], *An aircraft leaves Halifax starting due east and continuing on the great circle. Find its position and heading after it has flown 1000 nautical miles. Note that the latitude of Halifax is $44.67^\circ N$ and the longitude is $63.58^\circ W$.*

Solution: One can form a spherical triangle by three arcs of great circles. The first arc is already given by the traveled distance between Halifax and the destination. The other two arcs can be found by connecting the departure city Halifax (point C) and the destination point B to the North Pole because these two arcs are along meridians, which are great circles. The situation is depicted in figure 1.9. The angle b is the complementary angle of latitude and therefore

$$b = 90^\circ - 44.67^\circ = 45.33^\circ. \quad (1.25)$$

The distance traveled is 1000 nautical miles. One nautical mile on the surface of the Earth is approximately one minute of arc. There are 60 minutes of arc in each degree. Therefore, the angle a in degrees is

$$a = \frac{1000}{60} = 16.67^\circ. \quad (1.26)$$

Using the theorem of Pythagoras for spherical triangles in equation (1.21) yields

$$c = \arccos(\cos(45.33^\circ) \cos(16.67^\circ)) = 47.66^\circ. \quad (1.27)$$

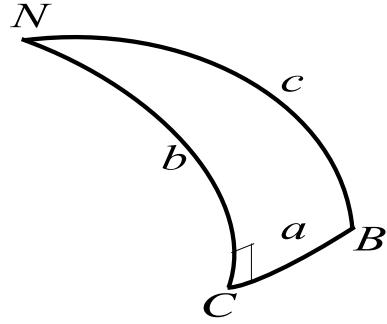


Figure 1.9: Navigation on Earth for example 1.4.1 (adapted from [6])

Therefore, the latitude ϕ_B at the destination B is the complementary angle of c , which is

$$\phi = 90^\circ - 47.66^\circ = 42.34^\circ. \quad (1.28)$$

To determine the longitude, one must first compute the angle at the North Pole vertex N . We can compute this angle using Napier's rule in equation (1.15) with A replaced by N as

$$N = \arccos(\tan(45.33^\circ) \cot(47.66^\circ)) = 22.81^\circ. \quad (1.29)$$

Therefore, the longitude of B is

$$\lambda_B = 63.58^\circ W - 22.81^\circ W = 40.77^\circ W. \quad (1.30)$$

Finally, to get the heading we use Napier's rule (1.20) with A replaced by N to yield

$$B = \arccos(\sin(22.81^\circ) \cos(45.33^\circ)) = 74.18^\circ. \quad (1.31)$$

Example 1.4.2 Assume that you are flying along a great circle from Montreal with latitude and longitude given by $(45.5^\circ N, 73.57^\circ W)$ to Lisboa with coordinates $(38.72^\circ N, 9.14^\circ W)$. If you are flying at a constant speed of 450

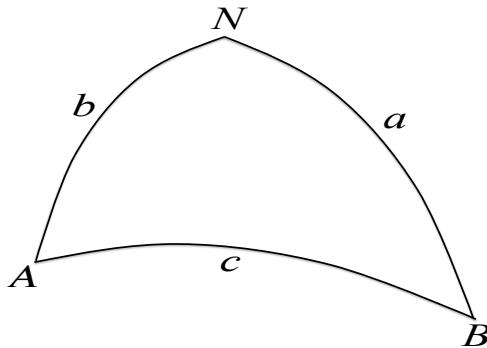


Figure 1.10: Navigation on Earth for example 1.4.2

knots and at a constant altitude of 30,000 feet how long will the flight be ?

Solution: To determine the time of flight we need to find first the distance. To do this we draw a spherical triangle as in figure 1.10, where A is Montreal, B is Lisboa, and N is the North Pole. As in example 1.4.1 we can compute the angles a , b , and N as

$$\begin{aligned} a &= 90.00^\circ - 38.72^\circ = 51.28^\circ, \\ b &= 90.00^\circ - 45.50^\circ = 44.50^\circ, \\ N &= 73.57^\circ - 9.14^\circ = 64.43^\circ. \end{aligned}$$

Using the cosine rule (1.23) yields

$$c = \arccos(\cos(44.5^\circ) \cos(51.28^\circ) + \sin(44.5^\circ) \sin(51.28^\circ) \cos(64.43^\circ)).$$

This gives $c = 47^\circ = 0.8201 \text{ rad}$. The radius of the Earth is $R = 6378137 \text{ m}$. Since each nautical mile corresponds to 1852 m (approximately one minute of arc) we have $R = 3444 \text{ nm}$, where nmi stands for nautical miles. The flight altitude is $h = 30000 \text{ ft} = 4.9 \text{ nmi}$. The distance is computed using equation (1.1) with a radius of $R + h = 3448.9 \text{ nautical miles}$, which yields

$$s = (R + h)c = (3448.9)(0.8201) = 2828 \text{ nmi}.$$

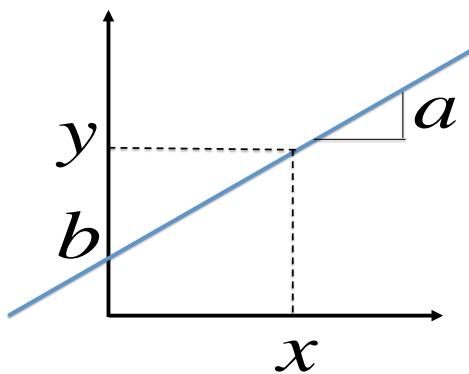


Figure 1.11: Line

Finally the time is computed as

$$\Delta t = \frac{s}{v} = \frac{2828}{450} = 6.28 \text{ hours.}$$

1.5 Analytical Geometry

Analytical geometry is the description of objects in an Euclidean space using mathematical equations. In this section we will focus on analytical geometry and will review the mathematical equations for lines and circles. To be able to describe geometric objects based on equations we will need a set of coordinate axes, called Cartesian coordinate axes after the French mathematician René Descartes. This section will focus on three geometric objects: lines, circles, and ellipses.

1.5.1 Lines

A set of Cartesian axes and a line are drawn in figure 1.11. Each point belonging to the line has an abscissa coordinate x along the horizontal axis and an ordinate coordinate y along the vertical axis. We can thus say that each point has Cartesian coordinates (x, y) . The origin of the Cartesian axes

lies at the intersection of the vertical and horizontal axes and has coordinates $(0, 0)$. The equation of the line is

$$y = ax + b. \quad (1.32)$$

The fact that the value of y is a function of the value of x is denoted in mathematics by the equation $y = f(x)$, a notation that is due to Leonard Euler. The function $f(x) = ax + b$ is called an affine function when $b \neq 0$. It is called a linear function when $b = 0$. The parameter b is called the ordinate of the line at the origin. The slope of the line is defined to be the tangent of the angle that the line makes with the horizontal axis. Therefore, based on the equation for the tangent (1.5), the slope can be obtained by the ratio

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (1.33)$$

for any pairs of points on the line with coordinates $(x_1, y_1) = (x_1, f(x_1))$ and $(x_2, y_2) = (x_2, f(x_2))$, where $x_2 > x_1$.

Example 1.5.1 What is the equation of the line that passes through the points with coordinates $(1, 1)$ and $(-1, -1)$?

Solution: Replacing the coordinates of both points in equation (1.32) yields the system of equations

$$\begin{cases} 1 = a(1) + b \\ -1 = a(-1) + b \end{cases} \Leftrightarrow \begin{cases} a = 1 - b \\ -1 = (1 - b)(-1) + b \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = 0 \end{cases}$$

The equation of the line (1.32) then becomes $y = x$.

1.5.2 Circles

A circle is defined to be the set of points that are at the same distance of a central point, called the center of the circle. A circle with center at the point with Cartesian coordinates (x_0, y_0) and radius r is shown in figure 1.12. The equation of this circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2. \quad (1.34)$$

Notice that in this case one cannot solve for the coordinate y as a single function of the coordinate x because of the squared terms. In fact, for each

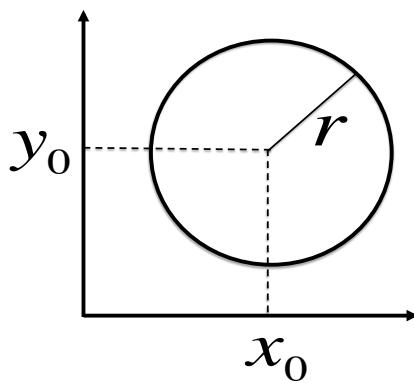


Figure 1.12: Circle

value of x there are two possible values of y . The way around this difficulty is to define two functions of x : one that describes the upper arc of the circle and the other that describes the lower arc of the circle, both arcs corresponding to a 180 degree rotation. Doing this yields

$$y_{up} = y_0 + \sqrt{r^2 - (x - x_0)^2} \quad (1.35)$$

$$y_{low} = y_0 - \sqrt{r^2 - (x - x_0)^2} \quad (1.36)$$

Example 1.5.2 *What is the equation of the circle with center at the point with coordinates $(1, 0)$ and radius 2?*

Solution: Replacing $x_0 = 1$, $y_0 = 0$, and $r = 2$ in equation (1.34) yields

$$(x - 1)^2 + y^2 = 4.$$

1.5.3 Ellipses

Returning to the equation of the circle, we note that it can be rewritten in a different form. If one divides both sides of equation (1.34) by r^2 it yields

$$\frac{(x - x_0)^2}{r^2} + \frac{(y - y_0)^2}{r^2} = 1. \quad (1.37)$$

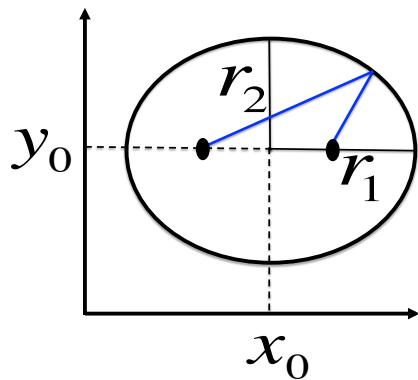


Figure 1.13: Ellipse

An ellipse is the set of points whose sum of the distances to two foci is constant, as indicated in figure 1.13 by the two blue lines. The equation of the ellipse is a generalization of equation (1.37), which is written as

$$\frac{(x - x_0)^2}{r_1^2} + \frac{(y - y_0)^2}{r_2^2} = 1. \quad (1.38)$$

This equation represents an ellipse with center at the point with coordinates (x_0, y_0) with major and minor axes along the x and y coordinates. Note that if $r_1 = r_2$ then equation (1.38) is the same as equation (1.37). If $r_1 > r_2$ then the length of the semi-major axis is r_1 , which corresponds to the distance to the origin of the points for which $y = y_0$. The length of the semi-minor axis is r_2 , which corresponds to the distance to the origin of the points for which $x = x_0$. This is the case depicted in figure 1.13.

1.5.4 Approximation of Curves by Linearization

For navigation on Earth one considers very frequently circular trajectories that are locally approximated by the tangent line to the circle, as shown in figure 1.14.. The tangent line is obtained by a limiting process. One starts with a secant line passing through the points P and Q_1 and then progressively moves the second point in the line closer to the point P . Figure 1.14 shows

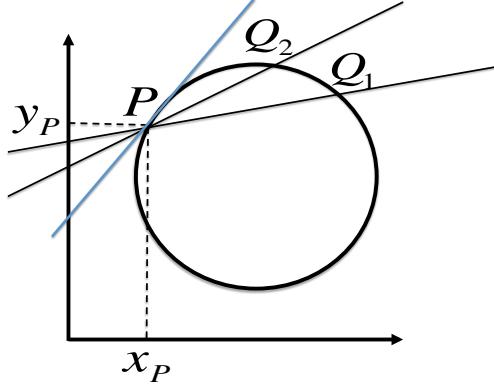


Figure 1.14: Tangent and secant lines to a circle

two secant lines corresponding to the points Q_1 and Q_2 . If one continues moving the second point of the secant line closer and closer to the point P then in the limit one gets the tangent line which is colored blue in figure 1.14. Notice that in the case shown in figure 1.14 the point Q_1 is on the upper 180 degree arc of the circle so the coordinates of this point are related by equation (1.35) as

$$f(x) = y_0 + \sqrt{r^2 - (x - x_0)^2}. \quad (1.39)$$

The slope of the secant line that intersects the circle at the point P with coordinates (x_P, y_P) and at any point Q with coordinates (x, y) is obtained from equation (1.33) as

$$a_Q = \frac{f(x) - f(x_P)}{x - x_P}. \quad (1.40)$$

Therefore, in the limit situation, when x converges to x_P (written $x \rightarrow x_P$) we can write the slope a of the tangent line as

$$a = \lim_{x \rightarrow x_P} \frac{f(x) - f(x_P)}{x - x_P} = f'(x_P) \quad (1.41)$$

When this limit exists it is called the derivative of the function $f(x)$ at the point $x = x_P$ and it is written as $f'(x_P)$ or as $df(x_P)/dx$. If f is a function

of several variables then the notation for the (partial) derivative with respect to each variable x_i is $\partial_{x_i} f$.

Changing variables from x to $\tilde{x} = x - x_P$, the vertical axis in figure 1.14 will move toward the left so that it passes through the point P . The reason for this change is that for $x = x_P$ we have $\tilde{x} = 0$. Therefore, in the new variables (\tilde{x}, y) the equation of the tangent line to the circle at the point x_P is the same as (1.32) with a given by (1.41) and b equal to $f(x_P)$, i.e.,

$$y = f(x_P) + f'(x_P)\tilde{x} = f(x_P) + f'(x_P)(x - x_P). \quad (1.42)$$

Computing the equation of the tangent line to a curve is called linearization. Note that when the tangent line is flat then the slope is zero, and therefore the derivative (1.41) is also zero. When the derivative of a function $f(x)$ is zero it means that the function is locally constant, and vice-versa. This happens at a stationary point of the function (maximum, minimum or saddle).

The following rules can be proved using the definition of derivative and will be used throughout the text,

$$(f + g)'(x) = f'(x) + g'(x), \quad (1.43)$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x), \quad (1.44)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \quad g(x) \neq 0, \quad (1.45)$$

$$(f^n)'(x) = n f^{n-1}(x) f'(x), \quad (1.46)$$

where $f(x)$ and $g(x)$ are any two differentiable functions. The chain rule is used to compute derivatives of functions that depend on a variable x that itself depends on another variable t and is written as

$$\frac{df(x(t))}{dt} = \frac{df(x)}{dx} \cdot \frac{dx}{dt}. \quad (1.47)$$

Some derivatives that we will need in this book are

$$\frac{d(at + b)}{dt} = a, \quad (1.48)$$

$$\frac{d(\sin(t))}{dt} = \cos(t), \quad (1.49)$$

$$\frac{d(\cos(t))}{dt} = -\sin(t), \quad (1.50)$$

$$\frac{d\left(\sqrt{r^2 \pm (t - t_0)^2}\right)}{dt} = \pm \frac{t - t_0}{\sqrt{r^2 \pm (t - t_0)^2}}. \quad (1.51)$$

Example 1.5.3 Write the equation of the line that is tangent to the circle of example 1.5.2 at the point $(x, y) = (0, 2)$. Find the set of all points of intersection of the line and the circle and prove that there is only one point.

Solution: The point $(0, 2)$ is located in the top part of the circle. Therefore, the variable y is given as a function $f(x)$ according to equation (1.39). Replacing t by x and setting $t_0 = 1, r = 2$ in (1.51) yields the derivative of $f(x)$ as

$$f'(x) = \frac{df}{dx} = -\frac{x-1}{\sqrt{4-(x-1)^2}} \quad (1.52)$$

At $x_P = 0$ the derivative is $f'(x_P) = f'(0) = \frac{1}{\sqrt{3}}$. Replacing the values $x_P = 0, f(x_P) = f(0) = \sqrt{3}$ and $f'(x_P) = \frac{1}{\sqrt{3}}$ in equation (1.42) yields

$$y = \sqrt{3} + \frac{1}{\sqrt{3}}x = \frac{1}{\sqrt{3}}(3+x).$$

To find the intersection of the line and the circle we solve the equations

$$\begin{cases} y &= \sqrt{4-(x-1)^2} \\ y &= \frac{1}{\sqrt{3}}(3+x) \end{cases} \Rightarrow 4-(x-1)^2 = \frac{1}{3}(3+x)^2 \Leftrightarrow 4x(1-x) = 0.$$

The solutions of $4x(1-x) = 0$ are $x = 0, x = 1$. Replacing $x = 0$ and $x = 1$ in the original set of equations we realize that only $x = 1$ is a solution of the original system because we squared the original equations. This means that the line and the circle intersect only at the point $(0, 2)$.

1.6 Navigation by Triangulation

Planar trigonometry and analytical geometry are used in terrestrial radio navigation by triangulation. For local area navigation one can approximate the curved Earth by a flat tangent plane. If the aircraft is over land there are several radio stations distributed on the ground. The location of the radio stations in the local horizontal plane is well known. Radial lines emanate from each station in all directions of this plane. The angle of a radial line is measured from north and is positive clockwise (as seen from above). If an aircraft is on a fixed radial corresponding to a given angle θ then its Line of Position (LOP) is a line. The angle θ of a radial line is measured using Very High Frequency Omnidirectional Range (VOR) equipment. A VOR

radio station broadcasts two signals: one omnidirectional, and another that rotates clockwise (as seen from above) at an angular frequency of 30 Hz. The two signals are in phase when the rotating signal passes through north. The phase difference between the two received signals yields the angle of the radial line in which the aircraft is located.

The distance of the aircraft to the station can be measured by Distance Measuring Equipment (DME). This system works by interrogation of the station when the flying aircraft emits a pulse pair. After a known delay, the station replies by transmitting another pulse pair at a frequency that is separated from the interrogation frequency by 63 MHz. The aircraft can determine the slant distance to the station by measuring the total time elapsed between sending the interrogating signal and receiving the station signal back, given that the propagation speed is known (speed of light). If an aircraft is at a constant distance from a radio station then its LOP is a circle. The measured distance or range is called ρ . By intersecting LOPs one can determine the position of the aircraft, which is called a position fix. This technique of finding a position fix is called $\rho - \theta$ navigation if the measurements come from a DME and a VOR. If one uses two VOR stations then it is called $\theta - \theta$ navigation. It is called $\rho - \rho - \rho$ navigation in the case of three DME stations.

Example 1.6.1 An aircraft receives measurements from two ground stations at the same time. Station 1 is located at $(0, 0)$ km and the VOR measurement indicates that the aircraft is in the 60 degree radial. Station 2 is located at $(100, 0)$ km and the collocated VOR/DME reading puts the aircraft in the 335 degree radial and at a distance of 50 km. Determine the distance of the aircraft from VOR station 1 and the position coordinates of the aircraft.

Solution: We denote the distance between the radio stations 1 and 2 as a , the distance of the aircraft to VOR station 1 as c , and the distance of the aircraft to station 2 as b . The angle of 335° counted from north, positive clockwise, yields an angle of 65° at the triangle vertex corresponding to station 2. Figure 1.15 shows the intersection of the two LOPs on the ground to determine a position fix. Since the aircraft is at the 60° radial of station 1, then the angle at the vertex corresponding to station 1 is equal to 30° . Using the law of cosines (1.8) yields

$$c = \sqrt{100^2 + 50^2 - 2(100)(50) \cos(65^\circ)} = 90.96 \text{ km.}$$

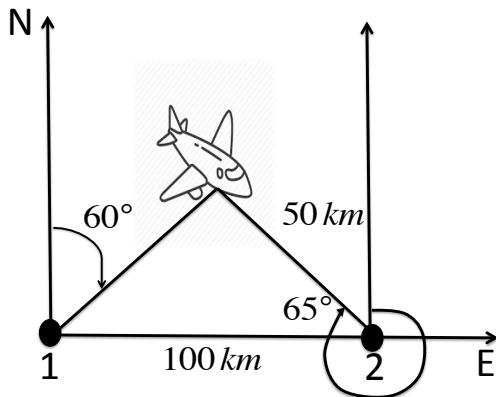


Figure 1.15: Local area navigation in example 1.6.1

The east-north position coordinates (x_E, x_N) are

$$\begin{aligned} x_E &= 90.96 \cos(30^\circ) = 78.77 \text{ km}, \\ x_N &= 90.96 \sin(30^\circ) = 45.48 \text{ km}. \end{aligned}$$

1.7 Notes and References

Section 1.1 is based on references [7] and [8]. Section 1.4 is a brief summary of the material presented in reference [6] that is relevant to navigation. One of the examples introduced in the same reference was used in section 1.4. A broad overview of navigation systems can be found in [9].

1.8 Problems

Problem 1.8.1 Draw a spherical Earth and the North Star Polaris in the sky above the North Pole. Using your drawing and trigonometry show that if you measure the elevation of Polaris you can determine your latitude, assuming that the north star is infinitely far above the North Pole.

Problem 1.8.2 At time t_1 an airborne VOR receiver shows that the aircraft is on the 30° radial from a VOR station. The pilot maintains a ground track heading of 120° with respect to magnetic north and 120 knots ground speed. At time t_2 , 10 minutes later than t_1 , the VOR receiver indicates that the aircraft is on the 90° radial from the same station. Assuming no errors, what is the pilots distance from the VOR station at time t_2 ? If the pilot had access to airspeed, can the range be determined from the VOR measurements? Can it be determined if the pilot had access to ground speed and ground track heading? Justify.

Problem 1.8.3 Two collocated VOR/DME stations A and B are at the same latitude and provide measurements to an aircraft at time t_1 . Station B lies 100 km west of station A. The VOR measurement from station A puts the aircraft on a 330° radial. The DME measurements indicate that the aircraft is at the same distance from both stations. What is the distance of the aircraft to stations A and B at time t_1 ?

Problem 1.8.4 How long will it take to fly from Calgary, Canada (Latitude: 51° north; Longitude: 114° west) to Karachi, Pakistan (Latitude: 25° north; Longitude: 66° east) in the shortest possible time assuming a constant altitude of 10 km and a constant speed of 800 km/h)?

Problem 1.8.5 [6] The mouth of the Amazon River and the city of Quito, Ecuador, are situated on the equator with the mouth of the Amazon river being $28^\circ 30'$ east of Quito. The port of Charleston, South Carolina, is directly north of Quito by approximately $32^\circ 48'$. Find to the nearest 10 nautical miles the distance of the port of Charleston to the mouth of the Amazon river.

Problem 1.8.6 An aircraft leaves Honolulu, Hawaii, US (Latitude: 21.31° north; Longitude: 157.87° west) traveling toward Tokyo, Japan (Latitude: 35.68° north; Longitude: 139.77° east) on a great circle route with a heading of 60.5° west of north. What will be the distance covered in this travel?

Chapter 2

Vector Calculus and Matrices

This chapter reviews vector calculus and vector notation that will be used in subsequent chapters and applies it to Newton's laws.

2.1 Vectors, Frames, and Vector Coordinates

Geometrically speaking a vector is an arrow in space. In other words, a vector is a geometrical object that has magnitude and direction (see figure 2.1). A unit vector is a vector with unit length. Any vector \vec{v} is written as

$$\vec{v} = v\vec{e}_v, \quad (2.1)$$

where $v = \|\vec{v}\|$ is the magnitude (or length) of \vec{v} and \vec{e}_v is a unit vector indicating the direction of \vec{v} .

Examples of vectors in navigation applications are the position, velocity, and acceleration of a vehicle. The position vector has a starting point and an end point and is represented by the vector \vec{s} . The starting point O is always a reference point relative to which the position is determined. The end point P is the point for which we want to determine the position (see figure 2.2). In this book we will denote a position vector with two capital letters as subscripts. The first letter indicates the point for which the position is measured and the second indicates the reference point for the measurement. The velocity vector indicates the direction of motion and its magnitude is equal to the speed. Note that the velocity of the same point can be a different vector depending on the observer relative to whom the velocity is measured. For example, if two people are seated inside a train and the person O measures

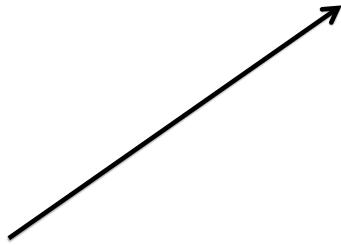


Figure 2.1: Vector

the velocity of the person P , then the result is the zero vector. However, if the person O is sitting at the station and measures the velocity of the person P who is seated in the train that passes by the station, then the velocity vector will not be zero. Therefore, the notation used in this book to indicate the velocity vector will have a capital letter as superscript to denote the observer relative to which the velocity is taken. The same notation is used for the acceleration vector. For the angular velocity vector of a body rotating relative to another body we will use two capital letters as superscripts: the first to denote the rotating body and the second to denote the reference body.

In summary, the notation used in this book is as follows:

- The letter s is used for position vectors. They have two subscripts: the first indicates the point for which the position is measured, and the second indicates a reference point relative to which the position is measured. The position of point P relative to point O is denoted \vec{s}_{PO} .
- The letter v is used for velocity vectors. They have one subscript that indicates the point for which the velocity is measured. They also have a superscript denoting the observer relative to whom the velocity is measured. The velocity of point P measured by observer O is denoted by \vec{v}_P^O .
- The letter a is used for acceleration and the same subscript-superscript

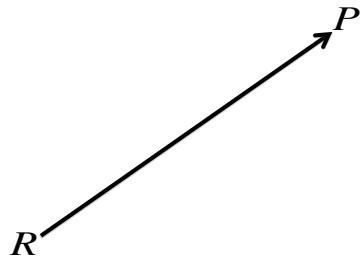


Figure 2.2: Position vector

notation of velocity vectors is used for the acceleration vector. The acceleration of point P measured by observer O is denoted by \vec{a}_P^O .

- The letter ω is used for angular velocity with two superscripts: the first indicates the body that is rotating and the second indicates the body relative to which the angular velocity is measured. When the angular velocity is measured relative to inertial space the second superscript can be dropped for ease of notation. The angular velocity of body B relative to body I is denoted by $\vec{\omega}^{BI}$.

There are relationships between some of these vectors. The velocity is the time rate of change of position, and the acceleration is the time rate of change of velocity. A time rate of change is computed by taking the derivative with respect to time. We saw in section 1.5 that the notation for derivative of a function $f(x)$ is $f'(x)$ or $df(x)/dx$. The latter is Leibnitz's notation for the derivative. However, if the independent variable represents time we will use the letter t instead of x . We will also use Newton's notation for time derivative, which places a dot on top of the vector. Therefore, $\vec{v}_P^O = \dot{\vec{s}}_{PO}$ and $\vec{a}_P^O = \ddot{\vec{v}}_P^O$. Later in section 2.7 the process of computing time derivatives of vectors will be explained in detail.

Notice that so far we have not talked about coordinates of a vector to emphasize that a vector is a geometric object and does not require coordinates to be described. To compute the coordinates of a vector one has to define first a reference frame consisting of a set of orthogonal axes. A reference frame is a mathematical description of a physical observer. Note that the same vector

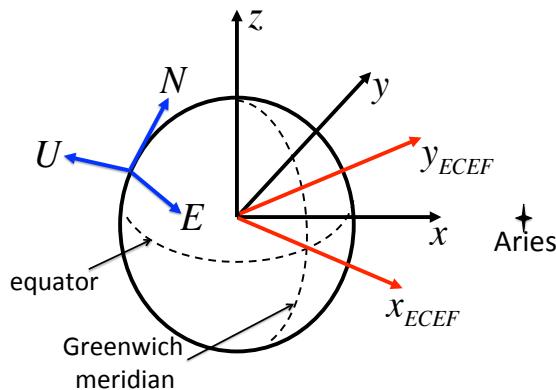


Figure 2.3: Earth reference frames

will have different coordinates in different reference frames. Therefore, the notation for coordinates must include a letter to denote the reference frame.

To write the navigation equations it is essential to have at least an inertial reference frame but there could be two or three different frames used in navigation applications. An inertial reference frame is a frame that is not accelerating relative to the distant stars. In other words, it is either fixed or in rectilinear motion at constant velocity. The Earth Centered Inertial (ECI) frame is an inertially fixed frame that has its origin placed at the center of mass of the Earth. The z axis lies along the mean polar axis of the Earth. The x and y axes lie in the equatorial plane of the Earth. The x axis points in the direction of the First Point of Aries and the y axis completes the right-handed triad (see dark arrows in figure 2.3). The Earth Centered Earth Fixed (ECEF) frame also has its origin placed at the center of mass of the Earth but it rotates with the Earth. The z axis passes through the North Pole and lies along the Earth's spin axis. The x axis connects the center of the Earth to the intersection of the equator and the Greenwich meridian and rotates with the Earth. The y axis completes the triad (see red arrows in figure 2.3). It is also useful to have a navigation frame defined at any point on the local tangent plane, which is called the East-North-Up (ENU) frame. The x axis points east, the y axis points north, and the z axis points up (see blue arrows in figure 2.3).

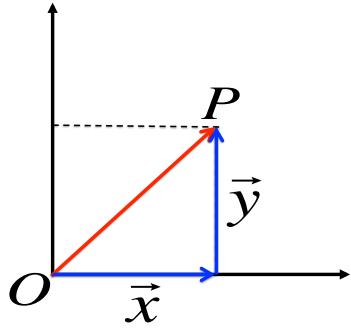


Figure 2.4: Absolute position vector and parallelogram law of addition

The coordinates of a vector \vec{v} in a given frame E will be denoted in this book by a column array and written as ${}^E[\vec{v}]$. When the frame of reference is clear because only one frame is being used then one may write $[\vec{v}]$ to keep the notation simpler. If the vector is the position of a point P relative to a point O then its coordinates in the frame where point O is fixed are denoted by ${}^O[\vec{s}_{PO}]$. As an example, any point P can lead to a position vector simply by using as a reference point the origin of the Cartesian coordinate system. The position of P relative to O is represented by the red vector in figure 2.4.

Going back to the representation of a vector according to equation (2.1) one can define a horizontal vector with magnitude x and horizontal direction \vec{e}_x , which can be written as $\vec{x} = x\vec{e}_x$. Similarly, one can define a vector with magnitude y and vertical direction \vec{e}_y leading to $\vec{y} = y\vec{e}_y$. The parallelogram rule for vector addition says that when the starting point of vector \vec{y} is placed at the tip of vector \vec{x} then their sum is a vector from the starting point of \vec{x} to the tip of \vec{y} . Using this rule, the vector \vec{s}_{PO} , which can be written as $\vec{s}_{PO} = \vec{x} + \vec{y} = x\vec{e}_x + y\vec{e}_y$ is represented by the red vector in figure 2.4. The coordinates of the position vector can be written as the column array

$${}^E[\vec{s}_{PO}] = x{}^E[\vec{e}_x] + y{}^E[\vec{e}_y] = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}. \quad (2.2)$$

We will use the superscript T to denote the transpose, which transforms a

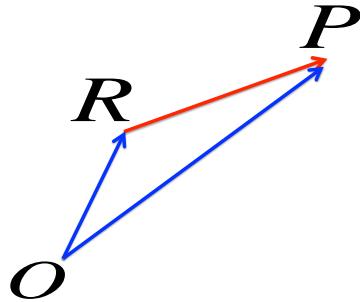


Figure 2.5: Relative position vector

column array into a row array. For example, the transpose of (2.2) is

$${}^E[\vec{s}_{PO}]^T = [x \quad y]. \quad (2.3)$$

The letter O will always be used in this book to denote the origin of a reference frame. When the reference point is not the origin then we use the letter R to denote the reference point. The relative position vector can then be written as $\vec{s}_{PR} = \vec{s}_{PO} - \vec{s}_{RO}$ (see figure 2.5). The coordinates of the relative position vector in any coordinate system E can be obtained from the coordinates of the absolute position vectors as

$${}^E[\vec{s}_{PR}] = {}^E[\vec{s}_{PO}] - {}^E[\vec{s}_{RO}]. \quad (2.4)$$

Example 2.1.1 Let $\vec{s}_{PO} = \vec{e}_x$, $\vec{s}_{RO} = \vec{e}_y$. Then ${}^E[\vec{s}_{PO}] = [1 \quad 0]^T$, ${}^E[\vec{s}_{RO}] = [0 \quad 1]^T$ and

$${}^E[\vec{s}_{PR}] = \begin{bmatrix} 1 - 0 \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Vector coordinates can also be stacked as columns in larger arrays called matrices. For example, a matrix with two columns corresponding to the coordinates of the vectors ${}^E[\vec{e}_x]$ and ${}^E[\vec{e}_y]$, respectively, is written as

$$\left[\begin{array}{cc} {}^E[\vec{e}_x] & {}^E[\vec{e}_y] \end{array} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is called the identity matrix. See section 2.3.3 for more details on matrices.

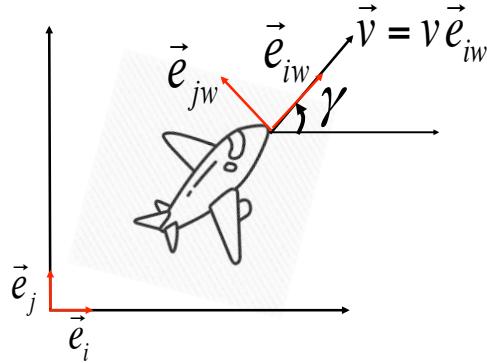


Figure 2.6: Change of coordinates (aircraft taken from en.wikipedia.org)

2.2 Change of Coordinates

As mentioned before, the same vector will have different coordinates in different reference frames. Therefore, it is important to have a process to change the coordinates of the same vector between different reference frames. As an example, let an inertial reference frame I be defined by the unit vectors \vec{e}_i and \vec{e}_j and assume that an aircraft A is flying relative to this frame of reference. The inertial velocity vector can be written as

$$\vec{v} = \vec{v}_A^I = v \vec{e}_{iw} \quad (2.5)$$

where v is the speed and \vec{e}_{iw} is the direction of the velocity vector. The vector \vec{v} makes an angle γ with the horizontal axis, which is called the flight path angle (see figure 2.12). What are the coordinates of the inertial velocity vector in the inertial frame of reference? To answer this question let us first define a reference frame that is usually called wind axes W and is represented by the vectors \vec{e}_{iw} and \vec{e}_{jw} , which are orthogonal to each other (see figure 2.12). In this frame the coordinates of the velocity vector are easy to determine as

$${}^W[\vec{v}] = \begin{bmatrix} v \\ 0 \end{bmatrix}. \quad (2.6)$$

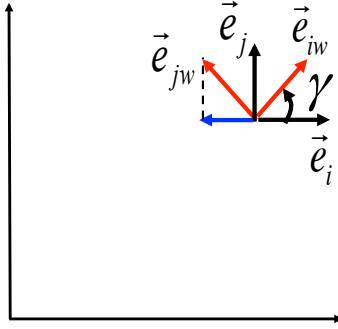


Figure 2.7: Relation between unit vectors

To obtain the coordinates of the inertial velocity vector in the inertial reference frame I one must use trigonometry relations for the angle γ as explained in section 1.3. Trigonometry will allow us to relate the unit vectors $\vec{e}_{iw}, \vec{e}_{jw}$ with the unit vectors \vec{e}_i, \vec{e}_j . To obtain these relations one simply translates the set of axes \vec{e}_i, \vec{e}_j so that their origin coincides with the origin of the set of axes $\vec{e}_{iw}, \vec{e}_{jw}$ (see figure 2.7). Then using trigonometry and the parallelogram rule for vector addition we can write

$$\vec{e}_{iw} = \cos \gamma \vec{e}_i + \sin \gamma \vec{e}_j \quad (2.7)$$

$$\vec{e}_{jw} = -\sin \theta \vec{e}_i + \cos \theta \vec{e}_j \quad (2.8)$$

The minus sign in the second equation is due to the fact that the projection of the vector \vec{e}_{jw} into the horizontal vector \vec{e}_i (the blue vector in figure 2.7) points in the opposite direction of the vector \vec{e}_i . Replacing (2.7) into (2.5) yields

$$\vec{v} = v \vec{e}_{iw} = v \cos \gamma \vec{e}_i + v \sin \theta \vec{e}_j \quad (2.9)$$

From (2.9) we see that

$${}^I[\vec{v}] = \begin{bmatrix} v \cos(\gamma) \\ v \sin(\gamma) \end{bmatrix}. \quad (2.10)$$

In matrix notation one can write (see section 2.3.3)

$$\begin{bmatrix} v \cos(\gamma) \\ v \sin(\gamma) \end{bmatrix} = v \begin{bmatrix} \cos(\gamma) \\ \sin(\gamma) \end{bmatrix} + 0 \begin{bmatrix} -\sin(\gamma) \\ \cos(\gamma) \end{bmatrix} = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}. \quad (2.11)$$

Although this was a specific example the following always holds

$${}^I[\vec{v}] = T_W^I {}^W[\vec{v}], \quad (2.12)$$

where in two dimensional space

$$T_W^I = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix}, \quad (2.13)$$

which is called a transformation matrix or a rotation matrix. Note that the notation is such that the superscript W of ${}^W[\vec{v}]$ cancels the subscript W of T_W^I to yield ${}^I[\vec{v}]$. The first column of T_W^I has the coordinates in frame I of the first vector \vec{e}_{iw} of the frame W . The second column has the coordinates in frame I of the second vector \vec{e}_{jw} of frame W . This is a general rule that holds true for any linear transformation.

Example 2.2.1 For the same problem depicted in figure 2.7 assume that a third axis is defined to be pointing out of the page. For this 3D extension of the problem, what would be the transformation matrix T_W^I ?

Solution: With the third axis pointing out of the plane we see that the coordinates of the vectors on the plane have a zero component along this axis. Furthermore, the third axis has the same representation in frame I that it has in frame W with coordinates $[0 \ 0 \ 1]^T$. Using the rule that each column is the transformation of the corresponding unit vector we would then have

$$T_W^I = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.14)$$

The inverse transformation of coordinates is obtained by the inverse matrix. The inverse of a matrix H can be computed using Laplace's formula

$$H^{-1} = \frac{1}{|H|} Co^T(H). \quad (2.15)$$

Each entry (i, j) of the cofactors matrix $Co(H)$ is

$$Co_{ij}(H) = (-1)^{(i+j)}|H_{ij}|, \quad (2.16)$$

where H_{ij} is the matrix obtained from H by removing row i and column j and $|H_{ij}|$ is its determinant. For a 2×2 matrix

$$H_{2 \times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (2.17)$$

the cofactor matrix is

$$Co(H_{2 \times 2}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad (2.18)$$

and the determinant is equal to

$$|H_{2 \times 2}| = ad - bc. \quad (2.19)$$

For a 3×3 matrix $H_{3 \times 3}$ the cofactor matrix is

$$Co(H_{3 \times 3}) = \begin{bmatrix} |H_{11}| & -|H_{12}| & |H_{13}| \\ -|H_{21}| & |H_{22}| & -|H_{23}| \\ |H_{31}| & -|H_{32}| & |H_{33}| \end{bmatrix}. \quad (2.20)$$

Each of the determinants in (2.20) are computed for 2×2 matrices. Determinants for matrices that are not 2×2 will be reviewed in section 2.11.

Example 2.2.2 Compute the inverse of the coordinate transformation (2.13).

Solution: Using (2.15), (2.18), and (2.19) with H replaced by the matrix (2.13) the inverse coordinate transformation is given by

$$T_I^W = [T_W^I]^{-1} = \frac{1}{|T_W^I|} Co^T(T_W^I) = \frac{1}{\cos^2 \gamma + \sin^2 \gamma} \begin{bmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{bmatrix} = [T_W^I]^T$$

where we used the trigonometric identity $\cos^2 \gamma + \sin^2 \gamma = 1$. The inverse of the matrix T_I^W is its transpose. This is always the case for matrices with columns that are the coordinates of an orthonormal set of vectors. Such matrices are called orthogonal. This topic will be reviewed in the next section.

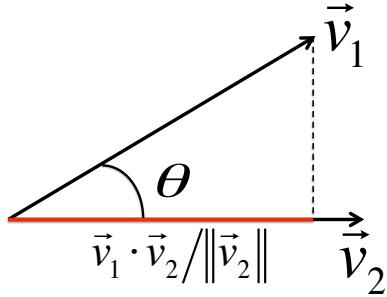


Figure 2.8: Angle between vectors

2.3 Vector Products and Vector Basis

There are three different vector products that will be considered in this book: the dot product, the cross product, and the dyadic product. Additionally, the dot and cross products will be combined in the scalar triple product.

2.3.1 Dot Product and Orthogonal Projections

The dot product is defined as

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos \theta \quad (2.21)$$

where θ is the smallest angle between the two vectors (see figure 2.8). When both vectors are the same then the dot product gives the square of the length of the vector. To see this we apply the formula (2.21) with $\vec{v}_1 = \vec{v}$ and $\vec{v}_2 = \vec{v}$ yielding the square of the norm of the vector \vec{v} , i.e.,

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2, \quad (2.22)$$

because $\cos \theta = \cos(0) = 1$ in this case. We also note from (2.21) that the dot product is commutative, i.e., $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1$. When one of the vectors has unit norm, for example if one replaces \vec{v}_2 by $\vec{e}_2 = \vec{v}_2 / \|\vec{v}_2\|$, then the expression (2.21) becomes

$$\vec{v}_1 \cdot \vec{e}_2 = \|\vec{v}_1\| \cos \theta. \quad (2.23)$$

From trigonometry we see that the dot product (2.23) yields the signed length of the projection of vector \vec{v}_1 in the direction of vector \vec{v}_2 , which is represented by the red line in figure 2.8. Using (2.22) the projection vector is written as

$$\vec{p}_{\vec{v}_1, \vec{v}_2} = (\vec{v}_1 \cdot \vec{e}_2) \vec{e}_2 = \left(\vec{v}_1 \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|} \right) \frac{\vec{v}_2}{\|\vec{v}_2\|} = (\vec{v}_1 \cdot \vec{v}_2) (\vec{v}_2 \cdot \vec{v}_2)^{-1} \vec{v}_2. \quad (2.24)$$

The dot product of two vectors with coordinates in the same frame ${}^E[\vec{v}_1]$ and ${}^E[\vec{v}_2]$ can also be computed as

$$\vec{v}_1 \cdot \vec{v}_2 = {}^E[\vec{v}_1]^T {}^E[\vec{v}_2] = \sum_{i=1}^n {}^E[\vec{v}_1]_i {}^E[\vec{v}_2]_i, \quad (2.25)$$

where n is the number of coordinates. Notice that if one combines equations (2.22) and (2.25) it yields

$$\|v\|^2 = \sum_{i=1}^n ({}^E[\vec{v}]_i)^2, \quad (2.26)$$

which is a generalization of Pythagoras Theorem (1.9) to more than two dimensions. Going back to the change of coordinates example in section 2.2, note that the vector \vec{v} must have the same length when computed in any of the coordinates (2.6) or (2.10). In fact, performing the computation of the norm for the coordinates (2.6) using (2.26) yields

$$\|\vec{v}\|^2 = v^2. \quad (2.27)$$

The same computation for the coordinates (2.10) yields

$$\|\vec{v}\|^2 = v^2 (\cos^2(\gamma) + \sin^2(\gamma)) = v^2. \quad (2.28)$$

The norm of a vector is thus invariant relative to the choice of coordinates. Two vectors are said to be orthogonal (perpendicular) if their dot product is zero. The dot product is zero because the angle between the two vectors is $\theta = \pi/2$ and $\cos \theta = 0$ in expression (2.21). When a set of vectors is such that the dot product of any two vectors is zero and the norm of each vector is one we call it an orthonormal set.

Example 2.3.1 *The unit vectors in a coordinate system form an orthonormal set because they have norm one and are orthogonal since*

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (2.29)$$

Using this relation determine the projection of the vector $\vec{v}_1 = \vec{e}_1 + 2\vec{e}_2$ onto the vector $\vec{v}_2 = -\vec{e}_1 + \vec{e}_2$?

Solution: The dot products that appear in equation (2.24) are

$$\begin{aligned}\vec{v}_1 \cdot \vec{v}_2 &= (\vec{e}_1 + 2\vec{e}_2) \cdot (-\vec{e}_1 + \vec{e}_2), \\ \vec{v}_2 \cdot \vec{v}_2 &= (-\vec{e}_1 + \vec{e}_2) \cdot (-\vec{e}_1 + \vec{e}_2).\end{aligned}$$

Applying the distributive property of the dot product over addition and using (2.29) yields

$$\vec{p}_{\vec{v}_1, \vec{v}_2} = \left[\frac{-\vec{e}_1 \cdot \vec{e}_1 + \vec{e}_1 \cdot \vec{e}_2 - 2\vec{e}_2 \cdot \vec{e}_1 + 2\vec{e}_2 \cdot \vec{e}_2}{\vec{e}_1 \cdot \vec{e}_1 - \vec{e}_1 \cdot \vec{e}_2 - \vec{e}_2 \cdot \vec{e}_1 + \vec{e}_2 \cdot \vec{e}_2} \right] (-\vec{e}_1 + \vec{e}_2) = -\frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2.$$

2.3.2 Basis Vectors and Linear Independence

A basis in an Euclidean space of dimension n is a set of n vectors $\vec{v}_1, \dots, \vec{v}_n$ such that any vector \vec{v} can be expressed as a linear combination of the basis vectors as

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n. \quad (2.30)$$

The set $\vec{e}_i, i = 1, \dots, n$, forms what is called an orthonormal basis because every pair of vectors is orthogonal and they all have unit norm. If their norm was not equal to one then the basis would be called orthogonal instead of orthonormal. Any vector \vec{v} can be expressed in an orthogonal basis $\vec{v}_1, \dots, \vec{v}_n$ as in equation (2.30) where the coordinates are given by the projection (2.24)

$$\alpha_i = \frac{\vec{v} \cdot \vec{v}_i}{\|\vec{v}_i\|^2}, i = 1, \dots, n. \quad (2.31)$$

This can be proved by simply taking the dot product of the vector \vec{v} in (2.30) with the vector \vec{v}_i , taking into account that $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$ and $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2$. Note that if the basis is orthonormal then \vec{v}_i is replaced by $\vec{e}_i = \vec{v}_i/\|\vec{v}_i\|$ with $\|\vec{e}_i\| = 1$. In this case, the coordinates (2.31) become simply $\alpha_i = \vec{v} \cdot \vec{e}_i, i = 1, \dots, n$.

We say that n vectors are linearly independent if the only solution of

$$\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0} \quad (2.32)$$

is $\alpha_1 = \dots = \alpha_n = 0$. Any basis for an Euclidean space of dimension n is formed by n vectors that are linearly independent.

2.3.3 Matrices, Matrix Multiplication, and Trace

We saw before that vectors can be stacked as columns in larger arrays called matrices. Alternatively, they can also be stacked as rows. Multiplication of two matrices is done by performing the dot product of the rows of the left matrix by the columns of the right matrix. For example, for matrices with two rows and two columns we can write

$$\begin{bmatrix} {}^E[\vec{x}_1]^T \\ {}^E[\vec{x}_2]^T \end{bmatrix} \cdot \begin{bmatrix} {}^E[\vec{y}_1] & {}^E[\vec{y}_2] \end{bmatrix} = \begin{bmatrix} {}^E[\vec{x}_1]^T \cdot {}^E[\vec{y}_1] & {}^E[\vec{x}_1]^T \cdot {}^E[\vec{y}_2] \\ {}^E[\vec{x}_2]^T \cdot {}^E[\vec{y}_1] & {}^E[\vec{x}_2]^T \cdot {}^E[\vec{y}_2] \end{bmatrix}. \quad (2.33)$$

The transpose operator can also be applied to matrices by writing each row of the original matrix as a column of the transposed version of the matrix. The transpose operation satisfies the following properties:

$$(A + B)^T = A^T + B^T, \quad (2.34)$$

$$(AB)^T = B^T A^T. \quad (2.35)$$

The trace of A is the sum of its main diagonal entries and is denoted $\text{Tr}(A)$. It satisfies the following properties,

$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B), \quad (2.36)$$

$$\text{Tr}(A^T) = \text{Tr}(A), \quad (2.37)$$

$$\text{Tr}(AA^T) = \text{Tr}(A^T A). \quad (2.38)$$

$$(2.39)$$

Example 2.3.2 Let ${}^E[\vec{x}_1]^T = [1 \ 2]$, ${}^E[\vec{x}_2]^T = [3 \ 4]$, ${}^E[\vec{y}_1]^T = [-1 \ 3]$, ${}^E[\vec{y}_2]^T = [3 \ -2]$. Computing (2.33), its transpose, and its trace yields

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 9 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 5 & 9 \\ -1 & 1 \end{bmatrix}, \quad \text{Tr}(A) = 6.$$

2.3.4 Cross Products and Areas

The cross product only exists in Euclidean spaces of dimension three. For vectors with only two coordinates one must add a third zero coordinate before performing the cross product. The cross product of two vectors $\vec{v}_1 = \vec{a}$ and

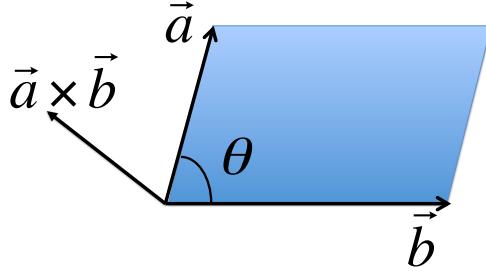


Figure 2.9: Cross product and area of parallelogram

$\vec{v}_2 = \vec{b}$ is denoted as $\vec{c} = \vec{a} \times \vec{b}$. The result is a vector \vec{c} that is orthogonal to the plane formed by the other two vectors. Its magnitude is

$$\|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta, \quad (2.40)$$

where θ is the smallest angle between the two vectors (see figure 2.9). This magnitude corresponds to the area of the parallelogram formed by the vectors \vec{a} and \vec{b} colored blue in figure 2.9. The direction of the cross product is given by the right hand rule. To apply the right hand rule we place the index of the right hand along the first vector \vec{a} , the middle finger along the second vector \vec{b} , and then the thumb will point in the direction of the cross product (see figure 2.10). From the right hand rule we can see that the cross product is anti-commutative, i.e., $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. We also note from (2.40) that the cross product of a vector by itself is zero, i.e., $\vec{a} \times \vec{a} = \vec{0}$ because $\sin(0) = 0$.

Example 2.3.3 *The unit vectors from the orthonormal coordinate system in an Euclidean space of dimension three satisfy the relation*

$$\vec{e}_i \times \vec{e}_j = \begin{cases} 0 & \text{if } i = j \\ \vec{e}_3 & \text{if } i = 1, j = 2 \\ -\vec{e}_2 & \text{if } i = 1, j = 3 \\ \vec{e}_1 & \text{if } i = 2, j = 3 \end{cases} \quad (2.41)$$

For the cases $i = 2, j = 1$ or $i = 3, j = 1$ or $i = 3, j = 2$ one can use the relation $\vec{e}_j \times \vec{e}_i = -\vec{e}_i \times \vec{e}_j$ and (2.41) to determine the cross product.

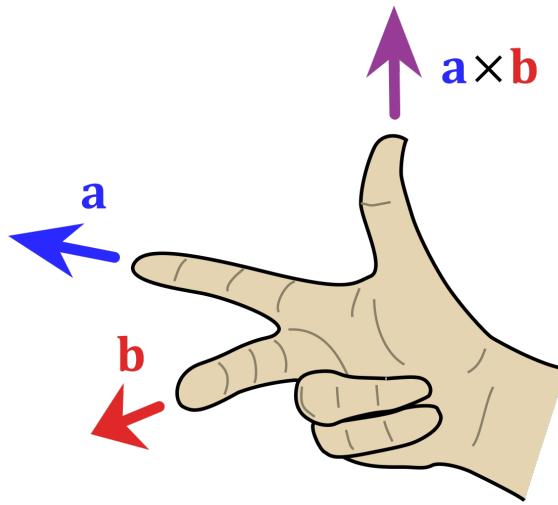


Figure 2.10: Right hand rule (taken from en.wikipedia.org)

If the coordinates of the vectors \vec{a} and \vec{b} are, respectively, ${}^E[\vec{a}]^T = [a_1 \ a_2 \ a_3]$ and ${}^E[\vec{b}]^T = [b_1 \ b_2 \ b_3]$, the coordinates of the cross product can be determined by the matrix multiplication

$${}^E[\vec{c}] = \Omega({}^E[\vec{a}]) \cdot {}^E[\vec{b}], \quad (2.42)$$

where

$$\Omega({}^E[\vec{a}]) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad (2.43)$$

Example 2.3.4 Let ${}^E[\vec{a}]^T = [1 \ 2 \ -1]$, ${}^E[\vec{b}]^T = [3 \ 4 \ 1]$. Then (2.42) becomes

$${}^E[\vec{c}] = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix}. \quad (2.44)$$

2.3.5 Triple Products and Volume

The dot product and the cross product satisfy the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}, \quad (2.45)$$

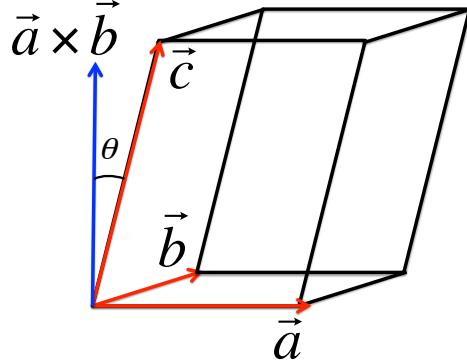


Figure 2.11: Volume of a parallelepiped

which can be verified by writing down the coordinates of the left and right hand sides of equation (2.45). The two products can also be combined in what is called the scalar triple product to compute the volume of a parallelepiped

$$V = A_{\text{base}}h, \quad (2.46)$$

where V is the volume, A_{base} is the area of the base, and h is the height. If the base is formed by the vectors \vec{a}, \vec{b} , and if the extra dimension is represented by the vector \vec{c} (see figure 2.11), then $h = \|\vec{c}\| \cos \theta$, $A_{\text{base}} = \|\vec{a} \times \vec{b}\|$ and one can write

$$V = \|\vec{a} \times \vec{b}\| \|\vec{c}\| \cos \theta = (\vec{a} \times \vec{b}) \cdot \vec{c}. \quad (2.47)$$

This entire operation can be done by stacking the coordinates of the three vectors \vec{a}, \vec{b} , and \vec{c} in a given frame E as the rows of a matrix and then computing its determinant, which can be written as

$$V = \begin{vmatrix} E[\vec{a}]^T \\ E[\vec{b}]^T \\ E[\vec{c}]^T \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \quad (2.48)$$

From (2.47), (2.42), (2.25), and (2.19) the determinant is

$$V = c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1). \quad (2.49)$$

Example 2.3.5 The volume of the parallelepiped formed by \vec{e}_1, \vec{e}_2 , and \vec{e}_3 is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

2.3.6 Dyads and Dyadic Product

The other vector product that we will use in this book is called the dyadic product. The dyadic product between vectors \vec{a} and \vec{b} is the dyad ${}^E[\vec{c}] = \vec{a}\vec{b}$. A dyad is a geometric object with magnitude and two directions, which are the directions of the two vectors that are paired in the dyad. The coordinates of the dyadic product are represented by the matrix

$${}^E[\vec{c}] = {}^E[\vec{a}] {}^E[\vec{b}]^T. \quad (2.50)$$

Example 2.3.6 Let ${}^E[\vec{a}]^T = [1 \ 2 \ -1]$, ${}^E[\vec{b}]^T = [3 \ 4 \ 1]$. Then (2.50) becomes

$${}^E[\vec{c}] = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [3 \ 4 \ 1] = \begin{bmatrix} 3 & 4 & 1 \\ 6 & 8 & 2 \\ -3 & -4 & -1 \end{bmatrix} \quad (2.51)$$

A dyad represents the pairing of two vectors in space (see figure 2.12). For example, for a dyad expressing the state of stress at a point, one vector represents a force and the other vector represents the normal to the surface on which the force is applied.

The product of two matrices was computed in section 2.3.3 using the dot product of the rows of the left matrix by the columns of the right matrix. It can equivalently be computed by the sum of the dyadic products of the columns of the left matrix by the rows of the right matrix. For example, for matrices with two columns and two rows we can write

$$[{}^E[\vec{y}_1] \ {}^E[\vec{y}_2]] \cdot \begin{bmatrix} {}^E[\vec{x}_1]^T \\ {}^E[\vec{x}_2]^T \end{bmatrix} = {}^E[\vec{y}_1] {}^E[\vec{x}_1]^T + {}^E[\vec{y}_2] {}^E[\vec{x}_2]^T. \quad (2.52)$$

Example 2.3.7 Compute the following product of matrices using the dyadic product.

$$\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} [4 \ 1] + \begin{bmatrix} 1 \\ -2 \end{bmatrix} [8 \ 2] = \begin{bmatrix} 16 & 4 \\ -20 & -5 \end{bmatrix}$$

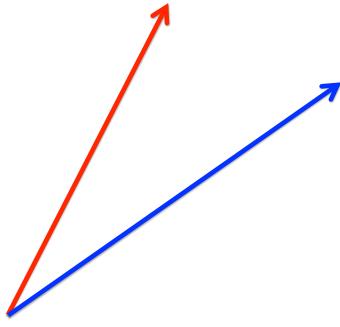


Figure 2.12: Dyad

To finish this section, it is important to recall that one can also compute vector products without resorting to coordinates, as it was done in example 2.3.1 for the dot product. This is done simply by applying the distributive property of multiplication over addition using the description of the vectors in a basis of the Euclidean space.

Example 2.3.8 Let a reference frame be given with unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and let $\vec{a} = \vec{e}_1 + 2\vec{e}_2, \vec{b} = 3\vec{e}_1 - \vec{e}_2$, and $\vec{c} = \vec{e}_1 - \vec{e}_3$. Then

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (\vec{e}_1 + 2\vec{e}_2) \cdot (3\vec{e}_1 - \vec{e}_2) = 3\vec{e}_1 \cdot \vec{e}_1 - \vec{e}_1 \cdot \vec{e}_2 + 6\vec{e}_2 \cdot \vec{e}_1 - 2\vec{e}_2 \cdot \vec{e}_2 = 1, \\ \vec{a} \times \vec{b} &= 3\vec{e}_1 \times \vec{e}_1 - \vec{e}_1 \times \vec{e}_2 + 6\vec{e}_2 \times \vec{e}_1 - 2\vec{e}_2 \times \vec{e}_2 = -7\vec{e}_3, \\ \vec{a} \vec{b} &= 3\vec{e}_1 \vec{e}_1 - \vec{e}_1 \vec{e}_2 + 6\vec{e}_2 \vec{e}_1 - 2\vec{e}_2 \vec{e}_2, \\ \vec{c} \cdot (\vec{a} \times \vec{b}) &= (\vec{e}_1 - \vec{e}_3) \cdot (-7\vec{e}_3) = -7\vec{e}_1 \cdot \vec{e}_3 + 7\vec{e}_3 \cdot \vec{e}_3 = 7.\end{aligned}$$

2.4 Orthogonal Projection on a Plane

Many times one needs to find the ground position of a flying aircraft. If the ground can be approximated by a horizontal plane, then the ground position is the projection of the position vector of the aircraft on the horizontal plane. As seen in figure 2.13, the projection on a plane can be computed by first projecting into the unit normal vector \vec{n} to the plane ($\vec{n} = \vec{e}_3$ in the figure),

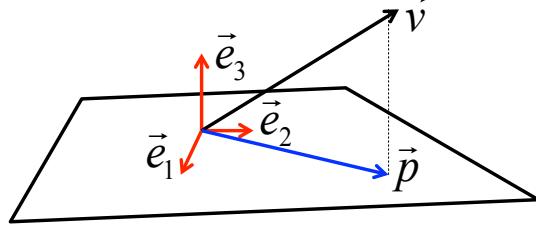


Figure 2.13: Projection of a vector on a plane

and then subtracting this projection from the original vector. To do this, one must first define the plane based on the unit normal vector \vec{n} as

$$S_c = \{\vec{x} \mid \vec{x} \cdot \vec{n} = c\}. \quad (2.53)$$

The symbol $\{$ can be read as "the set of" and the symbol $|$ means "such that". So one reads (2.53) as "the set of \vec{x} such that its dot product with \vec{n} is the constant value c ". When $c = 0$ one gets the set of vectors that are orthogonal to \vec{n} . For any other values of c it yields the plane that corresponds to a translation by a distance c of the set of vectors orthogonal to \vec{n} . The projection \vec{p} of a vector \vec{v} onto the plane S_c is computed using the formula (2.24) as

$$\vec{p} = \vec{v} - \vec{p}_{\vec{v}, \vec{n}} = \vec{v} - (\vec{v} \cdot \vec{n}) \vec{n} = \left(\overleftrightarrow{I} - \vec{n} \vec{n} \right) \cdot \vec{v} = \overleftrightarrow{P} \cdot \vec{v}, \quad (2.54)$$

where

$$\overleftrightarrow{P} = \overleftrightarrow{I} - \vec{n} \vec{n} \quad (2.55)$$

is called the projection tensor and \overleftrightarrow{I} is called the identity tensor. In 3D the identity tensor is written as $\overleftrightarrow{I} = \vec{e}_1 \vec{e}_1 + \vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3$ and has coordinates

$$\left[\overleftrightarrow{I} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.56)$$

The dot product of the dyad $\vec{n} \vec{n}$ and the vector \vec{v} is $(\vec{n} \vec{n}) \cdot \vec{v} = \vec{n}(\vec{n} \cdot \vec{v})$.

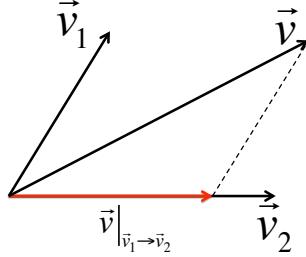


Figure 2.14: Oblique projection

Example 2.4.1 Compute the projection of the position vector $\vec{s}_{AO} = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$ on the horizontal plane defined by $S_0 = \{\vec{x} \mid \vec{x} \cdot \vec{e}_3 = 0\}$.

Solution: Using the formula (2.54) with \vec{v} replaced by \vec{s}_{AO} yields

$$\vec{p} = \vec{s}_{AO} - (\vec{s}_{AO} \cdot \vec{e}_3) \vec{e}_3 = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3 - 3\vec{e}_3 = \vec{e}_1 + 2\vec{e}_2.$$

2.5 Oblique Projection

Assume that a vector has been orthogonally projected onto a plane. One can then use a set of basis vectors on the plane to find the coordinates of the orthogonal projection. In the case of example 2.4.1 this basis is orthogonal (in fact orthonormal) consisting of the vectors \vec{e}_1 and \vec{e}_2 . However, it is not always the case that one has been given an orthonormal basis for the plane. If the original basis for the plane is not orthogonal one can create an orthogonal basis given two linearly independent vectors and then use the procedure outlined in section 2.3.1. Alternatively, one can simply determine the oblique (non-orthogonal) projection on two linearly independent basis vectors on the plane.

Figure 2.14 shows the oblique projection of vector \vec{v} along the direction of vector \vec{v}_1 onto vector \vec{v}_2 . This oblique projection is denoted by $\vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2}$. To determine a formula for the oblique projection we observe that the vector \vec{v} is written as (Figure 2.14)

$$\vec{v} = \vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1} + \vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = \vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1} + \lambda \vec{v}_2 \quad (2.57)$$

where the first term on the right hand side of equation (2.57) is the oblique projection of \vec{v} onto \vec{v}_1 , and the second term is the oblique projection of \vec{v} onto \vec{v}_2 . Let $\vec{n}_{\vec{v}_1}$ be a vector normal to \vec{v}_1 . Performing the dot product of both sides of the equation (2.57) with $\vec{n}_{\vec{v}_1}$ yields

$$\vec{v} \cdot \vec{n}_{\vec{v}_1} = 0 + \lambda \vec{v}_2 \cdot \vec{n}_{\vec{v}_1}. \quad (2.58)$$

From the linear independence of \vec{v}_1 and \vec{v}_2 the term $\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}$ is not zero. Solving equation (2.58) for λ then leads to

$$\lambda = \frac{\vec{v} \cdot \vec{n}_{\vec{v}_1}}{\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}}. \quad (2.59)$$

Therefore the formula for the oblique projection $\lambda \vec{v}_2$ of vector \vec{v} along the direction of vector \vec{v}_1 onto vector \vec{v}_2 is

$$\vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = (\vec{v} \cdot \vec{n}_{\vec{v}_1}) \frac{\vec{v}_2}{\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}}. \quad (2.60)$$

The oblique projection of vector \vec{v} along the direction of vector \vec{v}_2 onto vector \vec{v}_1 is obtained by replacing (2.60) into (2.57) and solving for $\vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1}$, yielding

$$\vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1} = \vec{v} - \vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = \vec{v} - (\vec{v} \cdot \vec{n}_{\vec{v}_1}) \frac{\vec{v}_2}{\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}}. \quad (2.61)$$

Notice that if \vec{v}_2 is orthogonal to \vec{v}_1 and if $\vec{n}_{\vec{v}_1}$ is chosen as the unit normal vector to \vec{v}_1 , then $\vec{n}_{\vec{v}_1}$ is a unit vector parallel to \vec{v}_2 . Therefore, one can write

$$\vec{n}_{\vec{v}_1} = \frac{\vec{v}_2}{\|\vec{v}_2\|}. \quad (2.62)$$

In this case, $\vec{v}_2 \cdot \vec{n}_{\vec{v}_1} = \|\vec{v}_2\|$ and the formula (2.60) can be rewritten as

$$\vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = \left(\vec{v} \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|} \right) \frac{\vec{v}_2}{\|\vec{v}_2\|}, \quad (2.63)$$

which, as expected, is the same as the orthogonal projection equation (2.24).

Example 2.5.1 Compute the oblique projection of the vector $\vec{v} = 3\vec{e}_1$ along the vector $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$ onto the vector $\vec{v}_2 = 4\vec{e}_1 - 2\vec{e}_2$.

Solution: To use the formula (2.60) we must first compute a unit normal vector to \vec{v}_1 as

$$\vec{n}_{\vec{v}_1} = \frac{1}{\sqrt{2}} (-\vec{e}_1 + \vec{e}_2).$$

The dot products in equation (2.60) are

$$\begin{aligned} (\vec{v} \cdot \vec{n}_{\vec{v}_1}) &= 3\vec{e}_1 \cdot \frac{1}{\sqrt{2}} (-\vec{e}_1 + \vec{e}_2) = -\frac{3}{\sqrt{2}}, \\ (\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}) &= 2(2\vec{e}_1 - \vec{e}_2) \cdot \frac{1}{\sqrt{2}} (-\vec{e}_1 + \vec{e}_2) = -3\sqrt{2}. \end{aligned}$$

Replacing these values in equation (2.60) yields

$$\vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = (\vec{v} \cdot \vec{n}_{\vec{v}_1}) \frac{\vec{v}_2}{\vec{v}_2 \cdot \vec{n}_{\vec{v}_1}} = \frac{1}{2}\vec{v}_2 = 2\vec{e}_1 - \vec{e}_2.$$

We notice that using equation (2.61) we can also get

$$\vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1} = 3\vec{e}_1 - (2\vec{e}_1 - \vec{e}_2) = \vec{e}_1 + \vec{e}_2.$$

The vector \vec{v} can therefore be written as

$$3\vec{e}_1 = \vec{v} = \vec{v}|_{\vec{v}_2 \rightarrow \vec{v}_1} + \vec{v}|_{\vec{v}_1 \rightarrow \vec{v}_2} = (\vec{e}_1 + \vec{e}_2) + (2\vec{e}_1 - \vec{e}_2).$$

2.6 Eigenvalues and Eigenvectors

In section 2.4 we found the projection of a vector \vec{v} on a plane and denoted the projection as the vector \vec{p} (see figure 2.13). What would happen if the vector \vec{p} was projected on the plane? Obviously the result would be again the vector \vec{p} . Therefore, we can say that the vector \vec{p} is left invariant under projections on the plane, or under the action of the projection tensor \overleftrightarrow{P} . Since $\vec{n} = \vec{e}_3$ in figure 2.13, the coordinates of the tensor (2.55) are

$$P = \left[\overleftrightarrow{P} \right] = \left[\overleftrightarrow{I} \right] - [\vec{e}_3][\vec{e}_3]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.64)$$

Any vector \vec{w} on the plane is left invariant under the projection. Therefore we can write

$$\overleftrightarrow{P} \cdot \vec{w} = \lambda \vec{w} \Leftrightarrow \left(\lambda \overleftrightarrow{I} - \overleftrightarrow{P} \right) \cdot \vec{w} = \vec{0}, \quad (2.65)$$

for any vector \vec{w} on the plane. The scalar λ is called an eigenvalue of \overleftrightarrow{P} . The vector \vec{w} satisfying equation (2.65) is called the eigenvector associated with the eigenvalue λ . If \vec{w} is a vector on the plane and \overleftrightarrow{P} is the projection tensor on that plane then $\lambda = 1$. However, equation (2.65) is valid for any \overleftrightarrow{P} , even if it is not a projection tensor. According to equation (2.65) the vector \vec{w} is not mapped to itself for the case of $\lambda \neq 1$. However, it is still mapped to a line in the direction of \vec{w} . This line remains invariant under the action of \overleftrightarrow{P} . Eigenvalues and eigenvectors are thus associated with invariant subspaces. Eigenvectors associated with an eigenvalue λ are defined as nonzero solutions of equation (2.65) for that value of λ . Therefore, resorting to coordinates and using (2.65) yields the condition

$$\left| \lambda \begin{bmatrix} \overleftrightarrow{I} \end{bmatrix} - \begin{bmatrix} \overleftrightarrow{P} \end{bmatrix} \right| = 0. \quad (2.66)$$

This constraint guarantees that the operator represented by $\lambda \overleftrightarrow{I} - \overleftrightarrow{P}$ is not invertible. Therefore, it is not possible for equation (2.65) to have a unique solution of $\vec{w} = \vec{0}$ when condition (2.66) is satisfied, which is the requirement for \vec{w} to be an eigenvector. Denoting the eigenvector associated with the eigenvalue λ as \vec{w}_λ , one can write equation (2.65) in coordinates as

$$\left(\lambda \begin{bmatrix} \overleftrightarrow{I} \end{bmatrix} - \begin{bmatrix} \overleftrightarrow{P} \end{bmatrix} \right) \cdot [\vec{w}_\lambda] = (\lambda I - P) [\vec{w}_\lambda] = 0. \quad (2.67)$$

Example 2.6.1 Find all eigenvalues and eigenvectors of the projection tensor (2.64).

Solution: Using (2.67), (2.64), and (2.56) yields

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| = 0 \Leftrightarrow \lambda(\lambda - 1)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1$. Replacing $\lambda = 0$ in equation (2.65) written in matrix coordinates yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = w_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the eigenvector associated with the eigenvalue $\lambda_1 = 0$ is $\vec{w}_{\lambda_1} = \vec{e}_3$. Geometrically this means that if a vector is on the invariant line in the direction of \vec{e}_3 it is projected to the zero vector because $\lambda_1 = 0$.

Replacing $\lambda = 1$ in equation (2.65) written in matrix coordinates yields

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the eigenvectors associated with the eigenvalues $\lambda_2 = \lambda_3 = 1$ are $\vec{w}_{\lambda_2} = \vec{e}_1$ and $\vec{w}_{\lambda_3} = \vec{e}_2$. The geometric interpretation is that any vector on the invariant plane with basis vectors \vec{e}_1 and \vec{e}_2 will be projected to itself because $\lambda_2 = \lambda_3 = 1$.

Note that in example 2.6.1 the projection tensor is symmetric. Furthermore, the eigenvalue $\lambda = 1$ has multiplicity two and it has two linearly independent eigenvectors. In this book we will always be finding eigenvalues of symmetric matrices. This means that whenever an eigenvalue has multiplicity larger than one there will be always as many linearly independent eigenvectors as the multiplicity of the eigenvalue. As a consequence, the symmetric matrix can be diagonalized. To see how this can be done, we note that there is one equation (2.67) for each eigenvalue λ and its corresponding eigenvector \vec{w}_λ . If we assume a space of dimension n then the n equations can be stacked together and written as

$$PW = W\Lambda \tag{2.68}$$

where $W = [[\vec{w}_{\lambda_1}] \dots [\vec{w}_{\lambda_n}]]$ and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}. \tag{2.69}$$

If in equation (2.68) we invert the matrix W we can see that the matrix P can be diagonalized by knowledge of the eigenvectors and eigenvalues as

$$W^{-1}PW = \Lambda. \tag{2.70}$$

Example 2.6.2 Write equation (2.68) explicitly for the example 2.6.1;

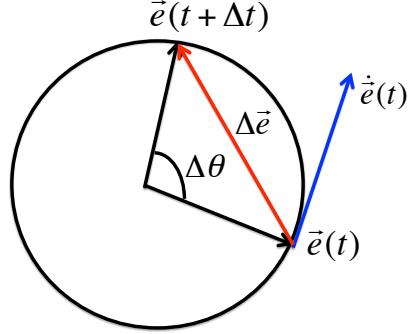


Figure 2.15: Rotation of a unit vector on a circle

Solution The eigenvalues and corresponding eigenvector coordinates are $\lambda_1 = 0$, $[\vec{w}_{\lambda_1}] = [0 \ 0 \ 1]^T$, $\lambda_2 = 1$, $[\vec{w}_{\lambda_2}] = [1 \ 0 \ 0]^T$, $\lambda_3 = 1$, $[\vec{w}_{\lambda_3}] = [0 \ 1 \ 0]^T$. Therefore, equation (2.68) is written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.7 Differentiation of Vectors

As we have seen in section 2.1 the velocity vector is the time rate of change of the position vector. It is therefore important to understand how to compute the derivative of vectors over time. We start by studying the case of a unit vector \vec{e} . A unit vector has a constant magnitude equal to one. Therefore, the only way that the unit vector can change is by changing direction, i.e., by rotating. Assume that the unit vector \vec{e} rotates in a circle as depicted in figure 2.15. The figure shows the position of the vector at two different instants of time that correspond to t and at to a later time $t + \Delta t$. Using the definition of derivative given in equation (1.41) and replacing the variable x by time t yields

$$\dot{\vec{e}}(t) = \frac{d\vec{e}(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{e}(t + \Delta t) - \vec{e}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{e}}{\Delta t}. \quad (2.71)$$

Note that $\Delta\vec{e}$ is represented by the red vector in figure 2.15, which is a secant line to the circle. Similarly to figure 1.14, as Δt converges to zero the secant line converges to the tangent line. Therefore the red vector converges to the blue vector in figure 2.15, which is the derivative of the vector $\vec{e}(t)$ with respect to time. It has magnitude and direction. According to (2.71) the magnitude is

$$\lim_{\Delta t \rightarrow 0} \frac{\|\Delta\vec{e}\|}{\Delta t}. \quad (2.72)$$

The norm of $\Delta\vec{e}$ can be approximated in a time Δt by the magnitude of the arc length. When Δt converges to zero this approximation is exact. Since the radius of the circle is the norm of the unit vector \vec{e} , which is one, according to (1.1) the magnitude of the arc length during an interval Δt is equal to $\Delta\theta$. Replacing this magnitude in (2.72) and taking the limit yields

$$\lim_{\Delta t \rightarrow 0} \frac{\|\Delta\vec{e}\|}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \dot{\theta}, \quad (2.73)$$

where Newton's dot notation for time derivative was used. Now that the magnitude was obtained, we can see from figure 2.15 that the direction of the blue vector is tangent to the circle. This direction can therefore be computed by the cross product of a unit vector \vec{e}_{out} pointing out of the paper with the vector \vec{e} . Therefore, using Newton's notation for the time derivative and combining the direction and the magnitude we obtain the formula

$$\dot{\vec{e}}(t) = \vec{\omega}^{\vec{e}}(t) \times \vec{e}(t), \quad (2.74)$$

where $\vec{\omega}^{\vec{e}}(t) = \dot{\theta}\vec{e}_{out}$ is the angular velocity of the vector $\vec{e}(t)$ relative to inertial space. In a more general case, the rotation of the vector $\vec{e}(t)$ will describe a cone as in figure 2.16. In this case one can write the vector $\vec{e}(t)$ as $\vec{e}(t) = \vec{e}_1(t) + \vec{e}_2(t)$. The two components of the vector $\vec{e}(t)$ are represented by the dashed vectors in figure 2.16. We note that the component $\vec{e}_2(t)$ stays fixed so that $\dot{\vec{e}}_2(t) = 0$. The only component that rotates is $\vec{e}_1(t)$, which rotates around a circle (the base of the cone). Therefore, using the property for the derivative of the sum of two vectors (1.43), one can write

$$\dot{\vec{e}}(t) = \dot{\vec{e}}_1(t) + \dot{\vec{e}}_2(t) = \dot{\vec{e}}_1(t) = \vec{\omega}^{\vec{e}}(t) \times \vec{e}_1(t) = \vec{\omega}^{\vec{e}}(t) \times \vec{e}(t) \quad (2.75)$$

where the last equality is due to the fact that the cross product $\vec{\omega}^{\vec{e}}(t) \times (\vec{e}_1(t) + \vec{e}_2(t))$ is distributive over the addition $\vec{e}_1 + \vec{e}_2$, and the fact that

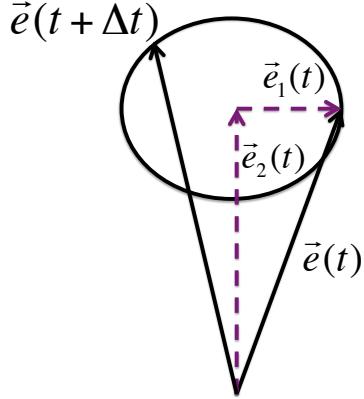


Figure 2.16: Rotation of a unit vector describing a cone

$\vec{\omega}^{\vec{e}}(t) \times \vec{e}_2(t) = \vec{0}$ because $\vec{\omega}^{\vec{e}}(t)$ is parallel to $\vec{e}_2(t)$. For a general vector $\vec{v}(t) = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$ one can use the rules for taking the derivative of the sum (1.43) and product (1.44) of different terms to write

$$\dot{\vec{v}}(t) = \left(\dot{v}_1\vec{e}_1 + v_1\dot{\vec{e}}_1\right) + \left(\dot{v}_2\vec{e}_2 + v_2\dot{\vec{e}}_2\right) + \left(\dot{v}_3\vec{e}_3 + v_3\dot{\vec{e}}_3\right). \quad (2.76)$$

Therefore, the derivative of a general vector can also be computed using equation (2.74) for the derivative of a unit vector.

Example 2.7.1 Consider an airplane A flying along an inertially fixed circle of radius r_e and center O in anti-clockwise direction. What are the position and velocity vectors of this vehicle ?

Solution: We first define a reference frame that rotates with the vehicle. The center of the frame lies at the center of the circle. The axis \vec{e}_r points from the center of the circle to the vehicle (see figure 2.17). The axis \vec{e}_θ is orthogonal to \vec{e}_r , and the axis \vec{e}_{out} points out of the paper and completes the triad. From figure 2.17 we see that the position of the aircraft is the vector

$$\vec{s}_{AO}(t) = r_e\vec{e}_r.$$

The inertial velocity of the aircraft is the time derivative of the position.

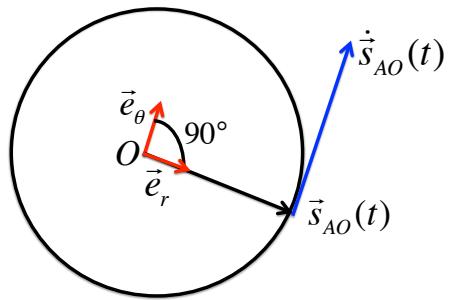


Figure 2.17: Circular motion

Using equation (2.74) it is written as

$$\vec{v}_A^I(t) = \dot{\vec{s}}_{AO}(t) = \vec{\omega}^{\vec{e}_r} \times (r_e \vec{e}_r) = \dot{\theta} \vec{e}_{out} \times (r_e \vec{e}_r) = r_e \dot{\theta} \vec{e}_\theta.$$

The angular velocity vector $\vec{\omega}^{\vec{e}_r}$ has magnitude $\dot{\theta}$ and has direction \vec{e}_{out} because the motion is anti-clockwise. One can see this by curling all the fingers of the right hand, except the thumb, in the anticlockwise direction. Then the thumb opens in the up direction. This is another form of the right hand rule and is depicted in figure 2.18.

2.8 Integration of Vectors

In example 2.7.1 we obtained the velocity vector starting from the position vector and taking the derivative over time. However, sometimes one measures the velocity vector of a vehicle and wants to estimate the position. In this case, one must integrate the velocity to obtain the position vector. Integrating is the inverse operation of differentiation, which was studied in section 2.7. To integrate a vector $\vec{v}(t) = \dot{\vec{s}}(t)$ from the initial time zero to a time t , one uses the Fundamental Theorem of Calculus that states that

$$\vec{s}(t) - \vec{s}(0) = \int_0^t \vec{v}(\tau) d\tau. \quad (2.77)$$



Figure 2.18: Another form of the right hand rule (taken from en.wikipedia.org)

Example 2.8.1 The vector $\vec{v}_A^I(t) = \vec{\omega}^{\vec{e}_r}(t) \times (r_e \vec{e}_r(t))$ is measured as the inertial velocity of an airplane in circular motion of radius r_e . What is the position vector?

Solution: We integrate the velocity vector using the result of equation (2.74)

$$\int_0^t \vec{v}_A^I(\tau) d\tau = \int_0^t [\vec{\omega}^{\vec{e}_r}(\tau) \times (r_e \vec{e}_r(\tau))] d\tau = \int_0^t \frac{d(r_e \vec{e}_r)}{d\tau} d\tau.$$

Since the expression after the last equality sign is the integral of a derivative then we use the Fundamental Theorem of Calculus to write

$$\int_0^t \frac{d(r_e \vec{e}_r)}{d\tau} d\tau = r_e \vec{e}_r(t) - r_e \vec{e}_r(0).$$

Putting it all together results in

$$\int_0^t \vec{v}_A^I(\tau) d\tau = \vec{s}_{AO}(t) - \vec{s}_{AO}(0),$$

where, as expected,

$$\vec{s}_{AO}(t) = r_e \vec{e}_r(t)$$

is the same position vector as the one defined in example 2.7.1.

2.9 Random Vectors

In certain applications the coordinates of a vector can be random variables. A random variable is a real-valued function of the outcome of a random experiment. For example, assume that the random experiment is tossing a coin. One could define a random variable w to have the value 1 if the outcome of tossing a coin is heads and the value -1 if the outcome is tails. This variable has a discrete set of possible values and is therefore a discrete random variable. An example of a continuous random variable would be the coordinate x of the position of a user as measured by a position sensor. The variable x can take on values in a continuous interval, i.e., $x \in [x_{\min}, x_{\max}]$. The measurement of the position is a random experiment because there is always random noise associated with a measurement. A random vector is a vector whose coordinates are random variables. For example the vector with coordinates x, y, z , that are measured by a position sensor subject to random measurement noise, whose array of coordinates is denoted by ν .

The expected (or mean) value of a continuous random vector is

$$\mu_\nu = E[\nu] = \int_{x_{\min}}^{x_{\max}} x f_\nu(x) dx, \quad (2.78)$$

where $f_\nu(x) \geq 0$ is the probability density function of ν , which satisfies

$$\int_{x_{\min}}^{x_{\max}} f_\nu(x) dx = 1. \quad (2.79)$$

The expected value operator is linear and therefore

$$E[A\nu_1 + B\nu_2] = AE[\nu_1] + BE[\nu_2], \quad (2.80)$$

$$E[A\nu_c \nu_c^T B^T] = AE[\nu_c \nu_c^T] B^T, \quad (2.81)$$

for all random vectors ν_1, ν_2, ν_c , and all constant matrices A, B . The matrix

$$\mathcal{K}_\nu = E[(\nu - \mu_\nu)(\nu - \mu_\nu)^T] \quad (2.82)$$

is called the covariance matrix of ν . Expanding (2.82) into four terms and applying the linearity property (2.80) yields

$$\mathcal{K}_\nu = E[\nu \nu^T] - E[\nu] E[\nu^T] \quad (2.83)$$

taking into account that $E[\mu_\nu] = E[E[\nu]] = E[\nu] = \mu_\nu$. The entry (i, j) of the covariance matrix is called the covariance between the random variables ν_i and ν_j , which is given by

$$\text{Cov}(\nu_i, \nu_j) = E[(\nu_i - E(\nu_i))(\nu_j - E(\nu_j))]. \quad (2.84)$$

We say that two random variables ν_i and ν_j are uncorrelated if the covariance between them is zero. Therefore, from (2.82) and (2.84) we see that the covariance matrix of a random vector with uncorrelated coordinates is diagonal. Each entry (i, i) in the main diagonal of the covariance matrix is

$$\sigma_i^2 = E[(\nu_i - E(\nu_i))^2] \quad (2.85)$$

and it is called the variance of the random variable ν_i . Similarly to (2.83) one can write

$$\sigma_i^2 = E[\nu_i^2] - E^2[\nu_i]. \quad (2.86)$$

Later in the book we will be using the Gaussian multivariate distribution with probability density function given by

$$f_\nu = p(\nu) = \frac{1}{(2\pi)^{\frac{m}{2}} \sqrt{|\mathcal{K}_\nu|}} e^{-\frac{1}{2}(\nu - \mu_\nu)^T \mathcal{K}_\nu^{-1} (\nu - \mu_\nu)}, \quad (2.87)$$

for a vector ν with m coordinates. This function depends only on the mean μ_ν and covariance matrix \mathcal{K}_ν .

Example 2.9.1 Let the random variable ν be the stopping angle of a wheel of fortune and assume that ν is uniformly distributed in the interval $[0, 2\pi)$. Compute the expected value and the variance of ν .

Solution: A random variable is a random vector with only one coordinate. Since ν is uniformly distributed in the interval $[0, 2\pi)$ then its probability density $f_\nu(x)$ is a constant c . This constant can be computed from (2.79) using as limits of integration $x_{\min} = 0$ and $x_{\max} = 2\pi$ as

$$f_\nu(x) = c = \frac{1}{2\pi}. \quad (2.88)$$

Replacing (2.88) in (2.78) yields the expected value

$$E[\nu] = \int_0^{2\pi} x \frac{1}{2\pi} dx = \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = \pi. \quad (2.89)$$

To find the variance we first compute

$$E[\nu^2] = \int_0^{2\pi} x^2 \frac{1}{2\pi} dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{4\pi^2}{3}. \quad (2.90)$$

Replacing (2.89) and (2.90) into equation (2.86) yields

$$\sigma_\nu^2 = \frac{4\pi^2}{3} - \pi^2 = \frac{\pi^2}{3}.$$

For a discrete random vector taking one of N possible values the expected value is

$$E[\nu] = \sum_{n=1}^N \nu_n P(\nu = \nu_n), \quad (2.91)$$

where $P(\nu = \nu_n) \in [0, 1]$ is the probability that the vector ν takes the value ν_n , which satisfies the property

$$\sum_{n=1}^N P(\nu = \nu_n) = 1. \quad (2.92)$$

Example 2.9.2 Let the value of the random variable x be the number on the face of a dice that is rolled on a casino table. Compute the expected value and the variance of x . How would your calculations change if instead of one dice there were two die being rolled at the same time, for which y_1 and y_2 are the random variables corresponding to the first and second dice, respectively? What is the probability that the sum of the numbers in the two die be equal to eight? How would the probability change if you knew that the sum was an even number?

Solution: The values taken by the random variable x are $x_n = n$ for $n = 1, \dots, 6$. Since all possible outcomes of rolling a dice have the same probability then $P(x = x_n) = \frac{1}{6}$ because of the constraint (2.92). Using (2.91) yields

$$E[x] = \sum_{n=1}^6 x_n P(x = x_n) = \frac{1}{6} \sum_{n=1}^6 n = \frac{7}{2} = 3.5. \quad (2.93)$$

Note that the expected value is not one of the possible values of the dice. From (2.86) and (6.35) the variance is

$$\sigma^2 = E[x^2] - E^2[x] = \frac{1}{6} \sum_{n=1}^6 n^2 - \frac{7^2}{2^2} = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

For two die let $y = [y_1 \ y_2]^T$ be the random vector whose coordinates are the outcomes of each dice and let $y_{ij} = [i \ j]^T$ for $i, j = 1, \dots, 6$. All 36 outcomes of rolling the die are equally probable with $P(y = y_{ij}) = \frac{1}{36}$. Using (2.91),

$$E[y] = \sum_{i,j=1}^6 y_{ij} P(y = y_{ij}) = \frac{1}{36} \sum_{i,j=1}^6 \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}. \quad (2.94)$$

Note that the mean of y_1 and y_2 are the same and equal to the mean of x because the two dice are independent. We now compute $E[yy^T]$ given by

$$E[yy^T] = \sum_{i,j=1}^6 y_{ij} y_{ij}^T P(y = y_{ij}) = \frac{1}{36} \sum_{i,j=1}^6 \begin{bmatrix} i^2 & ij \\ ji & j^2 \end{bmatrix} = \begin{bmatrix} \frac{91}{6} & \frac{49}{6} \\ \frac{49}{6} & \frac{91}{6} \end{bmatrix}. \quad (2.95)$$

From (2.83), (2.94), and (2.95) the covariance matrix of y is

$$\mathcal{K}_y = E[yy^T] - E[y]E[y^T] = \begin{bmatrix} \frac{91}{6} & \frac{49}{6} \\ \frac{49}{6} & \frac{91}{6} \end{bmatrix} - \begin{bmatrix} \frac{49}{4} & \frac{49}{4} \\ \frac{49}{4} & \frac{49}{4} \end{bmatrix} = \begin{bmatrix} \frac{35}{12} & 0 \\ 0 & \frac{35}{12} \end{bmatrix}.$$

The two variables y_1 and y_2 are uncorrelated (zero off diagonal terms) and their variances (main diagonal) are the same and equal to the variance of x . From all the 36 possible outcomes the five outcomes that lead to a sum of the numbers in the die equal to eight are $(y_1 = 5, y_2 = 3)$, $(y_1 = 3, y_2 = 5)$, $(y_1 = 2, y_2 = 6)$, $(y_1 = 6, y_2 = 2)$, $(y_1 = 4, y_2 = 4)$. Therefore, the probability of the sum being equal to eight is

$$P(y_1 + y_2 = 8) = \frac{5}{36}. \quad (2.96)$$

If we know that the sum of the numbers in the two die is even then there are 18 possible outcomes leading to that result. Therefore, the new probability of the sum being eight given that it is even is $\frac{5}{18}$ which is larger than (2.96).

As seen at the end of the previous example, the probability of the event "the sum of two die is equal to eight" changes when it is conditioned by the occurrence of the event "the sum is pair". This new information added by the conditioning event will also influence the expected value of the sum of the two die. For a discrete random vector ν taking one of N possible values, its conditional expected value given that another discrete random vector η takes the value $\bar{\eta}$ is

$$E[\nu | \eta = \bar{\eta}] = \sum_{n=1}^N \nu_n P_{\nu|\eta}(\nu = \nu_n | \eta = \bar{\eta}), \quad (2.97)$$

with the conditional probability given by

$$P_{\nu|\eta}(\nu = \nu_n | \eta = \bar{\eta}) = \frac{P(\nu = \nu_n, \eta = \bar{\eta})}{P(\eta = \bar{\eta})} \quad (2.98)$$

for $P(\eta = \bar{\eta}) \neq 0$. The symbol $|$ means "given". The numerator in (2.98) is called a joint probability. The comma in the joint probability means "and". When $P_{\nu|\eta}(\nu = \nu_n | \eta = \bar{\eta}) = P(\nu = \nu_n)$ for any ν_n and for any $\bar{\eta}$ we say that the random vectors ν and η are independent. The marginal probability can be computed from the joint probability as

$$P(\nu = \nu_i) = \sum_{n=1}^N P(\nu = \nu_i, \eta = \eta_n) = \sum_{n=1}^N P_{\nu|\eta}(\nu = \nu_i | \eta = \eta_n) P(\eta = \eta_n). \quad (2.99)$$

Example 2.9.3 Use the formula (2.98) to compute the probability that the sum of the numbers in two rolled die be equal to eight given that the sum is even. Compare this result with the one obtained in example 2.9.2.

Solution: Going back to example 2.9.2, let ν takes as values the sum of the two die, let $\nu_n = 8$, let η be a binary variable equal to one for an even sum and zero for an odd sum, and let $\bar{\eta} = 1$. Note that half of the 36 possible outcomes of rolling the two die have an even sum and half of the outcomes have an odd sum. Using the formula (2.98) leads to

$$P(\nu = 8 | \eta = 1) = \frac{P(\nu = 8, \eta = 1)}{P(\eta = 1)} = \frac{\frac{5}{36}}{\frac{1}{2}} = \frac{5}{18},$$

which is the same value as the one obtained in example 2.9.2.

The next example shows that the expected value can be conditioned on more than one random variable.

Example 2.9.4 Assume that one is drawing cards at random from a deck and placing them face up on a casino table without putting them back into the deck. Let the random variable x_k denote the score given to the k -th card drawn from the deck. If the card is a jack then the score is 11, if it is a queen the score is 12, if it is a king the score is 13, and if it is an ace the score is 1. All other cards will have a number which will be their score. Compute

the expected value of the score of the card drawn first. Compare this result with the expected value of the score of the card in the third draw given that the first two cards were an ace and a two, respectively.

Solution: There are 52 cards in a deck. Each card, be it a number or one of the face cards, appears 4 times. Therefore, the probability of the card score n , for any $n \in \{1, \dots, 13\}$, to be the first one to be drawn is

$$P(x_1 = n) = \frac{4}{52} = \frac{1}{13}. \quad (2.100)$$

The expected value is the average of all possible scores weighted by their probability and, since all outcomes are equally probable with probability (2.100),

$$E[x_1] = \frac{1}{13} \sum_{k=1}^{13} k = \frac{1}{13} \frac{1+13}{2} 13 = 7. \quad (2.101)$$

If the first two draws were an ace and a two, then there will be one less ace and one less two in the remaining 50 cards. Therefore, the probability of an ace appearing in the third drawn will be reduced to

$$P(x_3 = 1) = \frac{3}{50}. \quad (2.102)$$

This is also the probability for a two to appear in the third draw. The conditional expected value of the score in the third draw is then given by

$$E[x_3|x_2 = 2|x_1 = 1] = \frac{4}{50} \sum_{k=3}^{13} k + (1+2)\frac{3}{50} = \frac{261}{50} = 5.22, \quad (2.103)$$

Comparing (2.101) with (2.103) we see that the conditional expected value of the score after two cards were drawn out of the deck is smaller than the expected value of the first draw. Sometimes $E[x_3|x_2 = 2|x_1 = 1]$ is written as $E[x_3|x_2 = 2, x_1 = 1]$ where the comma after 2 denotes "and".

Similarly to (2.98), for continuous random vectors the conditional probability density function $f_{\nu|\eta}(x, y)$ is related to the joint probability density function $f_{\nu\eta}(x, y)$ by

$$f_{\nu|\eta}(x, y) = \frac{f_{\nu\eta}(x, y)}{f_\eta(y)} \quad (2.104)$$

for $f_\eta(y) \neq 0$. The marginal probability density function for ν can be computed from the joint probability density function as

$$f_\nu(x) = \int_{y_{\min}}^{y_{\max}} f_{\nu\eta}(x, y) dy = \int_{y_{\min}}^{y_{\max}} f_{\nu|\eta}(x, y) f_\eta(y) dy. \quad (2.105)$$

The conditional expected value of a continuous random vector ν given that the continuous random vector η takes the value $\bar{\eta}$ is defined as

$$E[\nu|\eta = \bar{\eta}] = \int_{x_{\min}}^{x_{\max}} x f_{\nu|\eta}(x, \bar{\eta}) dx. \quad (2.106)$$

This definition is similar to the expected value (2.78) using a different probability density function. The conditional expectation operator satisfies the linearity properties (2.80) and (2.81).

2.10 Newton's Laws

Mechanics is one of the most important applications of vectors. The physical laws that describe mechanical motion are Newton's laws. Newton's second law states that the total external force is equal to the time rate of change of linear momentum, i.e,

$$\vec{F}_{ext} = \dot{\vec{p}}^I \quad (2.107)$$

where I is an inertial reference frame and linear momentum is defined as

$$\vec{p}^I = m\vec{v}^I, \quad (2.108)$$

where m is the mass and \vec{v}^I is the inertial velocity. The total external force \vec{F}_{ext} includes the weight $m\vec{g}$ plus all other external forces \vec{F} . Equation (2.107) can therefore be rewritten as

$$\vec{F} + m\vec{g} = \dot{\vec{p}}^I. \quad (2.109)$$

When the mass is constant, Newton's law can be rewritten as

$$\vec{f} = \frac{\vec{F}}{m} = \vec{a}^I - \vec{g}, \quad (2.110)$$

where $\vec{a}^I = \dot{\vec{v}}^I$ is the inertial acceleration. The vector \vec{g} for motion of mass m around a central body with mass M is given by Newton's law of gravitation

$$\vec{g} = -\frac{GM}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM}, \quad (2.111)$$

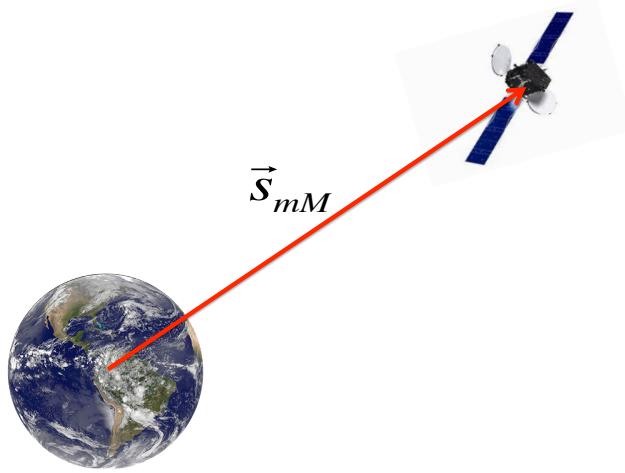


Figure 2.19: Newton’s gravitation law (pictures taken from en.wikipedia.org)

where $G = 6.6743 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the universal gravitational constant and \vec{s}_{mM} is the position of mass m with respect to mass M (see figure 2.19). One can define a new constant as $\mu = GM$ for ease of notation. The use of μ is a current practice in astrodynamics. Newton’s laws will be used later to describe orbital trajectories and for inertial navigation systems.

2.11 Notes and References

The notation used in this book adopts several elements of the notation proposed in references [10, 11]. In [10] vectors and tensors are used for robotics applications whereas [11] applies tensors and vectors to the modeling and simulation of aerospace vehicles. For a broad view of vectors and its applications to physics and engineering see [12].

2.12 Problems

Problem 2.12.1 *Figure 2.20 depicts a photovoltaic (PV) cell for the transformation of solar energy into electric energy, which is commonly used in*

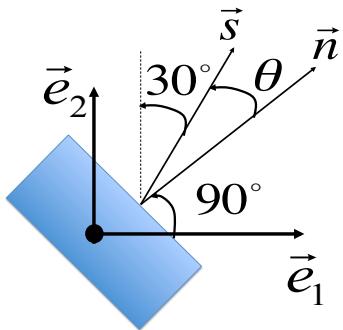


Figure 2.20: Photovoltaic cell of problem 2.12.1

spacecraft and satellites. The unit vector \vec{s} represents the direction from the PV cell to the sun, whereas the vector \vec{n} is the unit normal to the surface of the PV cell. Since the current generated in the PV cell depends on $\vec{s} \cdot \vec{n}$, one is usually interested in knowing the value of this dot product. Determine $\vec{s} \cdot \vec{n}$ and the angle of incidence θ for the situation depicted in figure 2.20 knowing that $\|\vec{s} \times \vec{n}\| = 0.5$.

Problem 2.12.2 In an inertial reference frame I with orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ there are two vectors described by

$$\begin{aligned}\vec{s}_1 &= -\vec{e}_1 + 2\vec{e}_2 - \vec{e}_3, \\ \vec{s}_2 &= 3\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3.\end{aligned}$$

Answer the following questions:

1. What is the norm of \vec{s}_1 ?
2. What are the coordinates of \vec{s}_2 in frame I ?
3. Compute $\vec{s}_1 \cdot \vec{s}_2$ and $\vec{s}_1 \times \vec{s}_2$.
4. What is the component of \vec{s}_1 along the direction of \vec{s}_2 ?
5. What is the projection of vector \vec{s}_1 into vector \vec{s}_2 ?

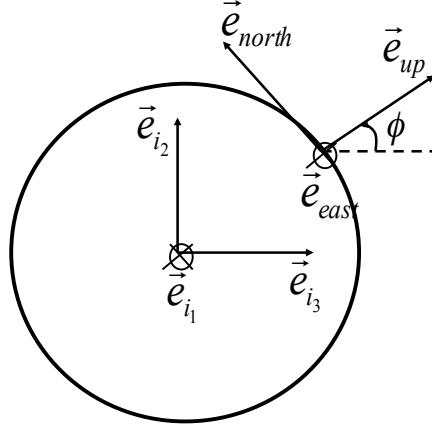


Figure 2.21: Reference frames for problem 2.12.3

6. Are \vec{s}_1 and \vec{s}_2 orthogonal vectors ?

Problem 2.12.3 Figure 2.21 represents a north-south cut view of the Earth at a given time. An inertial reference frame I with orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is attached to the center of the Earth. The frame E with orthonormal basis $(\vec{e}_{east}, \vec{e}_{north}, \vec{e}_{up})$ is attached to the rotating Earth at $\phi = 30^\circ$ latitude. For the same time instant shown in the figure compute the coordinate transformation matrices T_E^I and T_I^E .

Problem 2.12.4 A reference frame I with orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ has origin O . The frame E with orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$ also has origin O but is rotated by 60° around vector \vec{e}_3 . Answer the following questions:

1. Find the coordinates of the vector $\vec{v} = \vec{e}_1 + \vec{e}_2 - \vec{e}_3$ in frame E .
2. Find the coordinates of the vector $\vec{\omega} = \vec{i} - \vec{j} + 2\vec{e}_1$ in frame I .
3. Compute $\vec{v} \cdot \vec{\omega}$.
4. Find the coordinates of $\vec{v} \times \vec{\omega}$ in frame E .
5. Compute $\vec{\omega}\vec{v}$ and find its coordinates in frame I .

Chapter 3

Aircraft and Spacecraft Trajectories

Chapter 1 has reviewed vectors and applied them to Newton's laws. This chapter will use vectors parameterized by time to describe the trajectories of aircraft and spacecraft. Straight line trajectories are addressed in section 3.1 and curved trajectories are the subject of section 3.2. Section 3.3 focuses on orbital trajectories of spacecraft.

3.1 Straight Line Trajectories

Straight line trajectories of a point mass A must have a non-rotating velocity vector. Therefore, one can write

$$\vec{a}_A^I(t) = \dot{\vec{v}}_A^I(t) = k(t)\vec{v}_A^I(t), \quad (3.1)$$

for some real function $k(t)$, where $\vec{v}_A^I(t)$ is the inertial velocity vector of point mass A and $\vec{a}_A^I(t)$ is its inertial acceleration. One example of a straight line trajectory is the case of motion with a constant velocity vector \vec{v} . In the case of a constant velocity vector the equation (2.77) can be rewritten as

$$\vec{s}(t) = \vec{s}(0) + t\vec{v} \quad (3.2)$$

after one integrates the constant vector \vec{v} over time. For the case of the trajectory of point A the vector $\vec{s}(t)$ is written as $\vec{s}_{AO}(t)$, which is the position of a point A relative to a reference point O fixed in inertial space. Accordingly, the vector \vec{v} is the inertial velocity \vec{v}_A^I of point A . Equation (3.2) then

becomes

$$\vec{s}_{AO}(t) = \vec{s}_{AO}(0) + t\vec{v}_A^I. \quad (3.3)$$

From equation (3.3) we see that the kinematics equations relating position to velocity and velocity to acceleration are given by

$$\dot{\vec{s}}_{AO}(t) = \vec{v}_A^I, \quad (3.4)$$

$$\dot{\vec{v}}_A^I = 0. \quad (3.5)$$

Note that equation (3.5) is of the form (3.1) with $k(t) = 0$.

Example 3.1.1 Assume that an aircraft is climbing along a straight line trajectory on the longitudinal plane as depicted in figure 3.1. At time $t = 0$ the aircraft started its climbing from the origin of the inertial reference frame. The inertial climbing speed is 400 kmh^{-1} and the climbing angle is 25° . Write the equation of the trajectory of the aircraft as a function of time and also as the equation of a line on the longitudinal plane.

Solution: Using the unit vectors \vec{e}_x for the horizontal direction and \vec{e}_z for the vertical direction, the velocity vector of the aircraft in kmh^{-1} is obtained from the speed and climbing angle as

$$\vec{v}_A^I = 400 \cos\left(\frac{25\pi}{180}\right) \vec{e}_x + 400 \sin\left(\frac{25\pi}{180}\right) \vec{e}_y = 362.52\vec{e}_x + 169.05\vec{e}_z. \quad (3.6)$$

Replacing (3.6) and $\vec{s}_{AO}(0) = \vec{0}$ in equation (3.3), with time measured in hours, yields the equation of the trajectory as a function of time

$$\vec{s}_{AO}(t) = 362.52t\vec{e}_x + 169.05t\vec{e}_z \text{ km.}$$

The coordinates of the aircraft are

$$x = 362.52t, \quad (3.7)$$

$$z = 169.05t. \quad (3.8)$$

Solving equation (3.7) for t and replacing the result in equation (3.7) yields the equation of the line

$$z = \frac{169.05}{362.52}x.$$

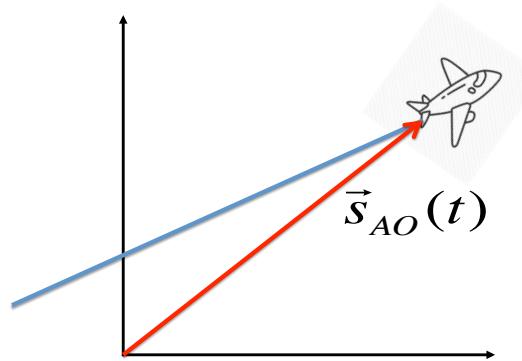


Figure 3.1: Straight line trajectory

3.2 Curved Trajectories

This section will address curved trajectories. We start by considering a spherical model for the Earth and then address the changes that must be made for an ellipsoidal model.

3.2.1 Spherical Model of the Earth

Figure 3.2 shows an aircraft flying at an altitude $r(t)$ above a spherical Earth with radius R . The figure depicts a cut in the direction north-south. The airplane is heading south and therefore the motion is clockwise. We assume that there is an inertial reference frame relative to which the motion is measured. The only axis of this reference frame represented in figure 3.2 is the vertical blue line denoting the radius R .

The first step to derive the kinematics equations for a curved trajectory is to choose a reference frame such that one can write the position vector in the simplest possible form. This happens when one of the axes of the reference frame is aligned with the position vector. We see in figure 3.2 that such a frame is formed by the set of axes $\vec{e}_r, \vec{e}_\theta$. The vector \vec{e}_{in} pointing into the

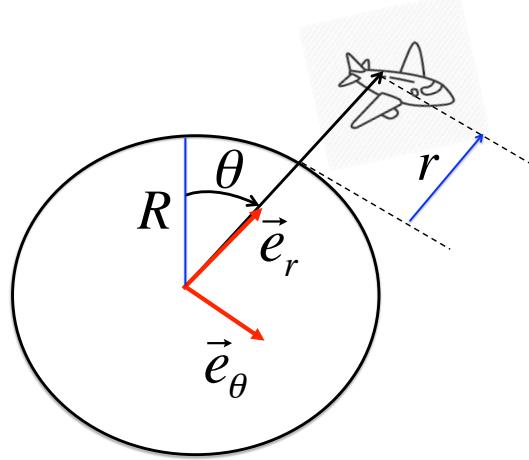


Figure 3.2: Curved trajectory above a spherical Earth

page completes the triad. Note that this reference frame rotates clockwise, as does the aircraft, so that the position vector is always aligned with \vec{e}_r . The position vector of the airplane A relative to the center of the circle O is

$$\vec{s}_{AO} = (R + r(t)) \vec{e}_r(t). \quad (3.9)$$

To compute the velocity vector we first determine the angular velocity vector. Its magnitude is the time rate of change of the angle described by the rotational motion. The direction is given by the right hand rule for clockwise motion, which is the converse direction of the one represented in figure 2.18. Therefore, the rotational velocity vector is $\vec{\omega}^e = \dot{\theta} \vec{e}_{in}$. Using this vector we apply the formula (2.74) to compute the derivative of the unit vectors $\vec{e}_r(t)$ and $\vec{e}_\theta(t)$ using the right hand rule (described in figure 2.10) to yield

$$\dot{\vec{e}}_r(t) = \dot{\theta} \vec{e}_\theta, \quad (3.10)$$

$$\dot{\vec{e}}_\theta(t) = -\dot{\theta} \vec{e}_r. \quad (3.11)$$

We can now take the derivative of the position vector (3.9) using the formula (1.44) for derivative of a product and equation (3.10) to write

$$\vec{v}_A^I(t) = \dot{\vec{s}}_{AO}(t) = \dot{r}(t) \vec{e}_r(t) + (R + r(t)) \dot{\vec{e}}_r(t) = v_r(t) \vec{e}_r(t) + v_\theta(t) \vec{e}_\theta(t), \quad (3.12)$$

where

$$v_r(t) = \dot{r}(t), \quad (3.13)$$

$$v_\theta(t) = (R + r(t)) \dot{\theta}(t). \quad (3.14)$$

Taking the derivative of the velocity vector (3.12) using the formulas for derivative of the sum (1.43) and product (1.44) gives the acceleration

$$\vec{a}_A^I(t) = \dot{\vec{v}}_A^I(t) = \dot{v}_r(t)\vec{e}_r(t) + v_r(t)\dot{\vec{e}}_r(t) + \dot{v}_\theta(t)\vec{e}_\theta(t) + v_\theta(t)\dot{\vec{e}}_\theta(t) \quad (3.15)$$

Replacing (3.10) and (3.11) in (3.15) yields

$$\vec{a}_A^I(t) = a_r(t)\vec{e}_r(t) + a_\theta(t)\vec{e}_\theta(t), \quad (3.16)$$

where

$$a_r(t) = \dot{v}_r(t) - v_\theta(t)\dot{\theta}(t), \quad (3.17)$$

$$a_\theta(t) = \dot{v}_\theta(t) + v_r(t)\dot{\theta}(t). \quad (3.18)$$

The kinematics equations are obtained by rewriting the equations (3.13), (3.14), (3.17), and (3.18) as

$$\dot{r}(t) = v_r(t), \quad (3.19)$$

$$\dot{\theta}(t) = \frac{v_\theta(t)}{R + r(t)}, \quad (3.20)$$

$$\dot{v}_r(t) = a_r(t) + \frac{v_\theta^2(t)}{R + r(t)}, \quad (3.21)$$

$$\dot{v}_\theta(t) = a_\theta(t) - \frac{v_r(t)v_\theta(t)}{R + r(t)}. \quad (3.22)$$

For a spacecraft trajectory under the effect of a Newtonian gravitational acceleration given by equation (2.19), the equations (3.21) and (3.22) become

$$\dot{v}_r(t) = -\frac{\mu}{(R + r(t))^2} + \frac{v_\theta^2(t)}{R + r(t)}, \quad (3.23)$$

$$\dot{v}_\theta(t) = -\frac{v_r(t)v_\theta(t)}{R + r(t)}, \quad (3.24)$$

where $\mu = GM$ is the gravitational parameter defined in chapter 1. One can change the independent variable from time t to the angle θ dividing each equation by $\dot{\theta}(t)$ given by (3.20), which yields

$$\frac{dv_r}{d\theta} = -\frac{\mu}{(R + r(t)) v_\theta} + v_\theta, \quad (3.25)$$

$$\frac{dv_\theta}{d\theta} = -v_r. \quad (3.26)$$

We will return to these equations in section 3.3 to show that under a conservation law we can integrate them.

Example 3.2.1 Assume that an aircraft is flying due south at 10,000 meters altitude and at a speed of 500 kilometers per hour. Consider a spherical Earth model with radius $R = 6,378,137$ meters. Write down the acceleration vector of the aircraft using the normal and tangential unit vectors as a reference frame.

Solution: Since the altitude is constant then $v_r(t) = \dot{v}_r(t) = 0$. Replacing $\dot{v}_r(t) = 0$ in equation (3.21) yields

$$a_r(t) = -\frac{v_\theta^2(t)}{R + r(t)} = -\frac{500^2}{6,378.137 + 10} = -39.135 \text{ kmh}^{-2} = -0.003 \text{ ms}^{-2}$$

Since the speed $v_\theta(t)$ is constant then $\dot{v}_\theta(t) = 0$. Given that $v_r(t) = \dot{v}_\theta(t) = 0$, then $a_\theta(t) = 0$ from (3.22). Therefore, from (3.16) we can write

$$\vec{a}_A^I(t) = a_r(t)\vec{e}_r(t) = -0.003\vec{e}_r(t) \text{ ms}^{-2}.$$

3.2.2 Ellipsoidal Model of the Earth

The ellipsoidal model of the Earth along a cut north-south in the direction β is shown in figure 3.3. Instead of a single radius of curvature R as in the spherical model, the ellipsoidal model has two radii of curvature. Recall that the meridians are curves connecting the North Pole to the South Pole. For travel along a meridian the radius of curvature R_m is called the meridian radius of curvature, which is the radius of the best fitting circle in the direction north-south (blue circle in figure 3.3). For travel east-west the radius of curvature R_p is called the prime radius of curvature. The value of R_p at a point is obtained by computing the normal to the ellipsoid at that point and

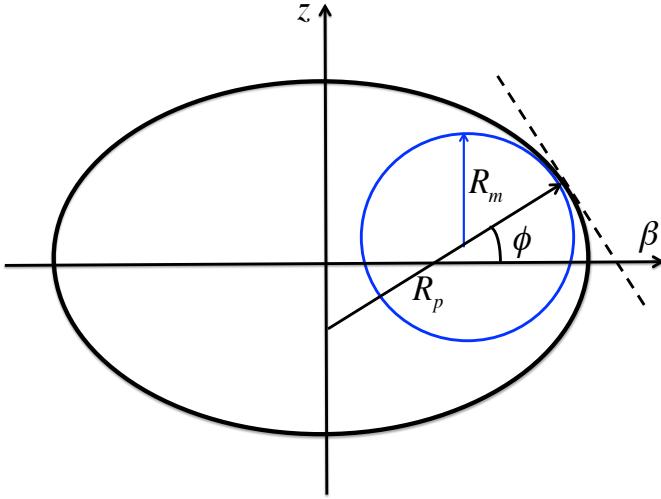


Figure 3.3: Ellipsoidal model of the Earth

then measuring the length of the normal line from the surface of the Earth to its intersection with the vertical z -axis (the Earth's axis of rotation). This is shown by the black arrow in figure 3.3 with R_p written below it. It is the radius of the best fitting circle in the direction east-west. In figure 3.3 the direction east is into the page and west is out of the page.

It can be seen from figure 3.3 that the radii of curvature depend on the latitude, being larger at the poles (flatter Earth) and smaller at the equator. The expressions for the radii of curvature are (see [13] for a proof)

$$R_m(\phi(t)) = \frac{r_e(1 - e^2)}{\left[1 - e^2 \sin^2(\phi(t))\right]^{\frac{3}{2}}}, \quad (3.27)$$

$$R_p(\phi(t)) = \frac{r_e}{\left[1 - e^2 \sin^2(\phi(t))\right]^{\frac{1}{2}}}. \quad (3.28)$$

For the World Geodetic System 1984 (WGS84) the values of the parameters in equations (3.27)–(3.28) are $r_e = 6,378,137\text{ m}$ for the semi-major axis, and $e = 0.081819191$ for the eccentricity of the ellipsoid.

To write the kinematics equations when one considers an ellipsoidal rather than spherical geometry the only difference is that the radius R would not

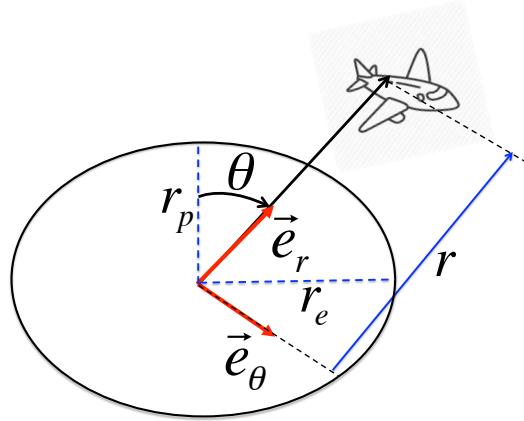


Figure 3.4: Curved trajectory above an ellipsoidal Earth

be constant anymore. However, observe that in the equations (3.19)–(3.22) the variable R always appears added to a time varying parameter $r(t)$, which was considered to be the altitude. Therefore, the same kinematics equations (3.21)–(3.22) can be used if we set $R = 0$ and interpret $r(t)$ as the distance of the aircraft to the origin of the reference frame (see figure 3.4).

3.3 Orbital Trajectories

Orbital trajectories are conic curves. They can be circles, ellipses, parabolas or hyperbolas. However, for spacecraft they are typically an ellipse or a circle. We have already reviewed the Cartesian equation for an ellipse in section 1.5. However, Cartesian coordinates are not the most appropriate to describe orbital trajectories. In this section we will first write down the equations of motion for a spacecraft in orbit and then use polar coordinates to describe elliptical orbits based on a set of conserved parameters.

Assuming that the spacecraft is not being thrusted, and using equation (2.19) for the gravitational acceleration between two masses (M and m) mov-

ing in an inertial reference frame with origin O , we can write the equations

$$\ddot{\vec{s}}_{mO} = -\frac{GM}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM}, \quad (3.29)$$

$$\ddot{\vec{s}}_{MO} = \frac{Gm}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM}. \quad (3.30)$$

Recall from figure 2.5 that the relative position vector is

$$\vec{s}_{mM} = \vec{s}_{mO} - \vec{s}_{MO}. \quad (3.31)$$

Therefore, subtracting (3.29) and (3.30) yields

$$\ddot{\vec{s}}_{mM} = -\frac{\mu}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM}, \quad (3.32)$$

where $\mu = G(M + m) \approx GM$ if M is much greater than m .

3.3.1 Conserved Trajectory Parameters

Equation (3.32) has important implications in the conservation of three parameters related to orbital trajectories. The first parameter is specific angular momentum, which is defined as

$$\vec{h} = \vec{s}_{mM} \times \dot{\vec{s}}_{mM}. \quad (3.33)$$

Taking the derivative of this parameter, and using (3.32) and the formula (1.44) for derivative of a product, as well as the fact that the cross product of a vector by itself is zero, yields

$$\dot{\vec{h}} = \dot{\vec{s}}_{mM} \times \dot{\vec{s}}_{mM} + \vec{s}_{mM} \times \left(-\frac{\mu}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM} \right) = \vec{0}, \quad (3.34)$$

which shows that \vec{h} is conserved, i.e., it is constant during the trajectory.

Another conserved parameter is the eccentricity vector defined as

$$\vec{e} = \frac{\dot{\vec{s}}_{mM} \times \vec{h}}{\mu} - \frac{\vec{s}_{mM}}{\|\vec{s}_{mM}\|}. \quad (3.35)$$

Taking the derivative over time of (3.35) using the rules for the derivative of the sum (1.43), product (1.44), and quotient (1.45) yields

$$\dot{\vec{e}} = \frac{\ddot{\vec{s}}_{mM} \times \vec{h} + \dot{\vec{s}}_{mM} \times \dot{\vec{h}}}{\mu} - \frac{\dot{\vec{s}}_{mM}}{\|\vec{s}_{mM}\|} + \frac{\vec{s}_{mM}}{\|\vec{s}_{mM}\|^2} \frac{d\|\vec{s}_{mM}\|}{dt}. \quad (3.36)$$

Using the chain rule (1.47), Pythagoras theorem (2.26), and the formula (1.46) for derivative of a power, one can compute the last term in (3.36) as

$$\frac{d\|\vec{s}_{mM}\|}{dt} = \frac{d\|\vec{s}_{mM}\|}{d\vec{s}_{mM}} \cdot \frac{d\vec{s}_{mM}}{dt} = \frac{\vec{s}_{mM}}{\|\vec{s}_{mM}\|} \cdot \dot{\vec{s}}_{mM}. \quad (3.37)$$

Using (3.33), (3.34), and (3.37), one can write (3.36) as

$$\dot{\vec{e}} = -\frac{\vec{s}_{mM} \times (\vec{s}_{mM} \times \dot{\vec{s}}_{mM})}{\|\vec{s}_{mM}\|^3} - \frac{\dot{\vec{s}}_{mM} \|\vec{s}_{mM}\|^2}{\|\vec{s}_{mM}\|^3} + \frac{\vec{s}_{mM} (\vec{s}_{mM} \cdot \dot{\vec{s}}_{mM})}{\|\vec{s}_{mM}\|^3} = \vec{0}. \quad (3.38)$$

The derivative (3.38) is zero because of the vector triple product identity (2.45). Therefore, from (3.38) we conclude that the eccentricity vector \vec{e} is also conserved when the spacecraft is not being thrusted.

The specific mechanical energy is defined as

$$\epsilon = \frac{\|\dot{\vec{s}}_{mM}\|^2}{2} - \frac{\mu}{\|\vec{s}_{mM}\|}. \quad (3.39)$$

Computing the derivative of the specific mechanical energy with respect to time using the formulas for the derivative of a sum (1.43), quotient (1.45), and power (1.46), and taking (3.32) and (3.37) into account yields

$$\dot{\epsilon} = \frac{\|\dot{\vec{s}}_{mM}\| \dot{\vec{s}}_{mM} \cdot \ddot{\vec{s}}_{mM}}{\|\dot{\vec{s}}_{mM}\|} + \frac{\mu \vec{s}_{mM} \cdot \dot{\vec{s}}_{mM}}{\|\vec{s}_{mM}\|^3} = 0. \quad (3.40)$$

Therefore, the specific mechanical energy is also conserved when the spacecraft is not being thrusted.

3.3.2 Orbital Trajectory Equations

We are now ready to derive the orbital trajectory equations in polar coordinates and to relate the equations to the three conserved parameters described in section 3.3.1. Figure 3.5 depicts the case of an elliptic orbital trajectory around the Earth. The perigee is the closest point and the apogee is the farthest point to the Earth while orbiting. There are two important differences between figures 3.5 and 3.4. The first difference is that the ellipse in figure 3.5 represents the trajectory of a satellite where the Earth is represented by one of the two foci of the ellipse, whereas in figure 3.4 the ellipse represents

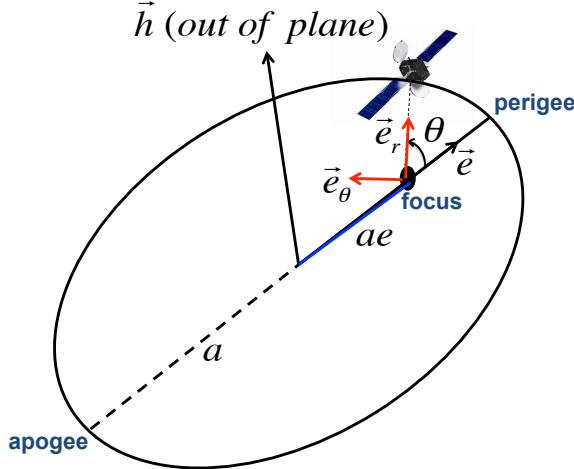


Figure 3.5: Elliptic orbit trajectory

the geometry of the Earth. The second difference is that the origin of the rotating reference frame is at one of the two foci of the ellipse in figure 3.5, whereas it is at the geometric center of the ellipse in figure 3.4. As before, we will consider the variable $r(t)$ to denote the distance from the origin of the coordinate system to the point mass m representing the satellite. Therefore, the position vector is written as

$$\vec{s}_{mM} = r(t)\vec{e}_r(t). \quad (3.41)$$

Similar to the derivation of equations (3.13)–(3.14), the inertial velocity vector is written as

$$\dot{\vec{s}}_{mM} = v_r(t)\vec{e}_r(t) + v_\theta(t)\vec{e}_\theta(t), \quad (3.42)$$

where

$$v_r(t) = \dot{r}(t), \quad (3.43)$$

$$v_\theta(t) = r(t)\dot{\theta}(t), \quad (3.44)$$

and $\theta(t)$ is called the true anomaly. Using the right hand rule, the specific angular momentum (3.33) is then

$$\vec{h} = (r(t)\vec{e}_r(t)) \times (\dot{r}(t)\vec{e}_r(t) + r(t)\dot{\theta}(t)\vec{e}_\theta(t)) = r^2(t)\dot{\theta}(t)\vec{e}_{out}, \quad (3.45)$$

where \vec{e}_{out} is pointing out of the page. From conservation of the specific angular momentum we can write

$$h = \|\vec{h}\| = r^2(t)\dot{\theta}(t) = r(t)v_\theta(t) = \text{constant}. \quad (3.46)$$

Using (3.45) and (3.46) the eccentricity vector (3.35) is

$$\vec{e} = \frac{(\dot{r}(t)\vec{e}_r(t) + r(t)\dot{\theta}(t)\vec{e}_\theta(t)) \times (h\vec{e}_{out})}{\mu} - \vec{e}_r, \quad (3.47)$$

which using (3.43) and (3.44) can be simplified to

$$\vec{e} = \frac{(r(t)\dot{\theta}(t)h - \mu)\vec{e}_r - h\dot{r}(t)\vec{e}_\theta(t)}{\mu} = \frac{(hv_\theta(t) - \mu)\vec{e}_r(t) - hv_r(t)\vec{e}_\theta(t)}{\mu}. \quad (3.48)$$

To determine the orbital position in polar coordinates we perform the dot product of the eccentricity vector (3.48) with the position vector (3.41) as

$$\mu er(t) \cos(\theta(t)) = \mu \vec{e} \cdot \vec{s}_{mM} = r(t) (r(t)\dot{\theta}(t)h - \mu) = h^2 - \mu r(t), \quad (3.49)$$

where we used equation (3.46) in the last equality. Defining

$$R_0 = h^2/\mu \quad (3.50)$$

and solving (3.49) for $r(t)$ the polar equation for the orbit position is

$$r(t) = \frac{R_0}{1 + e \cos(\theta(t))}. \quad (3.51)$$

The orbit is an ellipse when $0 < e < 1$, a circle when $e = 0$, a parabola when $e = 1$, and an hyperbola when $e > 1$.

The expression for the orbit specific mechanical energy can be obtained from the position as follows. First we compute $v_r(t)$ applying the chain rule of differentiation (1.47) to equation (3.51) using (3.50) and (3.51) to yield

$$v_r(t) = \frac{dr(t)}{d\theta(t)} \frac{d\theta(t)}{dt} = \frac{R_0 e \sin(\theta)}{(1 + e \cos(\theta))^2} \frac{h(1 + e \cos(\theta))^2}{R_0^2} = \frac{\mu}{h} e \sin(\theta(t)). \quad (3.52)$$

One can then determine $v_\theta(t) = r(t)\dot{\theta}(t)$ from equations (3.46), (3.50), and (3.51), which gives

$$v_\theta(t) = \frac{h}{r(t)} = \frac{h(1 + e \cos(\theta(t)))}{R_0} = \frac{\mu}{h} (1 + e \cos(\theta(t))). \quad (3.53)$$

Returning to the kinematics equations (3.25) and (3.26), and using the conservation of specific angular momentum from equation (3.46) with $r(t)$ replaced by $R + r(t)$, we see that the kinematics equations become affine. Note that (3.52) and (3.53) are solutions of the kinematics affine differential equations. The orbit speed squared can be obtained from (3.52) and (3.53) as

$$v^2(t) = v_r^2(t) + v_\theta^2(t) = \frac{\mu^2}{h^2} (1 + e^2 + 2e \cos(\theta(t))). \quad (3.54)$$

The specific mechanical energy is computed using equations (3.39), (3.50), (3.51) and (3.54), yielding

$$\epsilon = \frac{\|\dot{\vec{s}}_{mM}\|^2}{2} - \frac{\mu}{\|\vec{s}_{mM}\|} = -\frac{\mu}{2R_0} (1 - e^2) = -\frac{\mu}{2a}, \quad (3.55)$$

where

$$a = \frac{R_0}{1 - e^2} \quad (3.56)$$

for $e \neq 1$. Note that for $\theta(t) = 0$ the equation (3.5) should yield $r(t) = a(1-e)$ for an elliptic orbit as shown in figure 3.4, which leads to the equation (3.56). For elliptic orbits, a is the length of the semi-major axis. For circular orbits, $e = 0$ and $a = R_0$ is the radius.

Taking into account (3.50), (3.52), and (3.53), the eccentricity vector (3.48) can be written in polar coordinates as

$$\vec{e} = e \cos(\theta(t)) \vec{e}_r(t) - e \sin(\theta(t)) \vec{e}_\theta(t), \quad (3.57)$$

which represents a vector of magnitude e making an angle $-\theta(t)$ with the unit vector $\vec{e}_r(t)$. This implies that the eccentricity vector is aligned with the semi-major axis as shown in figure 3.5. Note also that the vector \vec{e} in (3.57) is on the orbital plane and the vector \vec{h} in (3.45) is orthogonal to the orbital plane. Therefore

$$\vec{h} \cdot \vec{e} = 0. \quad (3.58)$$

Example 3.3.1 A satellite is orbiting Earth with a perigee radius of $r_p = 15,000 \text{ km}$ and an apogee radius of $r_a = 25,000 \text{ km}$. If $r(0) = 20,000 \text{ km}$ what is the speed $v(0)$?

Solution: Recall the definition of ellipse in chapter 1 as the set of points whose sum of the distances to two foci is constant. When the point in the ellipse is the perigee or the apogee, one can see geometrically from figure 3.5 that the sum of the distances to the two foci is equal to $2a$. One can then write

$$2a = r_p + r_a = 15,000 + 25,000 = 40,000 \text{ km}.$$

Therefore, the length of the semi-major axis is $a = 20,000 \text{ km}$, which implies that the specific mechanic energy of the orbit given by equation (3.55) is

$$\epsilon = -\frac{\mu}{2a} = \frac{3.99 \times 10^{14} \text{ m}^3 \text{s}^{-1}}{4 \times 10^7 \text{ m}} = -9.975 \times 10^6 \text{ m}^2 \text{s}^{-2}. \quad (3.59)$$

Evaluating equation (3.55) at $t = 0$ and solving for $v(0)$ using the given $r(0)$ and the value of ϵ from (3.59) yields

$$v(0) = \sqrt{2 \left(\epsilon + \frac{\mu}{r(0)} \right)} = \sqrt{2 \left(-9.975 + \frac{39.9}{2} \right) \times 10^6} = 4.466 \text{ km/s}.$$

3.3.3 Orbital Velocity for Circular Orbits

For circular orbits the distance to the center of the coordinate system is constant, and therefore $r(t) = r$ is constant, which implies $v_r = \dot{v}_r(t) = 0$. Replacing $\dot{v}_r(t) = 0$ in the kinematics equation (3.23) yields

$$v_\theta(t) = \sqrt{\frac{\mu}{R+r}}. \quad (3.60)$$

Example 3.3.2 A satellite has a circular orbit radius of 26,000 km. What is the speed of the satellite ?

The speed is obtained by setting $R+r = 26,000 \text{ km}$ in equation (3.60) yielding

$$v_\theta(t) = \sqrt{\frac{3.99 \times 10^{14} \text{ m}^3 \text{s}^{-2}}{2.6 \times 10^7 \text{ m}}} = 3.917 \text{ km/s}.$$

3.4 Escape Velocity

The escape velocity is the minimum velocity that a rocket must have leaving the surface of the Earth so that it escapes gravitational attraction and cannot come back unless external energy is added. To escape gravitational attraction the rocket would have to reach an infinite distance of the Earth. Additionally, to have the minimum velocity at departure it would have to arrive at infinity with zero speed. Since it is assumed that no external force is acting on the rocket we can use conservation of specific mechanical energy. Equating the specific mechanic energy at departure with the specific mechanic energy at infinity, which should be zero, and using equation (3.39) yields

$$\epsilon_0 = \epsilon_\infty \Leftrightarrow \frac{v_e^2}{2} - \frac{\mu}{R} = 0, \quad (3.61)$$

where v_e is the escape velocity and R is the radius of the Earth. Solving (3.61) for the escape velocity v_e yields

$$v_e = \sqrt{\frac{2\mu}{R}}. \quad (3.62)$$

Example 3.4.1 What is the escape velocity for planet Earth ?

Solution: For planet Earth we have $R = 6,378,137 \text{ m}$ and $\mu = 3.99 \times 10^{14} \text{ m}^3\text{s}^{-2}$. Replacing these values in equation (3.62) yields

$$v_e = \sqrt{\frac{2 \times 3.99 \times 10^{14}}{6,378,137}} = 11.2 \text{ km/s.}$$

3.5 Notes and References

Detailed information on flight vehicles can be found in [14]. References [15] and [16] have a broad view of space mechanics and orbital trajectories.

3.6 Problems

Problem 3.6.1 An inertial reference frame has an orthonormal basis (\vec{e}_1, \vec{e}_2) and origin I . Three airplanes A, B, C are flying parallel line trajectories. The

position of airplane A relative to I as a function of time t is given by

$$\vec{s}_{AI} = t\vec{e}_1 - t\vec{e}_2 \text{ [km].}$$

Airplanes B, C fly trajectories that are parallel to the trajectory of A. The distance between the trajectories of airplanes A and B is 500 meters with B flying above A. The distance between the trajectories of airplanes A and C is also 500 meters with C flying below A. At time t = 0 all three airplanes have the coordinate along \vec{e}_1 equal to zero. Write down the equations of the straight line trajectories of B and C.

Problem 3.6.2 In the same reference frame of problem 3.6.1, two airplanes D and E are flying straight line trajectories. The position of airplane D relative to I in kilometers as a function of time t is given by

$$\vec{s}_{DI} = t\vec{e}_1 + 2t\vec{e}_2 \text{ [km].}$$

The initial position of the airplane E relative to I is

$$\vec{s}_{EI} = 0.5t\vec{e}_2 \text{ [km].}$$

The inertial velocity of airplane E is constant and given by

$$\vec{v}_E^I = \vec{e}_1 - \vec{e}_2 \text{ [kms}^{-1}\text{].}$$

Will airplanes D and E ever collide ? Justify your answer.

Problem 3.6.3 In a fixed reference frame I, two airplanes D and E are flying straight line trajectories. Assume that time is measured in hours and position is measured in kilometers. The position of airplane D relative to I as a function of time t is given by

$$\vec{s}_{DI}(t) = t\vec{e}_1 + 2(1-t)\vec{e}_2.$$

Find the equation of the trajectory of aircraft E that satisfies the following constraints:

- It starts at the origin for zero time.
- It is orthogonal to the trajectory of D.

- One hour after it started it intercepts the trajectory left behind by D .

Problem 3.6.4 [17] A reference frame I with orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ has origin O . An automobile A and an airplane P have the following trajectories:

$$\begin{aligned}\vec{s}_{AO} &= 30t\vec{e}_1, \\ {}^I[\vec{s}_{PO}]^T &= [1000 \ 90t \ 1000].\end{aligned}$$

The automobile has a reference frame A attached to it. Answer the following questions:

1. What is the velocity of the airplane measured in the automobile's reference frame with coordinates expressed in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$?
2. How close does the airplane get to the automobile ?

Problem 3.6.5 Assume that an aircraft is flying due south at 10000 meters altitude and at a speed of 500 kilometers per hour. Consider a spherical Earth model with radius equal to 6378.137 kilometers with gravitational acceleration equal to 9.8 meters per second squared. Write down the acceleration vector of the aircraft using the normal and tangential unit vectors as a reference frame.

Problem 3.6.6 [18] A synchronous Earth satellite appears stationary from Earth while describing its orbit over the equator. This means that it rotates at the same speed of the Earth, which is one full rotation in 24 hours. It also rotates in the same direction as the Earth, i.e., west-to-east. What are the altitude and the velocity of a synchronous Earth satellite ?

Problem 3.6.7 What is the escape velocity in the moon ?

Chapter 4

Dead Reckoning and Inertial Navigation

This chapter is about dead reckoning and inertial navigation systems. These systems measure either velocity or acceleration and then integrate it one or two times, respectively, to determine the position. The chapter starts by first deriving the dead reckoning equations in section 4.1 and then the inertial navigation kinematics equations are addressed in section 4.2. The chapter closes with a discussion of the inertial navigation mechanization equations in section 4.3.

4.1 Dead Reckoning

Dead reckoning is a method to compute or to deduce the current position of a vehicle knowing the previous position and integrating the velocity vector measurement. The next sections will derive the dead reckoning equations for a spherical model and for an ellipsoidal model of the Earth.

4.1.1 Equations for a Spherical Earth Model

For a spherical Earth model, all trajectories going either south or north occur along the meridians, which are circles whose radius is the radius R of the Earth's sphere. Meridians are therefore called great circles. For travel at an altitude $r(t)$ the radius of the circular trajectory is $R + r(t)$. The trajectories going either east or west occur along circles of constant latitude.

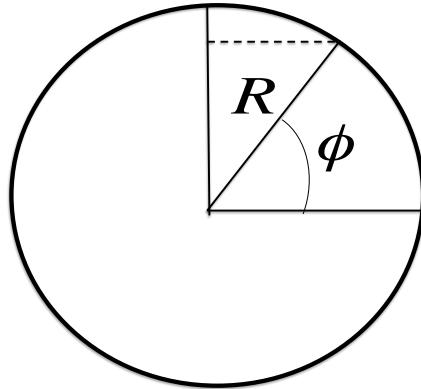


Figure 4.1: Radius of constant latitude circle (dashed line)

The radius of a constant latitude circle is $R \cos \phi(t)$, where $\phi(t)$ is the latitude (see dashed line in figure 4.1). For travel at an altitude $r(t)$ the radius of the circular trajectory is $(R + r(t)) \cos \phi(t)$. We have seen in section 3.2 that circular trajectories satisfy the equation

$$v_\theta(t) = R_e \dot{\theta}(t), \quad (4.1)$$

where R_e is the radius of the circular trajectory, $v_\theta(t)$ is the tangential velocity, and $\theta(t)$ is the angular displacement of the trajectory at time t . When determining positions for travel on a spherical model of the Earth one uses spherical coordinates of latitude $\phi(t)$, longitude $\lambda(t)$, and radius $R + r(t)$ (or simply altitude $r(t)$). The latitude is positive north with zero latitude at the equator. The longitude is positive east with zero longitude at the Greenwich meridian. Therefore, given these conventions and based on equation (4.1), the dead reckoning equation for traveling in the direction north-south is

$$v_N(t) = (R + r(t)) \dot{\phi}(t), \quad (4.2)$$

where $v_N(t)$ is the velocity north. The equation for traveling in the direction east-west is

$$v_E(t) = (R + r(t)) \cos \phi(t) \dot{\lambda}(t), \quad (4.3)$$

where $v_E(t)$ is the velocity east.

4.1.2 Equations for Ellipsoidal Earth Model

Rewriting equations (4.2)–(4.3) using $R_m(\phi(t))$ and $R_p(\phi(t))$ from (3.27)–(3.28), the dead reckoning equation for traveling in the direction north-south for ellipsoidal geometry is

$$v_N(t) = (R_m(\phi(t)) + r(t)) \dot{\phi}(t), \quad (4.4)$$

and the equation for traveling in the direction east-west is

$$v_E(t) = (R_p(\phi(t)) + r(t)) \cos \phi(t) \dot{\lambda}(t). \quad (4.5)$$

Example 4.1.1 An airplane is flying out of Montreal due east on a constant latitude circle. The speed is 400 knots and the altitude is 5.4 nautical miles. If the total flight time is 6 hours what is the longitude of the destination?

Solution: Montreal's latitude is $45.5^\circ N$ and the longitude is $73.6^\circ W$. Since the airplane is flying east we need to compute the prime radius of curvature

$$R_p(45.5^\circ) = \frac{6378137}{[1 - (0.081819191)^2 \sin^2(45.5^\circ)]^{1/2}} = 6389026 \text{ m} = 3450 \text{ n.m.}$$

Using equation (4.5)

$$\dot{\lambda}(t) = \frac{400}{(3450 + 5.4) \cos(45.5^\circ)} = 0.165158 \text{ rad/sec.}$$

Integrating this equation using the Fundamental Theorem of Calculus (2.77) between $t = 0$ and $t = 6$ hours yields

$$\lambda(6) = \lambda(0) + 0.165158(6) = -\frac{73.6\pi}{180} + 0.99 = -0.2936 \text{ rad} = 16.82^\circ W$$

4.2 Inertial Navigation Systems

There are two types of inertial navigation systems: gimbaled and strap-down. A gimbaled INS consists of a stable or gimbaled platform with three accelerometers and three gyros mounted in three orthogonal axes. The platform is maintained isolated from the vehicle and is torqued so that the axes

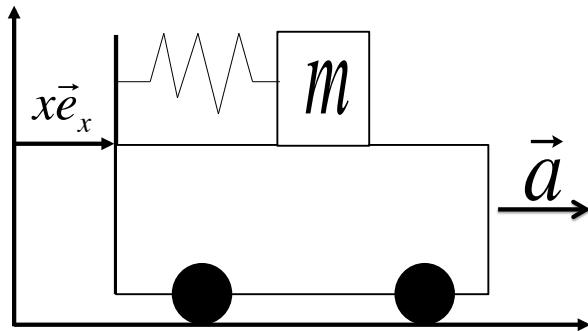


Figure 4.2: Accelerometer

of the three gyros keep a fixed orientation in inertial space. A measure of the attitude of the vehicle is obtained by reading the three gimbal angles. Gimbaled INS is less used in current applications than strapdown systems due to the wear and tear of the mechanical linkages associated with gimbaled systems. A strapdown INS straps the three accelerometers and three gyros directly to the vehicle. Instead of mechanically torquing a platform and reading three gimbal angles, the orientation of the vehicle is measured by the three gyros. The measurement of the gyros and accelerometers is fused in a computer or a microprocessor, which then estimates the position and orientation of the vehicle. This chapter will address only strapdown INS. The next two sections discusses the sensor models.

4.2.1 Accelerometer Model

The technology to build accelerometers is vast. Although it is not always the case that an accelerometer is built with a spring, for visualization purposes we will consider a mass-spring system. Assume that a proof mass is mounted on top of a vehicle connected to a spring as in figure 4.2. As the vehicle moves the spring attached to the mass will extend or compress, unless an external

force \vec{F} is applied to it to keep the mass steady. Neglecting friction, Newton's law yields equation (2.110), which is repeated here for convenience,

$$\vec{f} = \frac{\vec{F}}{m} = \vec{a}^I - \vec{g}. \quad (4.6)$$

Since the proof mass m is known, measuring the specific force \vec{f} is equivalent to measuring the difference $\vec{a}^I - \vec{g}$. Unfortunately the measured specific force \vec{f}_m is corrupted by a measurement error, which is typically in the form of an additive bias (constant) vector term \vec{b} . Denoting the axis along which the accelerometer is mounted as \vec{e}_m , the measurement equation is the projection

$$a_m = \vec{f}_m \cdot \vec{e}_m = (\vec{a}^I - \vec{g}) \cdot \vec{e}_m + \vec{b} \cdot \vec{e}_m. \quad (4.7)$$

The acceleration measurement is usually displayed in number of g 's. Therefore, a measurement of zero would imply an acceleration of the vehicle in free fall if no measurement errors were present.

From equation (4.7) we see that to recover the inertial acceleration \vec{a}^I after measuring the specific acceleration \vec{f} it is important to have a model of the gravitational acceleration \vec{g} . One such model is Newton's gravitation law from equation (2.19), repeated here for convenience,

$$\vec{g} = -\frac{GM}{\|\vec{s}_{mM}\|^3} \vec{s}_{mM}. \quad (4.8)$$

The magnitude of the gravitational acceleration at an altitude r above the surface of the Earth is

$$g = \frac{GM}{(R+r)^2}, \quad (4.9)$$

where R is the radius of the Earth. Since at the surface of the Earth the magnitude of gravitational acceleration is $g_0 = 9.8 \text{ ms}^{-2}$, one can divide equation (4.9) by g_0 to obtain an expression for the magnitude of gravitational acceleration at an altitude r as a function of g_0 in the form

$$g = \left(\frac{R_0}{R_0 + r} \right)^2 g_0. \quad (4.10)$$

Example 4.2.1 For the accelerometer in figure 4.2 assume that the measurement of acceleration has a bias (constant) error equal to b_x and a scaling

factor of $(1 + k)$. Write down the kinematic equations and integrate them after replacing the true acceleration by the measured acceleration. How does the position error grow with time if $b_x = 0.1\text{ms}^{-2}$ and $k = 0$? In the additional case of the position and velocity errors at the initial time being zero, what would be the position error after two seconds?

Solution: As indicated in figure 4.2 the position vector is $\vec{s} = x\vec{e}_x$. Since \vec{e}_x is not rotating, then $\dot{\vec{e}}_x = \vec{0}$ and we can write

$$\begin{aligned}\dot{\vec{s}} &= \dot{x}\vec{e}_x = v\vec{e}_x = \vec{v}, \\ \dot{\vec{v}} &= \dot{v}\vec{e}_x = a\vec{e}_x = \vec{a}.\end{aligned}$$

Therefore, the kinematics equations are

$$\dot{x} = v, \quad (4.11)$$

$$\dot{v} = a. \quad (4.12)$$

The variables x and v are called state variables because the state of a mechanical system can be determined by knowledge of its position and velocity. Replacing the true acceleration a by the measured acceleration a_m , and using $\vec{e}_m = \vec{e}_x$ in the measurement equation (4.7), yields

$$\dot{\hat{x}} = \hat{v}, \quad (4.13)$$

$$\dot{\hat{v}} = (1 + k)a + b_x, \quad (4.14)$$

because $\vec{g} \cdot \vec{e}_x = 0$ given that \vec{g} acts along the vertical and \vec{e}_x is the horizontal unit vector. Note that new (hat) variable names must be used since the right hand side of equation (4.14) is different from the one in equation (4.12), which leads to different values of the state variables. Integrating equation (4.12) using the Fundamental Theorem of Calculus (2.77) (with v replaced by $(1 + k)a + b_x$ and s replaced by v) and equation (1.48) yields

$$\hat{v}(t) = \hat{v}(0) + b_x t + (1+k) \int_0^t a(\tau) d\tau = \hat{v}(0) + b_x t + (1+k)(v(t) - v(0)). \quad (4.15)$$

Defining the error $\delta v(t) = \hat{v}(t) - v(t)$ one can rewrite equation (4.15) as

$$\delta v(t) = \delta v(0) + b_x t + k(v(t) - v(0)). \quad (4.16)$$

Integrating equation (4.16) using (1.46) and (1.48) yields

$$\int_0^t \delta v(\tau) d\tau = (\delta v(0) - kv(0)) t + b_x \frac{t^2}{2} + k \int_0^t v(\tau) d\tau. \quad (4.17)$$

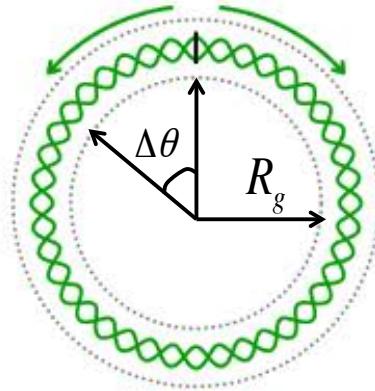


Figure 4.3: Ring laser gyro (adapted from en.wikipedia.org)

Using the Fundamental Theorem of Calculus (2.77) again gives

$$\delta x(t) = \delta x(0) + (\delta v(0) - kv(0)) t + b_x \frac{t^2}{2} + k(x(t) - x(0)), \quad (4.18)$$

where $\delta x(t) = \hat{x}(t) - x(t)$. Replacing the values of $b_x = 0.1ms^{-2}$ and $k = 0$ in (4.18) the position error in meters is

$$\delta x(t) = \delta x(0) + \delta v(0)t + 0.05t^2, \quad (4.19)$$

which yields an error of 20 cm after two seconds when $\delta x(0) = 0, \delta v(0) = 0$. We see from (4.19) that the position error grows quadratically with time, which is one of the drawbacks of INS systems. If the scaling error parameter k was not zero, we see from equation (4.18) that the error term $k(x(t) - x(0))$ would grow with the distance travelled. For large distances this error could also be significant, even when k is small and b_x is zero.

4.2.2 Gyroscope Model

Inertial navigation gyros can be manufactured using different technologies. One of the most common is the ring laser gyro. In a ring laser gyro two laser beams of the same wavelength λ are emitted around a ring in opposite

directions. They interfere with each other and form a standing wave pattern as shown in green color in figure 4.3. The standing wave is fixed in inertial space while the vehicle where the gyro is mounted rotates by an angle $\Delta\theta$ in a time interval Δt . From equation (1.1) the number of half wave fringes N_f swept in the rotation verifies the equation

$$N_f \frac{\lambda}{2} = R_g \Delta\theta, \quad (4.20)$$

where R_g is the radius of the ring. If the perimeter of the ring is L_g then

$$N_t \frac{\lambda}{2} = 2\pi R_g = L_g, \quad (4.21)$$

where N_t is the total number of fringes in the perimeter L_g . Dividing (4.20) by (4.21) yields

$$\frac{N_f \frac{\lambda}{2}}{N_t \frac{\lambda}{2}} = \frac{R_g \Delta\theta}{L_g}. \quad (4.22)$$

Solving (4.22) for N_f yields

$$N_f = \frac{N_t R_g \Delta\theta}{L_g}. \quad (4.23)$$

Replacing N_t from (4.21) in (4.23) and solving for $\Delta\theta$ yields

$$\Delta\theta = \frac{\lambda L_g}{4 A_g} N_f, \quad (4.24)$$

where $A_g = \pi R_g^2$ is the area of the ring. From equation (4.24) we see that a measurement of N_f yields a value for $\Delta\theta$ because the parameters λ , A_g , and L_g are known. With a value of $\Delta\theta$ measured in an interval Δt one can have an estimate of the magnitude of the angular rate as

$$\omega = \frac{\Delta\theta}{\Delta t}. \quad (4.25)$$

Since an estimate of the angular rate is provided these ring laser gyros are called rate gyros. A rate gyro measures the projection of the angular velocity vector $\vec{\omega}^{BI}$ of body B with respect to inertial space along the axis \vec{e}_m in which the gyro is mounted. For a ring laser gyro the measurement axis \vec{e}_m is orthogonal to the ring. Unfortunately, as in the case of accelerometers the

measurement ω_m of the gyro is corrupted by measurement error, which typically appears in the form of an additive bias b_g . The measurement equation for the rate gyroscope is then

$$\omega_m(t) = \vec{\omega}^{BI} \cdot \vec{e}_m + b_g. \quad (4.26)$$

Example 4.2.2 Assume that a ring laser gyro is attached to the ground at a latitude of 45 degrees. The gyro is mounted such that the ring is orthogonal to the vertical. If the measurement has a bias of one tenth of a degree per hour what would be the estimated angle $\Delta\hat{\theta} = \hat{\theta} - \hat{\theta}_0$ after six minutes?

Solution: Since the gyro is attached to the Earth it measures the angular velocity of the Earth along the vertical line at 45 degrees of latitude. The rotation rate of Earth is 360 degrees every 24 hours along the axis \vec{e}_N from the South Pole to the North Pole. Therefore, the Earth's angular velocity vector is $\vec{\omega}^{BI} = 15\vec{e}_N$ in degrees per hour. The direction of the vertical line at 45 degrees latitude makes an angle of $90 - 45 = 45$ degrees with the rotation axis of the Earth. Using equation (4.26) then gives

$$\omega_m(t) = 15\vec{e}_N \cdot \vec{e}_m + 0.1 = 15 \cos(45^\circ) + 0.1 \text{ } {}^\circ/\text{h}.$$

Integrating this equation yields

$$\Delta\hat{\theta}(t) = \left(\frac{15}{\sqrt{2}} + 0.1 \right) t = 10.707t \text{ } {}^\circ/\text{h}.$$

Replacing time by 0.1 hours gives

$$\Delta\hat{\theta}(0.1) = 10.707(0.1) = 1.07^\circ.$$

4.2.3 Kinematics and Coriolis Theorem

The kinematics equations used for INS navigation and the accelerometer specific force equation will be reviewed in this section. We recall that a simple 1D case of INS kinematics equations was already considered in example 4.2.1. The reader is invited to review it first before proceeding with the material of this section. The next example looks at a case of trajectories of a flying aircraft above the Earth on a longitudinal plane.

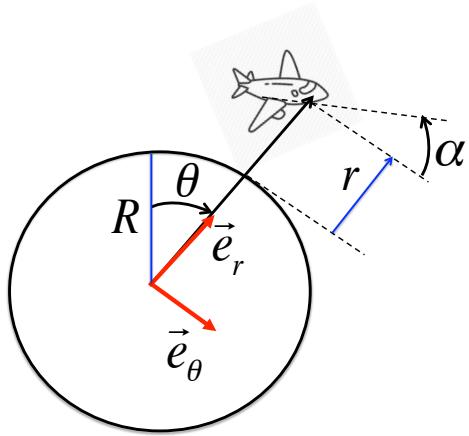


Figure 4.4: Curved trajectory above a spherical Earth

Example 4.2.3 Find the kinematics equations for the system in figure 4.4 assuming that the aircraft flies at constant altitude and that the Earth's angular rate is small enough when compared with $\dot{\theta}(t)$. Use $\vec{e}_\theta, \vec{e}_r, \vec{e}_{out}$, as the navigation frame.

Solution: In section 3.2 the kinematics equations for a point mass describing a trajectory at an altitude $r(t)$ above a spherical model of the Earth were derived as

$$\dot{r}(t) = v_r(t), \quad (4.27)$$

$$\dot{\theta}(t) = \frac{v_\theta(t)}{R + r(t)}, \quad (4.28)$$

$$\dot{v}_r(t) = a_r(t) + \frac{v_\theta^2(t)}{R + r(t)}, \quad (4.29)$$

$$\dot{v}_\theta(t) = a_\theta(t) - \frac{v_r(t)v_\theta(t)}{R + r(t)}, \quad (4.30)$$

where $r(t)$ is the altitude, $\theta(t)$ is the angular displacement, $v_r(t)$ is the radial inertial velocity, $v_\theta(t)$ is the tangential inertial velocity, $a_r(t)$ is the radial inertial acceleration, $a_\theta(t)$ is the tangential inertial acceleration, and R is the

spherical radius of the Earth (see figure 4.4). For INS navigation we need to consider a rigid body instead of a point mass. Therefore, one must add the extra degree of freedom of body rotation to the equations (4.27)–(4.30). To do this we add the pitch angle of the aircraft represented by the angle α in figure 4.4, which depicts trajectories on a longitudinal plane that cuts the Earth along a north-south direction. Note that we must consider both the changes of the angle θ and the changes of the angle α to compute the angular velocity of the aircraft relative to inertial space. From figure 4.4 we see that the aircraft rotates clockwise by θ and anticlockwise by α . Considering the rotating frame $\vec{e}_\theta, \vec{e}_r, \vec{e}_{out}$, where \vec{e}_{out} is out of the page, as well as the right hand rule in figure 2.18, the angular velocity vector of the airplane relative to inertial space is

$$\vec{\omega}^{BI} = \omega \vec{e}_{out} = (\dot{\alpha} - \dot{\theta}) \vec{e}_{out}. \quad (4.31)$$

Therefore, we should add the equation

$$\omega = \dot{\alpha} - \dot{\theta} \quad (4.32)$$

to the equations (4.27)–(4.30) to form the INS kinematics equations.

Having reviewed the 1D case in example 4.2.1 and the 2D case in example 4.2.3, we are now ready to derive the general 3D form of the INS kinematics equations. To do this we start by recalling from section 3.2 that the first step in deriving the kinematics equations for motion of a vehicle B relative to an inertially fixed point O is to pick a reference frame. We should choose a reference frame A that follows the body in its rotation due to the curvature of the Earth. This choice allows us to write the position vector aligned with one of the reference frame's unit vectors $e_i(t)$ in the form

$$\vec{s}_{BO}(t) = r(t) \vec{e}_i(t). \quad (4.33)$$

The next step is to take the derivative of the position vector (4.33) over time to get the inertial velocity vector using the rule (1.44) for derivative of the product to write

$$\dot{\vec{s}}_{BO}(t) = \dot{r}(t) \vec{e}_i(t) + r(t) \dot{\vec{e}}_i(t). \quad (4.34)$$

Using the rule for derivative of unit vectors (2.74) to compute the second term of (4.34) yields

$$\dot{\vec{s}}_{BO}(t) = \dot{r}(t) \vec{e}_i(t) + r(t) \vec{\omega}^{AI}(t) \times \vec{e}_i(t). \quad (4.35)$$

Since $r(t)$ is a scalar, using (4.33) equation (4.35) can be rewritten as

$$\dot{\vec{s}}_{BO}(t) = \dot{r}(t)\vec{e}_i(t) + \vec{\omega}^{AI}(t) \times (r(t)\vec{e}_i(t)) = \dot{r}(t)\vec{e}_i(t) + \vec{\omega}^{AI}(t) \times \vec{s}_{BO}(t). \quad (4.36)$$

This is an interesting result. It tells us that the derivative of a vector \vec{s}_{BO} in inertial space is equal to the sum of two terms. The second term is a cross product of the angular velocity of reference frame A with the vector itself. The first term is the derivative of the vector \vec{s}_{BO} as measured by an observer that moves solidary with the reference frame A . In this case the observer does not see the unit vectors rotating because he/she is rotating with them at the same rate. This result is called Coriolis Theorem and is written as

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} \Big|_I = \frac{d\vec{r}}{dt} \Big|_A + \vec{\omega}^{AI} \times \vec{r}, \quad (4.37)$$

where \vec{r} is any arbitrary vector. The subscripts A and I denote the reference frames with which the observer taking the derivative should be solidary. Comparing equations (4.36) and (4.37) we can see that the result (4.37) is valid when $\vec{r} = \vec{s}_{BO}(t)$, A is the rotating frame, and I is the inertial frame. However, it can be shown that the Coriolis Theorem is applicable to an arbitrary vector \vec{r} and to any two arbitrary reference frames A and I , not necessarily a rotating frame and an inertial frame. Using the Coriolis Theorem the general velocity equation will be derived in the next section.

4.2.4 INS Velocity Equation and Specific Force

We will denote frame I as the inertial reference frame and frame A will be changed to E , which is attached to the Earth. Using these reference frames, we can rewrite equation (4.37) for $\vec{r} = \vec{s}_{BO}(t)$ as

$$\frac{d\vec{s}_{BO}}{dt} \Big|_I = \vec{v}^I(t) = \vec{v}^E(t) + \vec{\omega}^{EI}(t) \times \vec{s}_{BO}(t), \quad (4.38)$$

where $\vec{v}^I(t)$ is the inertial velocity of point B and

$$\vec{v}^E(t) = \frac{d\vec{s}_{BO}(t)}{dt} \Big|_E \quad (4.39)$$

is the velocity of point B relative to Earth. Taking the derivative of (4.38) with respect to time using the addition and product rules (1.43)-(1.44) yields

$$\vec{a}^I(t) = \frac{d\vec{v}^I(t)}{dt} \Big|_I = \frac{d\vec{v}^E(t)}{dt} \Big|_I + \vec{\omega}^{EI}(t) \times \frac{d\vec{s}_{BO}(t)}{dt} \Big|_I, \quad (4.40)$$

where we used the fact that the angular velocity of the Earth $\vec{\omega}^{EI}(t)$ does not change with time and therefore it has a zero time derivative. Replacing (4.38) into (4.40) yields

$$\vec{a}^I(t) = \frac{d\vec{v}^E(t)}{dt} \Big|_I + \vec{\omega}^{EI}(t) \times \vec{v}^E(t) + \vec{\omega}^{EI}(t) \times (\vec{\omega}^{EI}(t) \times \vec{s}_{BO}(t)). \quad (4.41)$$

Using equation (4.6) to replace $\vec{a}^I(t)$ in (4.41) and solving for the derivative of $\vec{v}^E(t)$ yields

$$\frac{d\vec{v}^E(t)}{dt} \Big|_I = \vec{f} + \vec{G} - \vec{\omega}^{EI}(t) \times \vec{v}^E(t), \quad (4.42)$$

where

$$\vec{G} = \vec{g} - \vec{\omega}^{EI}(t) \times (\vec{\omega}^{EI}(t) \times \vec{s}_{BO}(t)) \quad (4.43)$$

is the sum of the gravitational acceleration and the centrifugal acceleration and is called the apparent gravity. The objective of the INS kinematics equations is to compute the derivative of $\vec{v}^E(t)$ in the navigation frame N . Using Coriolis theorem to relate the derivative in the navigation frame N with the derivative in the inertial frame I yields

$$\frac{d\vec{v}^E(t)}{dt} \Big|_N = \frac{d\vec{v}^E(t)}{dt} \Big|_I + \vec{\omega}^{IN}(t) \times \vec{v}^E(t). \quad (4.44)$$

From equations (4.44) and (4.42) one can write

$$\frac{d\vec{v}^E(t)}{dt} \Big|_N = \vec{f} + \vec{G} + (\vec{\omega}^{IN}(t) - \vec{\omega}^{EI}(t)) \times \vec{v}^E(t). \quad (4.45)$$

Since angular velocities are additive one can write

$$\vec{\omega}^{EN} + \vec{\omega}^{NI} = \vec{\omega}^{EI}, \quad (4.46)$$

which is equivalent to

$$\vec{\omega}^{IN} = -\vec{\omega}^{NI} = \vec{\omega}^{EN} - \vec{\omega}^{EI} = -\vec{\omega}^{NE} - \vec{\omega}^{EI}. \quad (4.47)$$

Replacing (4.47) in (4.45) yields

$$\frac{d\vec{v}^E(t)}{dt} \Big|_N = \vec{f} + \vec{G} - (\vec{\omega}^{NE}(t) + 2\vec{\omega}^{EI}(t)) \times \vec{v}^E(t), \quad (4.48)$$

which is called the INS velocity equation. Solving this equation for the specific force \vec{f} gives

$$\vec{f} = \frac{d\vec{v}^E(t)}{dt} \Big|_N - \vec{G} + (\vec{\omega}^{NE}(t) + 2\vec{\omega}^{EI}(t)) \times \vec{v}^E(t). \quad (4.49)$$

Example 4.2.4 Find the specific force for the system in figure 4.4 assuming that the aircraft flies at constant altitude and that the Earth's angular rate is small enough when compared with $\dot{\theta}(t)$. Use $\vec{e}_\theta, \vec{e}_r, \vec{e}_{out}$, as the navigation frame.

Solution: If the Earth's angular rate can be neglected then

$$\vec{\omega}^{EI} \approx \vec{0}. \quad (4.50)$$

Using $v_r(t) = 0$ for constant altitude and the navigation reference frame $\vec{e}_\theta, \vec{e}_r, \vec{e}_{out}$, one can write

$$\vec{\omega}^{NE}(t) = -\dot{\theta}(t)\vec{e}_{out}, \quad (4.51)$$

$$\vec{G}(t) = -g(t)\vec{e}_r(t), \quad (4.52)$$

$$\vec{v}^E(t) = v_\theta(t)\vec{e}_\theta(t). \quad (4.53)$$

From equation (4.43) we see that $\vec{G}(t) = \vec{g}(t)$ under the assumption (4.50). Replacing (4.50), (4.51), (4.52), and (4.53), in equation (4.49) yields

$$\vec{f}(t) = \dot{v}_\theta(t)\vec{e}_\theta(t) + \left(g(t) - \dot{\theta}(t)v_\theta(t) \right) \vec{e}_r(t). \quad (4.54)$$

We notice that the same result could have been obtained by the formula (4.6) using the inertial acceleration coordinates $a_r(t)$ and $a_\theta(t)$ given in equations (4.29)–(4.30) with $v_r(t) = 0$, as well as $\vec{g}(t) = \vec{G}(t)$ and equation (4.28).

4.2.5 INS Attitude Equations and Angular Velocity

The INS velocity equation derived in the previous section is obtained from the kinematics equations of a point mass. As we did before in the 2D trajectory example we must now consider a rigid body instead of a point mass and add a set of equations for the attitude of the vehicle. The reader is invited at this point to review section 2.2 where the description of the attitude of a rigid body using transformation matrices was addressed. Let T_E^I be the

transformation matrix from the frame E attached to the Earth to the inertial frame I . Writing equation (4.38) in coordinates we get

$$\frac{d^I[\vec{s}_{BO}]}{dt} = T_E^I \frac{d^E[\vec{s}_{BO}]}{dt} + T_E^I \Omega(^E[\vec{\omega}^{EI}])^E[\vec{s}_{BO}], \quad (4.55)$$

where the matrix Ω is given in (2.43) and the dependence of the vectors with time was omitted for simplicity. From (2.12) we know that

$${}^I[\vec{s}_{BO}] = T_E^{IE}[\vec{s}_{BO}]. \quad (4.56)$$

Taking the derivative of (4.56) with respect to time using the rule (1.44) for derivative of the product yields

$$\frac{d^I[\vec{s}_{BO}]}{dt} = \frac{dT_E^I}{dt} {}_E[\vec{s}_{BO}] + T_E^I \frac{d^E[\vec{s}_{BO}]}{dt}. \quad (4.57)$$

Comparing equations (4.57) and (4.55) we conclude that

$$\frac{dT_E^I}{dt} = T_E^I \Omega(^E[\vec{\omega}^{EI}]), \quad (4.58)$$

which is the transformation matrix equation. In addition to equation (4.58) one can write the angular velocity vector of the rigid body as

$$\vec{\omega}^{BI} = \vec{\omega}^{BN} + \vec{\omega}^{NE} + \vec{\omega}^{EI}. \quad (4.59)$$

In INS one is interested in tracking the angular velocity of the body relative to the navigation reference frame. Thus, solving equation (4.59) for $\vec{\omega}^{BN}$,

$$\vec{\omega}^{BN} = \vec{\omega}^{BI} - (\vec{\omega}^{NE} + \vec{\omega}^{EI}), \quad (4.60)$$

which is the INS attitude equation. To track the transformation matrix of the body frame to the navigation frame one uses equation (4.58) with E replaced by B and I replaced by N to write

$$\frac{dT_B^N}{dt} = T_B^N \Omega({}^B[\vec{\omega}^{BN}]), \quad (4.61)$$

where $\vec{\omega}^{BN}$ is given by (4.60).

Example 4.2.5 For the same system as in example 4.2.4 write down the attitude equation and the transformation matrix equation.

Solution: This example corresponds to rotation on a plane orthogonal to the vector \vec{e}_{out} . Therefore, from the right hand rule in figure 2.18, the direction of the angular velocity of the aircraft in inertial space is \vec{e}_{out} . We can then write

$$\vec{\omega}^{BI} = \omega \vec{e}_{out}. \quad (4.62)$$

The angular velocity of the aircraft relative to the navigation frame corresponds to the motion in pitch and therefore it is equal to (see figure 4.4)

$$\vec{\omega}^{BN} = \dot{\alpha} \vec{e}_{out}. \quad (4.63)$$

Replacing (4.50), (4.51), (4.62), and (4.63) in equation (4.60) yields

$$\dot{\alpha} \vec{e}_{out} = \omega \vec{e}_{out} - \left(-\dot{\theta} \vec{e}_{out} + \vec{0} \right). \quad (4.64)$$

Note that equation (4.64) is equivalent to equation (4.32). From (2.13) the transformation matrix T_B^N can be written as (see figure 4.4)

$$T_B^N = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \quad (4.65)$$

which satisfies equation (4.61) with

$$\Omega(^B[\vec{\omega}^{BN}]) = \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (4.66)$$

4.3 Mechanization Equations

The mechanization equations are the equations implemented inside a computer or a microprocessor to estimate the position and orientation of a user. These equations are based on the kinematics model and the measurements of the accelerometers and the gyroscopes. They are obtained by replacing each component $a(t)$ of the inertial acceleration by the corrected measurement $a_m(t) - c_a(t)$, and by replacing each component of the inertial angular velocity $\omega(t)$ by the corrected measurement $\omega_m(t) - c_\omega(t)$, where $c_a(t)$ and $c_\omega(t)$ are the correcting terms. The objective of the correcting terms is to make sure that the acceleration and angular velocity that are integrated numerically in the computer are as close as possible to the real values.

Example 4.3.1 For the example 4.2.1 write down the mechanization equations. Compute the position error after two seconds using an estimate of the bias of $\hat{b}_x = 0.09ms^{-2}$ and assuming that the initial errors in position and velocity are zero. How does this error compare with the one in example 4.2.1?

Solution: Going back to equations (4.13)–(4.14) we now add as a correcting term c_a an estimate \hat{b}_x of the bias leading to

$$\begin{aligned}\dot{\hat{x}} &= \hat{v}, \\ \dot{\hat{v}} &= a + b_x - \hat{b}_x.\end{aligned}$$

Integrating these equations by following the same steps as in example 4.2.1 one can write the position error in meters as

$$\delta x(t) = \delta x(0) + \delta v(0)t + \delta b_x \frac{t^2}{2}, \quad (4.67)$$

where $\delta b_x = b_x - \hat{b}_x$ is the bias estimation error. Assuming that the initial errors in position and velocity are zero, and replacing the bias estimation error by its value $\delta b_x = 0.01ms^{-2}$, the position error after two seconds is equal to 2cm. The correcting terms allowed a 90% reduction of the error after two seconds when compared with the error of example 4.2.1.

4.4 Notes and References

For more information on inertial navigation systems please see references [9] and [19]. Reference [20] details the derivation of kinematics and dynamics model equations.

4.5 Problems

Problem 4.5.1 For the INS system with platform tilt on a flat Earth shown in figure 4.5, which has one accelerometer and one gyro, answer the following questions:

1. Assume that the accelerometer has a relative scaling error of k , and a bias error of b . Find an expression for the specific force that is measured by the accelerometer.

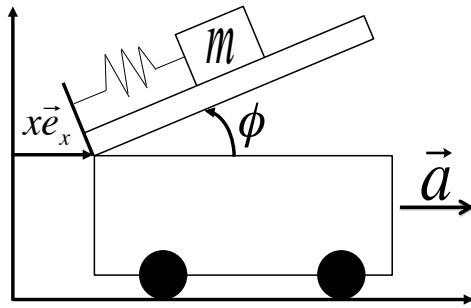


Figure 4.5: INS with plattform tilt for problem 4.5.1

2. Assume that the gyro has a drift of ϵ . Find an expression for the time rate of change of the tilt angle of the platform as measured by the gyro.
3. Write down the error equations that can be used while one is navigating
4. Assuming small angles and a small k , write down the solution of the equations found in item 3. From this solution what can you conclude about inertial navigation systems and its use for a long period of time? Justify your answer.

Problem 4.5.2 An aircraft is flying east as shown in the figure 4.6, which depicts the Earth seen from above. North is represented by the letter N . The distance d in figure 4.6 is given by $d = h \cos(\phi)$, where $\phi = 45^\circ$ degrees north is the latitude. The flight started at the Greenwich Meridian. Assume that the altitude h is constant, x is the distance travelled by the vehicle, v is the speed, ω is the angular velocity relative to inertial space, the Earth rotates at a constant rate, and the longitude angle is λ . Assume also that the Earth is spherical. Answer the following questions:

1. What are the kinematic equations in the ECEF frame $(\vec{e}_1, \vec{e}_2, \vec{e}_{\text{north}})$?
2. An accelerometer is mounted in the x_B axis, which is defined as usual from the center of mass of the vehicle to its nose. The measurement of

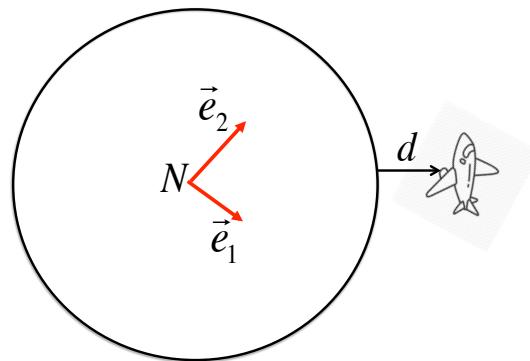


Figure 4.6: Flight over curved Earth in INS problem 4.5.2

acceleration has a small bias b_a . A gyro is mounted for measurements of rotations around $\vec{e}_{z_B} = -\vec{e}_{\text{north}}$. The measurement of the gyro has a bias b_g . Under these assumptions, write down the equations for the measured acceleration f_{meas} and the measured angular velocity ω_{meas} .

3. Assume that you have an estimate of both the gyro and accelerometer biases. Based on the equations written in item 1 and 2, write the mechanization equations as a function of the measurements.

Problem 4.5.3 An aircraft takes off from a position with 0° longitude and 10° latitude. It flies for one hour with a heading of 180° and a speed of 500 knots at an altitude of 5 nautical miles. Then, it turns to a heading of 270° and flies for an extra five hours at the same speed and altitude. Answer the following questions:

1. Find the final position in latitude and longitude assuming the aircraft is flying on a spherical Earth.
2. Find the final position in latitude and longitude using the WGS-84 reference ellipsoid.
3. Compute the difference between your two previous answers. What can you conclude?

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4. *Compute the distance traveled over the surface of the Earth assuming a spherical geometry.*

Chapter 5

Global Positioning System

This chapter is about satellite navigation systems. Current satellite navigation systems include the Global Positioning System GPS (American), GLONASS (Russian), Galileu (European), and BeiDou (Chinese). GPS was first launched to space in the late 1970s and became fully operational in 1994. GLONASS was contemporary of GPS but had incomplete coverage of the globe until the mid 2000s. Galileu was the first global satellite system of the 21st century. It was launched first in 2005 and it reached full operational capability in 2019. BeiDou is the other global satellite system of the 21st century. It was launched first in 2015 and it finished its full deployment in 2020. There exist also two regional satellite systems: QZSS (Japan) and IRNSS or NavIC (India). With so many different satellite systems in place it is impossible to cover all of them. This chapter focuses on the Global Positioning System (GPS). The chapter starts by discussing the GPS segments and signals in section 5.1. Sections 5.2 and 5.3 will then address the pseudorange equation and its linearization. The weighted least squares solution for the position of the user is discussed in section 5.4, followed by dilution of precision in section 5.5. The chapter closes by addressing the carrier phase equation in section 5.7, the determination of attitude by differential GPS in section 5.8, and the problem of integer ambiguity in section 5.7.

5.1 GPS Segments and Signals

GPS signals are broadcast in three frequencies: L1, L2, and L5. The L1 carrier signal is broadcast at a frequency of 1572.42 MHz (19 cm wavelength)

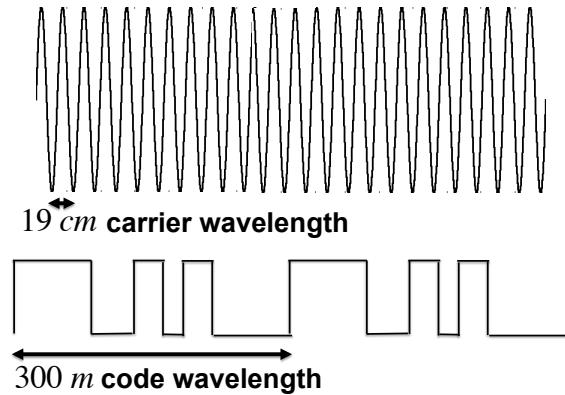


Figure 5.1: GPS code and L1 carrier phase signals

and is modulated by a Code Division Multiple Access (CDMA) Course Acquisition (CA) code an a precise positioning sevice P(Y) code. The L2 carrier frequency is 1227.60 MHz (around 24 cm wavelength) and was originally modulated only by the P(Y) code. Initially civilian users could only receive the signal L1, but since 2005 a second civil signal was added to L2. The L5 carrier frequency is 1176 MHz . The first GPS satellite with an L5 transmitter was launched in 2010. Civil navigation messages started being broadcast in this signal in 2014. Each GPS satellite has a different code signal that is modulated on a carrier wave using a technique called Binary Phase Shift Keying (BPSK). The code signals have a wavelength of approximately 300 meters as shown in figure 5.1.

GPS is managed by the United States Department of Defense (DoD), although it has been open for civilian use since the 1980s. It has three segments: user, space, and control. The user segment consists of all receivers in aircraft, boats, cars, cellular phones, etc. The space segment consists of a constellation of satellites in 6 orbital planes with an inclination of 55 degrees relative to the equator. Each satellite orbits the Earth at approximately twice a day with the reliable operation of 24 satellites starting in 1994. The control segment consists of 10 monitoring stations and one master station located at Schriever's Airforce Base near Colorado Springs in the United States. The monitoring stations download the GPS navigation message and transmit it

to the master station. The master station processes the navigation data, makes corrections and re-uploads the corrected data.

5.2 GPS Pseudorange Equation

GPS measurements are based on time of arrival (TA) of the signal, which is measured by the GPS receiver. The navigation message for each satellite k is stamped with the time of transmission so the transit time is computed as

$$\Delta t^k = t_{rc}^k - t_{tr}^k = \frac{d_u^k}{v_g}, \quad (5.1)$$

where the superscript k indicates the satellite number, t_{rc}^k is the time of arrival at the receiver, t_{tr}^k is the time of transmission of the signal, d_u^k is the distance from the user to satellite k , and v_g is the propagation velocity or group velocity. Since d_u^k and v_g are not known they are replaced by estimates. The estimate of v_g is the speed of light in vacuum $c = 3 \times 10^8 \text{ m/s}$. Although this is the propagation speed in vacuum, the speed in the atmosphere will be lower because of the troposphere and ionosphere delays. Therefore, taking this into account one can rewrite equation (5.1) as

$$\Delta t^k = t_{rc}^k - t_{tr}^k = \frac{\hat{d}_u^k + E^k}{c} + I_u^k + T_u^k, \quad (5.2)$$

where \hat{d}_u^k is the estimate of the distance of the user to satellite k , E^k is the error in that estimate (called ephemeris error), I_u^k is the ionosphere delay, and T_u^k is the troposphere delay. The arrival time that is measured at the GPS receiver will have three additive errors: a bias b_u , a random error ϵ_u^k , and multipath error M_u^k . Therefore, the measurement equation is

$$t_u^k = t_{rc}^k + b_u + M_u^k + \epsilon_u^k. \quad (5.3)$$

The time of transmission is recorded by high precision Cesium-Rubidium atomic clocks with a bias error B^k and a random error η^k for each satellite k . Therefore, the recorded time of transmission is

$$t^k = t_{tr}^k + B^k + \eta^k. \quad (5.4)$$

Using equations (5.4), (5.3), and (5.2), the measured transit time is

$$\Delta t_u^k = t_u^k - t^k = \frac{\hat{d}_u^k + E^k}{c} + b_u - B^k + I_u^k + T_u^k + M_u^k + \epsilon_u^k - \eta^k. \quad (5.5)$$

The actual measurement of the GPS receiver is a pseudorange defined to be equal to the measured transit time multiplied by the speed of light in vacuum. Therefore, multiplying equation (5.5) by c yields

$$\rho_u^k = c\Delta t_u^k = \hat{d}_u^k + E^k + cb_u - cB^k + cI_u^k + cT_u^k + cM_u^k + c\epsilon_u^k - c\eta^k = \hat{d}_u^k + cb_u + \nu^k, \quad (5.6)$$

where

$$\nu^k = E^k - cB^k + cI_u^k + cT_u^k + cM_u^k + c\epsilon_u^k - c\eta^k \quad (5.7)$$

is the term that collects all error sources except the receiver clock bias cb_u . The estimated distance of the user to satellite k is written as

$$\hat{d}_u^k = \sqrt{(\hat{x}_1^k - x_1)^2 + (\hat{x}_2^k - x_2)^2 + (\hat{x}_3^k - x_3)^2}, \quad (5.8)$$

where $\hat{x}^k = {}^E[\vec{s}]^T = [\hat{x}_1^k \ \hat{x}_2^k \ \hat{x}_3^k]$ are the coordinates of the estimated position of the satellite k and $r = {}^E[\vec{r}]^T = [x_1 \ x_2 \ x_3]$ are the coordinates of the user position. Both coordinates are expressed in a reference frame E centered at the center of the Earth and rotating with it, which is denoted as ECEF (Earth Centered Earth Fixed) frame.

5.3 GPS Linearized Pseudorange Equation

The GPS pseudorange equations are nonlinear because of the square root in the expression (5.8) for the estimated distance from the user to each satellite. In order to solve for the user's position the first step will be to linearize these equations around an assumed user position (for example the previous position known from the user). Since we will approximate the graph of the function (5.8) by its tangent plane (a first order Taylor series), the partial derivative of \hat{d}_u^k with respect to each variable $x_i, i = 1, 2, 3$, is needed. A square root corresponds to a power of 0.5. Therefore the partial derivatives can be computed using the rule (1.46) for the derivative or (1.51) leading to

$$\partial_{x_i} \hat{d}_u^k = - \frac{\hat{x}_i^k - x_i}{\sqrt{(\hat{x}_1^k - x_1)^2 + (\hat{x}_2^k - x_2)^2 + (\hat{x}_3^k - x_3)^2}}, \quad i = 1, 2, 3. \quad (5.9)$$

From (5.9) we see that the gradient vector of \hat{d}_u^k , whose coordinates are the three partial derivatives, is given by

$$\nabla \hat{d}_u^k = -\vec{e}_k, \quad (5.10)$$

where \vec{e}_k is the unit vector in the direction from the user to satellite k . Therefore, using equation (1.42) for each coordinate, one can write the linear approximation of the function \hat{d}_u^k at the point with coordinates $r_0 = [\vec{r}_0^T]^E = [x_0 \ y_0 \ z_0]^T$ in vector notation as

$$\hat{d}_u^k \approx \hat{d}_u^k(r_0) - \vec{e}_k \cdot (\vec{r} - \vec{r}_0). \quad (5.11)$$

Replacing \hat{d}_u^k in equation (5.6) by (5.11) and rearranging yields

$$\rho_u^k - \hat{d}_u^k(r_0) = -\vec{e}_k \cdot (\vec{r} - \vec{r}_0) + cb_u + \nu^k, \quad k = 1, \dots, n_s, \quad (5.12)$$

where n_s is the number of satellites. Equations (5.12) can be written in matrix form as

$$\delta\rho = H\delta r + \nu. \quad (5.13)$$

The parameter $\delta\rho$ is called pseudorange corrected for estimated distance and

$$\delta r = [\vec{r} - \vec{r}_0]^T \quad cb_u, \quad (5.14)$$

$$\nu = [\nu^1 \ \dots \ \nu_{n_s}^n]^T, \quad (5.15)$$

$$\delta\rho = \left[(\rho_u^1 - \hat{d}_u^1(r_0)) \ \dots \ (\rho_u^{n_s} - \hat{d}_u^{n_s}(r_0)) \right]^T = \rho - \hat{d}_u(r_0). \quad (5.16)$$

The matrix H is given by

$$H = \begin{bmatrix} -{}^E[\vec{e}_1]^T & 1 \\ \dots & \dots \\ -{}^E[\vec{e}_{n_s}]^T & 1 \end{bmatrix}. \quad (5.17)$$

Example 5.3.1 For the GPS geometry represented in figure 5.2 write down the matrix H .

Solution: The unit vector to satellite 2 has coordinates ${}^E[\vec{e}_2]^T = [0 \ 1]$. The unit vectors to satellites 1 and 3 have coordinates

$${}^E[\vec{e}_1]^T = [\cos(30^\circ) \ -\sin(30^\circ)],$$

$${}^E[\vec{e}_3]^T = [-\cos(30^\circ) \ -\sin(30^\circ)].$$

Replacing the coordinates in (5.17) yields

$$H = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & -1 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \end{bmatrix}. \quad (5.18)$$

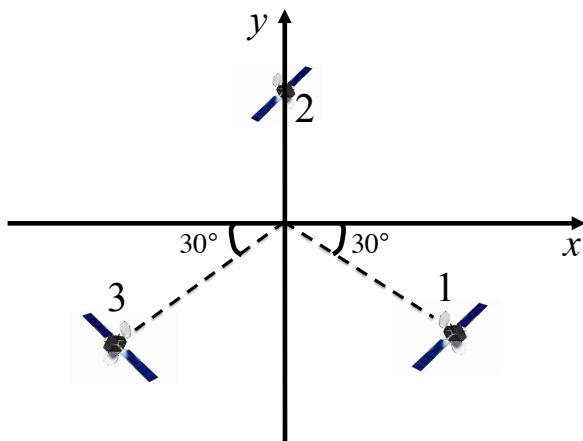


Figure 5.2: GPS geometry for example 5.3.1

5.4 Weighted Least Squares Solution

After obtaining the linearized pseudorange equation we now search for a position solution. The system of equations (5.13), which is valid for 3D positioning, has 4 unknowns: x, y, z , and cb_u . The number of equations is equal to the number of satellites, which is typically larger than 4. With more equations than unknowns it is not possible to find an exact solution. In this section we will compute what is called the weighted least squares solution $\delta\hat{r}$, which solves the optimization problem

$$\begin{aligned} \min_{\delta\hat{r}, \nu} \quad & \nu^T L \nu \\ \text{s.t.} \quad & \delta\rho = H\delta\hat{r} + \nu \end{aligned} \tag{5.19}$$

for a given positive definite symmetric matrix¹ L . The optimization problem (5.19) has a vector equality constraint. Solving the constraint for ν yields

$$\nu = \delta\rho - H\delta\hat{r}. \tag{5.20}$$

¹with positive eigenvalues

Replacing (5.20) into the objective function $\nu^T L \nu$ yields the equivalent unconstrained optimization problem

$$\min_{\delta\hat{r}} (\delta\rho - H\delta\hat{r})^T L (\delta\rho - H\delta\hat{r}). \quad (5.21)$$

The matrix L is typically chosen to be diagonal with diagonal elements l_1, l_2, \dots, l_n , which are the eigenvalues of L . In this case one must minimize the objective function

$$f_o(\delta\hat{r}) = l_1 (\delta\rho - H\delta\hat{r})_1^2 + \dots + l_n (\delta\rho - H\delta\hat{r})_n^2. \quad (5.22)$$

The name "least squares" becomes then very clear because the function (5.24) is a sum of squares that must be minimized.

To find the solution of the minimization (5.23) we first expand the expression for the objective function using the rules (2.34)–(2.35) for the transpose of the sum and product of matrices, which yields the equivalent problem

$$\min_{\delta\hat{r}} f(\delta\hat{r}), \quad (5.23)$$

where

$$f(\delta\hat{r}) = \delta\rho^T L \delta\rho - \delta\rho^T L H \delta\hat{r} - \delta\hat{r}^T H^T L \delta\rho + \delta\hat{r}^T H^T L H \delta\hat{r}. \quad (5.24)$$

A necessary condition for a minimum is that the slope of the tangent plane to the graph of $f(\delta\hat{r})$ be zero in all directions. This condition can be enforced by computing the partial derivatives of $f(\delta\hat{r})$ with respect to the coordinates of $\delta\hat{r}$ and equating them to zero, which yields

$$\partial_{\delta\hat{r}} f = 2H^T L H \delta\hat{r} - 2H^T L \delta\rho = 0. \quad (5.25)$$

The matrix $H^T L H$ is invertible assuming that H has full rank², and solving (5.25) for $\delta\hat{r}$ yields

$$\delta\hat{r} = (H^T L H)^{-1} H^T L \delta\rho. \quad (5.26)$$

When $L = I$, where I is the diagonal identity matrix (with all its diagonal entries equal to one) one calls the matrix $(H^T H)^{-1} H^T$ the left pseudoinverse of matrix H because their product yields the identity matrix. When H is a square invertible matrix then $(H^T H)^{-1} = H^{-1}(H^T)^{-1}$ and the left pseudoinverse becomes simply the inverse matrix, which can be computed by the Laplace formula (2.15).

²the columns of H are linearly independent vectors as per condition (2.32)

Example 5.4.1 Going back to example 5.3.1 assume that the user is located somewhere close to the origin and that the three GPS satellites have the following pseudorange measurements:

$$\begin{aligned}\rho_1 &= 2000 \text{ km}, \\ \rho_2 &= 1500 \text{ km}, \\ \rho_3 &= 2000 \text{ km}\end{aligned}$$

The magnitude of the x coordinate of the position of satellite 1 and 3 is 1000 km. The y coordinate of the position of satellite 2 is also 1000 km. Compute the least squares solution using $L = I$.

Solution: If the user is close to the origin we choose the origin as the linearization point r_0 . The distance of each satellite to the origin is

$$\begin{aligned}\hat{d}_u^1(0) &= \frac{1000}{\cos(30^\circ)}, \\ \hat{d}_u^2(0) &= 1000, \\ \hat{d}_u^3(0) &= \frac{1000}{\cos(30^\circ)},\end{aligned}$$

and

$$\delta\rho = \rho - \hat{d}_u(0) = \begin{bmatrix} 2000 - \frac{2000}{\sqrt{3}} \\ 1500 - 1000 \\ 2000 - \frac{2000}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2000 \left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \\ 500 \\ 2000 \left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \end{bmatrix}. \quad (5.27)$$

In this case there are three satellites and three unknowns x, y , and cb_u . Thus, the matrix H in (5.18) is 3×3 and the cofactor matrix (2.20) is

$$Co(H) = \frac{1}{2} \begin{bmatrix} -3 & \sqrt{3} & \sqrt{3} \\ 0 & -2\sqrt{3} & \sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} \end{bmatrix}.$$

The inverse of H computed using Laplace's formula (2.15) is

$$H^{-1} = \frac{1}{\frac{3\sqrt{3}}{2}} \left(\frac{1}{2}\right) \begin{bmatrix} -3 & 0 & 3 \\ \sqrt{3} & -2\sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix}. \quad (5.28)$$

Using (5.27) and (5.28) the least squares solution is

$$\delta\hat{r} = H^{-1}\delta\rho = \frac{1}{3\sqrt{3}} \begin{bmatrix} 0 \\ 4000(\sqrt{3}-1) - 1000\sqrt{3} \\ 4000(\sqrt{3}-1) + 500\sqrt{3} \end{bmatrix} = \frac{1}{3\sqrt{3}} \begin{bmatrix} 0 \\ 3000\sqrt{3} - 4000 \\ 4500\sqrt{3} - 4000 \end{bmatrix} \text{ km.}$$

5.5 Dilution of Precision

So far we have not considered that we know any statistics of the error ν . However, in practice we have some information about such statistics. Recall that each component k of the error is given by equation (5.7). If we assume that there are estimates of each source of error denoted by $\hat{E}^k, \hat{c}B^k, c\hat{I}_u^k, c\hat{T}_u^k, \hat{c}M_u^k$, then these estimates can be subtracted on the left and right hand sides of equation (5.6) to yield

$$\rho_c^k = \hat{d}_u^k + cb_u + \nu_c^k, \quad (5.29)$$

where ρ_c^k is the corrected pseudorange to satellite k and

$$\nu_c^k = \Delta E^k - c\Delta B^k + c\Delta I_u^k + c\Delta T_u^k + c\Delta M_u^k + c\epsilon_u^k - c\eta^k \quad (5.30)$$

is the new GPS error, where $\Delta w^k = w^k - \hat{w}^k$ for any variable w . After performing these corrections, since the signals come from similar satellites that are physically separated from each other, it is reasonable to assume

$$\begin{aligned} E[\nu_c^k] &= 0, \\ E[\nu_c^i \nu_c^j] &= \sigma^2 \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Dirac delta function defined in (2.29), σ^2 is the variance of the pseudorange measurement, and $E[.]$ is the expected value operator defined in (2.78) that satisfies the linearity properties (2.80) and (2.81). The vector of expected values $\mu_{\nu_c} = E[\nu_c]$ is the mean of the random vector ν_c and $\mathcal{K}_{\nu_c} = E[(\nu_c - \mu_{\nu_c})(\nu_c - \mu_{\nu_c})^T]$ is its covariance matrix.

Performing the same linearization process to the corrected pseudorange equation (5.29) that led to equation (5.13) yields

$$\delta\rho_c = H\delta r + \nu_c, \quad (5.31)$$

where $\delta\rho_c$ is the corrected pseudorange vector and ν_c is the error term with first and second order statistics given by

$$E[\nu_c] = 0, \quad (5.32)$$

$$E[\nu_c \nu_c^T] = \sigma^2 I, \quad (5.33)$$

where I is the identity matrix. The GPS measurement errors with these statistics will cause position errors with different statistics. The mapping from measurement errors to position errors is called Dilution of Precision (DOP). To derive the position error statistics we first compute the mean of $\delta\rho_c$, which from (5.29) and (2.80) is equal to

$$E[\delta\rho_c] = HE[\delta r] + E[\nu_c]. \quad (5.34)$$

From (5.34) and (5.32) we conclude that

$$E[\delta\rho_c] = HE[\delta r]. \quad (5.35)$$

Taking the expected value of equation (5.26) with $\delta\rho$ replaced by $\delta\rho_c$ and using (2.80) yields

$$E[\delta\hat{r}] = (H^T LH)^{-1} H^T L E[\delta\rho_c]. \quad (5.36)$$

Replacing (5.35) into (5.37) and using (2.80) gives

$$E[\delta\hat{r} - \delta r] = E[\delta\hat{r}] - E[\delta r] = 0, \quad (5.37)$$

which implies that $\delta\hat{r}$ is an unbiased estimate of the true value δr . The position error is

$$\delta\hat{r} - \delta r = (H^T LH)^{-1} H^T L \delta\rho_c - \delta r. \quad (5.38)$$

Replacing (5.31) into (5.38) yields

$$\delta\hat{r} - \delta r = (H^T LH)^{-1} H^T L \nu_c. \quad (5.39)$$

Using (5.39), (2.81), and (2.35), the covariance of the position error is

$$E[(\delta\hat{r} - \delta r)(\delta\hat{r} - \delta r)^T] = (H^T LH)^{-1} H^T L E[\nu_c \nu_c^T] H (H^T LH)^{-1}. \quad (5.40)$$

From (5.33) and (5.40), one can rewrite the position error covariance as

$$\mathcal{K} = E[(\delta\hat{r} - \delta r)(\delta\hat{r} - \delta r)^T] = (H^T LH)^{-1} \sigma^2. \quad (5.41)$$

We see from (5.41) that the entries of \mathcal{K} are equal to σ^2 multiplied by dilution of precision (DOP) factors. Therefore, one calls $(H^T LH)^{-1}$ the DOP matrix, which for positioning in 3D has the form

$$(H^T LH)^{-1} = \begin{bmatrix} DOP_x^2 & \times & \times & \times \\ \times & DOP_y^2 & \times & \times \\ \times & \times & DOP_z^2 & \times \\ \times & \times & \times & DOP_t^2 \end{bmatrix}. \quad (5.42)$$

Based on the DOP factors one can define the following parameters

$$TDOP = \sqrt{DOP_t^2}, \quad (5.43)$$

$$VDOP = \sqrt{DOP_z^2}, \quad (5.44)$$

$$HDOP = \sqrt{DOP_x^2 + DOP_y^2}, \quad (5.45)$$

$$PDOP = \sqrt{DOP_x^2 + DOP_y^2 + DOP_z^2}, \quad (5.46)$$

$$GDOP = \sqrt{DOP_x^2 + DOP_y^2 + DOP_z^2 + DOP_t^2}. \quad (5.47)$$

Example 5.5.1 Compute the PDOP and the standard deviation of the position error in coordinate x for the problem of example 5.3.1. Assume a 1σ User Equivalent Range Error (UERE) of 10 m.

Solution: In example 5.3.1 we had $L = I$. From (5.18) we get

$$H^T H = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and

$$(H^T H)^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}. \quad (5.48)$$

Using (5.48) and equation (5.46) yields

$$PDOP = \sqrt{\frac{2}{3} + \frac{2}{3} + \frac{1}{3}} = \sqrt{\frac{5}{3}}.$$

The variance of the position error in coordinate x is

$$\sigma_x^2 = DOP_x^2 \sigma^2 = \frac{2}{3}(10)^2 = \frac{200}{3} \text{ m}^2,$$

and the standard deviation is

$$\sigma_x = \sqrt{\frac{200}{3}} \text{ m.}$$

5.6 Error Ellipses

An error ellipse is an ellipse for which there is a pre-defined probability that the position error lies inside it. There is a different error ellipse for each value of the pre-defined probability. When talking about error ellipses we are implicitly assuming that the position variables are x, y , which are located on a plane. In 3D a similar analysis can be done with error ellipsoids instead of error ellipses. For simplicity we will focus the discussion of this section only on error ellipses. The reader may want to review the material about the equation of the ellipse (1.38) in chapter 1 before reading the rest of this section. We define the 2D position error on the xy plane as

$$\epsilon_{xy} = [\delta\hat{x} - \delta x \quad \delta\hat{y} - \delta y]^T, \quad (5.49)$$

which corresponds to the first two entries of the 4D position error $\delta\hat{r} - \delta r$, which was defined in section 5.5. We recall from (5.37) that the mean value of the position error is zero, i.e., $E[\epsilon_{xy}] = 0$. From (5.41) the covariance matrix of the 4D position error $\delta\hat{r} - \delta r$ is

$$\mathcal{K} = E[(\delta\hat{r} - \delta r)(\delta\hat{r} - \delta r)^T] = (H^T L H)^{-1} \sigma^2 = \begin{bmatrix} \mathcal{K}_{\delta\hat{x}\delta\hat{y}} & X \\ X^T & \mathcal{K}_{\delta\hat{z}\delta\hat{b}} \end{bmatrix}, \quad (5.50)$$

where

$$\mathcal{K}_{\delta\hat{x}\delta\hat{y}} = E \left[\begin{bmatrix} \delta\hat{x} - \delta x \\ \delta\hat{y} - \delta y \end{bmatrix} \begin{bmatrix} \delta\hat{x} - \delta x \\ \delta\hat{y} - \delta y \end{bmatrix}^T \right] = \begin{bmatrix} \sigma_{\delta\hat{x}}^2 & \rho\sigma_{\delta\hat{x}}\sigma_{\delta\hat{y}} \\ \rho\sigma_{\delta\hat{y}}\sigma_{\delta\hat{x}} & \sigma_{\delta\hat{y}}^2 \end{bmatrix} \quad (5.51)$$

is the covariance matrix of the 2D position error with correlation coefficient ρ , X is a 2×2 matrix, and $\mathcal{K}_{\delta\hat{z}\delta\hat{b}}$ is the covariance matrix of the error in the variables $\delta\hat{z}$ and $c\delta\hat{b}$. To determine an ellipse for the 2D position error we will assume that the error follows a bivariate Gaussian distribution with probability density (2.87) for $m = 2$ written as

$$p(\epsilon_{xy}) = \frac{1}{2\pi\sqrt{|\mathcal{K}_{\delta\hat{x}\delta\hat{y}}|}} e^{-\frac{1}{2}\epsilon_{xy}^T \mathcal{K}_{\delta\hat{x}\delta\hat{y}}^{-1} \epsilon_{xy}}. \quad (5.52)$$

There are two cases to be considered: uncorrelated and correlated variables.

5.6.1 Uncorrelated Variables

If the variables x and y are uncorrelated then the correlation coefficient ρ is zero. Replacing $\rho = 0$ in (5.51) and taking the inverse yields

$$\mathcal{K}_{\delta\hat{x}\delta\hat{y}}^{-1} = \begin{bmatrix} \sigma_{\delta\hat{x}}^{-2} & 0 \\ 0 & \sigma_{\delta\hat{y}}^{-2} \end{bmatrix}. \quad (5.53)$$

The probability density (5.52) then becomes

$$p(\epsilon_{xy}) = \frac{1}{2\pi\sigma_{\delta\hat{x}}\sigma_{\delta\hat{y}}} e^{-\frac{1}{2}\left[\left(\frac{\delta\hat{x}-\delta x}{\sigma_{\delta\hat{x}}}\right)^2 + \left(\frac{\delta\hat{y}-\delta y}{\sigma_{\delta\hat{y}}}\right)^2\right]}. \quad (5.54)$$

The exponent of (5.54) must be constant for the probability density to be constant. Level sets of constant probability are thus the ellipses with equation

$$\left(\frac{\delta\hat{x}-\delta x}{\sigma_{\delta\hat{x}}}\right)^2 + \left(\frac{\delta\hat{y}-\delta y}{\sigma_{\delta\hat{y}}}\right)^2 = N^2. \quad (5.55)$$

For each value of N the ellipse will have semi-major and semi-minor axes $N\sigma_{\delta\hat{x}}$ and $N\sigma_{\delta\hat{y}}$. If $\sigma_{\delta\hat{y}} > \sigma_{\delta\hat{x}}$ then the semi-major axis is equal to $N\sigma_{\delta\hat{y}}$. Otherwise, it is equal to $N\sigma_{\delta\hat{x}}$. This ellipse is thus called a N -sigma ellipse. Integrating the probability density function on an N -sigma ellipse yields the probability P of the error lying inside the ellipse. To do this we define

$$\delta\bar{x} = \frac{\delta\hat{x}-\delta x}{\sigma_{\delta\hat{x}}}, \quad (5.56)$$

$$\delta\bar{y} = \frac{\delta\hat{y}-\delta y}{\sigma_{\delta\hat{y}}}. \quad (5.57)$$

$$(5.58)$$

In the new coordinates $\delta\bar{x}, \delta\bar{y}$, the equation (5.55) becomes

$$\delta\bar{x}^2 + \delta\bar{y}^2 = N^2, \quad (5.59)$$

which is the equation of a circle with radius N . We can change to polar coordinates

$$\delta\bar{x} = r \cos \theta, \quad (5.60)$$

$$\delta\bar{y} = r \sin \theta, \quad (5.61)$$

and compute the probability through the integral

$$\mathcal{P} = \int_0^{2\pi} \int_0^N p(\epsilon_{xy}) r dr d\theta = 1 - e^{-\frac{N^2}{2}}. \quad (5.62)$$

For a 1-sigma ellipse the probability is $\mathcal{P} = 0.39$ or 39%. This should be contrasted with the value of 68%, which is the probability for a single Gaussian variable to lie within plus or minus one standard deviation from its mean.

Example 5.6.1 *Draw the 1-sigma ellipse for the example 5.5.1.*

Solution: From equations (5.48), (5.50), and (5.51) with $L = I$ and $\sigma = 10$ meters we see that

$$\mathcal{K}_{\delta\hat{x}\delta\hat{y}} = \begin{bmatrix} \frac{200}{3} & 0 \\ 0 & \frac{200}{3} \end{bmatrix}.$$

Therefore, the equation of the 1-sigma ellipse corresponds to the choice $N = 1$ in equation (5.55), which yields

$$\frac{(\delta\hat{x} - \delta x)^2}{\frac{200}{3}} + \frac{(\delta\hat{y} - \delta y)^2}{\frac{200}{3}} = 1.$$

The 1-sigma ellipse, which is a circle in this case, is shown in figure 5.3.

5.6.2 Correlated Variables

If the variables are correlated then we change variables to a new set of uncorrelated variables. One can then use the result of section 5.6.1. The right change of variables is the one that diagonalizes the covariance matrix. From section 2.6 we know that the change of variables to diagonalize $\mathcal{K}_{\delta\hat{x}\delta\hat{y}}$ is

$$\begin{bmatrix} \delta\hat{x} - \delta x \\ \delta\hat{y} - \delta y \end{bmatrix} = W \begin{bmatrix} \delta\tilde{x} \\ \delta\tilde{y} \end{bmatrix}, \quad (5.63)$$

where W is the matrix whose columns are the eigenvectors of $\mathcal{K}_{\delta\hat{x}\delta\hat{y}}$. Computing the covariance matrix in the new variables using (2.81) yields

$$\mathcal{K}_{\delta\tilde{x}\delta\tilde{y}} = E \left[\begin{bmatrix} \delta\tilde{x} \\ \delta\tilde{y} \end{bmatrix} \begin{bmatrix} \delta\tilde{x} \\ \delta\tilde{y} \end{bmatrix}^T \right] = W^{-1} E \left[\begin{bmatrix} \delta\hat{x} - \delta x \\ \delta\hat{y} - \delta y \end{bmatrix} \begin{bmatrix} \delta\hat{x} - \delta x \\ \delta\hat{y} - \delta y \end{bmatrix}^T \right] W^{-T}, \quad (5.64)$$

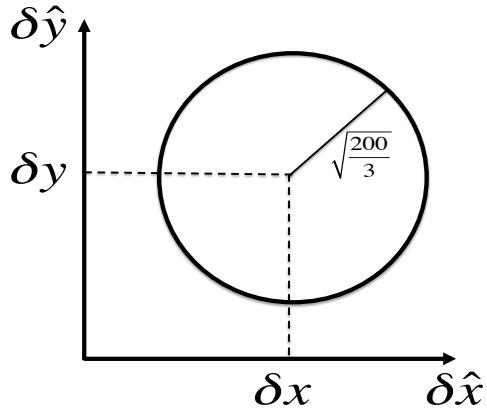


Figure 5.3: Error ellipse of example 5.6.1

where $W^{-T} = (W^{-1})^T$. Since the matrix of eigenvectors W for a symmetric matrix is orthogonal if one normalizes the eigenvectors to have unit norm, then its inverse is $W^{-1} = W^T$, as shown in example 2.2.2. Using this property in equation (5.64), as well as equations (5.51) and (2.70) with $P = \mathcal{K}_{\delta\hat{x}\delta\hat{y}}$ yields the diagonal covariance matrix

$$\mathcal{K}_{\delta\tilde{x}\delta\tilde{y}} = \Lambda. \quad (5.65)$$

Example 5.6.2 Assume the satellite geometry shown in figure 5.4. Using $\sigma = 10$ meters draw the 1-sigma ellipse.

Solution: For the geometry in figure 5.4, following the same process of examples 5.3.1 and 5.5.1 we get

$$H = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad (5.66)$$

$$H^T H = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & \frac{\sqrt{2}-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{3}{2} & \frac{\sqrt{2}-1}{\sqrt{2}} \\ \frac{\sqrt{2}-1}{\sqrt{2}} & \frac{\sqrt{2}-1}{\sqrt{2}} & 3 \end{bmatrix}, \quad (5.67)$$

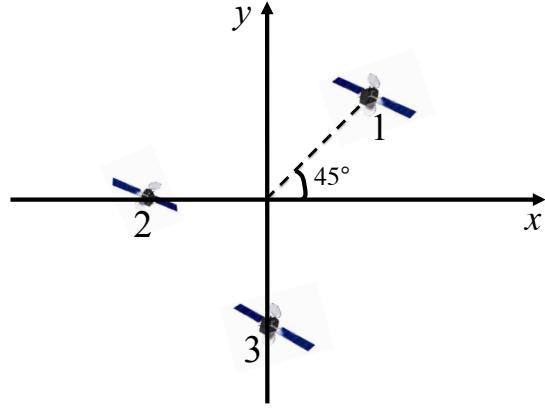


Figure 5.4: Satellite geometry of example 5.6.2

$$(H^T H)^{-1} = \frac{1}{3+2\sqrt{2}} \begin{bmatrix} 3+\sqrt{2} & -\sqrt{2} & -\frac{\sqrt{2}-1}{\sqrt{2}} \\ -\sqrt{2} & 3+\sqrt{2} & -\frac{\sqrt{2}-1}{\sqrt{2}} \\ -\frac{\sqrt{2}-1}{\sqrt{2}} & -\frac{\sqrt{2}-1}{\sqrt{2}} & 2 \end{bmatrix}. \quad (5.68)$$

From (5.50) and (5.68) we conclude that

$$\mathcal{K}_{\delta\hat{x}\delta\hat{y}} = \frac{\sigma^2}{3+2\sqrt{2}} \begin{bmatrix} 3+\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 3+\sqrt{2} \end{bmatrix}. \quad (5.69)$$

Computing the eigenvalues and eigenvectors of this matrix using the process outlined in section 2.6 yields

$$\mathcal{K}_{\delta\tilde{x}\delta\tilde{y}} = \frac{\sigma^2}{3+2\sqrt{2}} \begin{bmatrix} 3 & 0 \\ 0 & 3+2\sqrt{2} \end{bmatrix}, \quad (5.70)$$

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \quad (5.71)$$

From (5.71) we can conclude that the directions of the axes of the 1-sigma ellipse are plus or minus 45° relative to the horizontal axis. The standard

deviations are obtained from (5.70) and $\sigma = 10$ m as

$$\begin{aligned}\sigma_{\delta\tilde{x}} &= 10\sqrt{\frac{3}{3+2\sqrt{2}}} \\ \sigma_{\delta\tilde{y}} &= 10\end{aligned}$$

Therefore, the equation of the 1-sigma ellipse corresponds to the choice $N = 1$ in equation (5.55), which using the tilde variables yields

$$\frac{\delta\tilde{x}^2}{\frac{300}{3+2\sqrt{2}}} + \frac{\delta\tilde{y}^2}{100} = 1.$$

Using (5.63) and (5.71) the equation of the ellipse in original variables is

$$\frac{(\Delta x + \Delta y)^2}{\frac{300}{3+2\sqrt{2}}} + \frac{(\Delta y - \Delta x)^2}{100} = 2,$$

where $\Delta x = \delta\hat{x} - \delta x$ and $\Delta y = \delta\hat{y} - \delta y$. The 1-sigma ellipse is plotted in figure 5.5.

5.7 GPS Carrier Phase Equation

The pseudorange measurements are based on the GPS code signal. In some of the first GPS receivers the accuracy of the position estimate was close to 10% of the length of this signal. Since the code length is around 300 meters (see figure 5.1), then the accuracy of stand alone position estimation based on the code phase was close to 30 meters. Good receivers can be designed nowadays with an error in position closer to 1% to 2% of the code length. Still the position error can be at best anywhere between 3 and 6 meters.

The carrier phase on the other hand has a wavelength of 19 cm (see figure 5.1). Therefore, position estimation based on carrier phase can in theory yield centimeter or millimeter level accuracy. However, as we will see later there is an added issue of integer ambiguity related to carrier phase measurements. The main difference in the measurement equation of carrier phase relative to the equation for the pseudorange measurement is in the ionospheric delay.

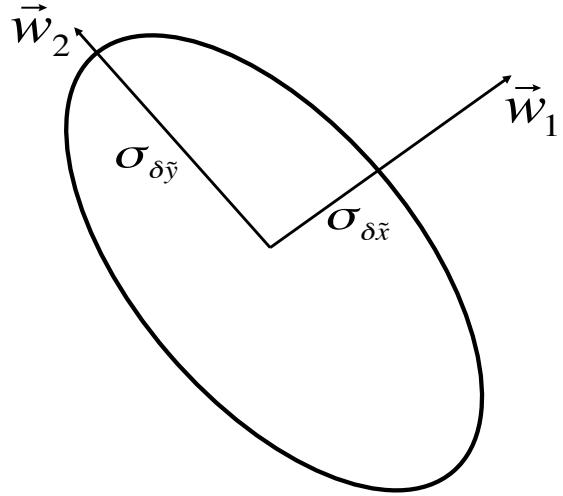


Figure 5.5: Error ellipse of example 5.6.2

The ionospheric delay is determined based on what is called the Total Electron Content (TEC), which is the total number of electrons per cross section area present along the path of the GPS signal as it crosses the ionosphere. The ionosphere is an atmospheric layer that is partially ionized by the sun's ultraviolet radiation. It is a spherically stratified plasma that is dispersive (unlike the troposphere). This implies that different frequencies propagate at different velocities. The GPS code propagates at what is called the group velocity v_g . The more electrons encountered in the path of propagation of the code, the more it will be delayed, thus decreasing the group velocity. It can be shown that the ionospheric delay for the code from any satellite k , which is called group delay, is equal to

$$d_g = cI_u^k = 40.3 \frac{TEC}{f^2}, \quad (5.72)$$

where f is the frequency of the GPS signal and TEC is the total electron content per meters squared in TEC units (one TEC unit is 10^{16} electrons per meters squared). This is why a positive ionospheric delay cI_u^k was used in equation (5.6) for the code phase. Unlike the code signal, the carrier phase propagates at the phase velocity v_ϕ . Since the group and phase velocities in

the ionosphere are related by the equation

$$c = \sqrt{v_g v_\phi} = 3 \times 10^8 \text{ ms}^{-1}, \quad (5.73)$$

the effect of the TEC on the phase velocity v_ϕ is the converse of the effect on the group velocity v_g . In other words, the phase velocity v_ϕ increases as the TEC increases, which leads to a negative ionospheric delay given by

$$d_\phi = -40.3 \frac{TEC}{f^2} = -cI_u^k, \quad (5.74)$$

where f is the frequency of the GPS signal. Note that v_ϕ is larger than the speed of light because of the negative delay. However, this does not contradict relativity's theory because no information is carried at the speed v_ϕ . We also observe from (5.72) and (5.74) that propagation in the ionosphere leads to an advance on the carrier phase measurement by an equal amount to the delay on the code measurement.

The equation for the carrier phase measurement can now be obtained using the same method followed to derive the pseudorange equation (5.6), provided one changes the sign of the ionosphere delay. Replacing cI_u^k by $-cI_u^k$ in equation (5.6) leads to the carrier phase measurement equation

$$(\phi_u^k + \bar{N}_u^k) \lambda_\phi = d_u^k + cb_u - cB^k - cI_u^k + cT_u^k + cM_u^k + c\epsilon_u^k - c\eta^k, \quad (5.75)$$

where \bar{N}_u^k is the number of full cycles of the carrier with wavelength $\lambda_\phi = 19 \text{ cm}$, and ϕ_u^k is the additional (fractional) phase beyond the number of full cycles. Comparing with equation (5.6) we note that the real distance d_u^k to the satellite k was used in equation (5.75) in place of the estimated distance \hat{d}_u^k plus the ephemeris error E^k . The reason for this change will become clear when we discuss single differencing in the next section.

5.8 Differential GPS and Attitude

Many of the GPS error terms do not change in a local region. Therefore, these error terms can be removed by subtracting the equations for the GPS measurements of two receivers that are located near to each other. This is called differential GPS. An additional radio transmitter that broadcasts the measurements of one of the receivers to the other receiver is needed for differential GPS. Upon receiving this information, which is called the

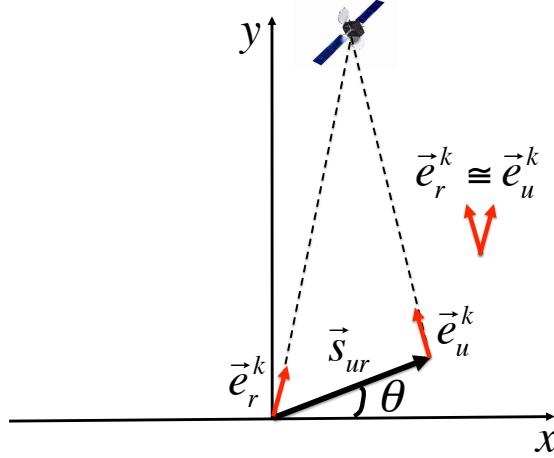


Figure 5.6: Differential GPS

correction, one can subtract the equations of the measurements of the two receivers, which is called single differencing. The typical application of single differencing is when one receiver is the moving user and the other is a reference stationary receiver called the base station. The single difference always eliminates the satellite clock bias provided that the corrections are received promptly. If the user is close to the reference station then the ionospheric advance and the tropospheric delay will also be canceled by the single difference. The single differencing can be written as

$$\Delta\bar{\phi}_{ur}^k = (\phi_u^k - \phi_r^k)\lambda_\phi = \Delta d_{ur}^k + c\Delta b_{ur} + \Delta N_{ur}^k\lambda_\phi + c(\Delta M_{ur}^k + \Delta\epsilon_{ur}^k), \quad (5.76)$$

where $\Delta b_{ur} = b_u - b_r$, $\Delta N_{ur}^k = \bar{N}_r^k - \bar{N}_u^k$, $\Delta M_{ur}^k = M_u^k - M_r^k$, $\Delta\epsilon_{ur}^k = \epsilon_u^k - \epsilon_r^k$,

$$\Delta d_{ur}^k = d_u^k - d_r^k = \vec{s}_{uk} \cdot \vec{e}_k^u - \vec{s}_{rk} \cdot \vec{e}_k^r \approx \vec{s}_{ur} \cdot \vec{e}_k^u = -\vec{s}_{ur} \cdot \vec{e}_u^k = -\vec{s}_{ur} \cdot \vec{e}_k \quad (5.77)$$

because the unit vectors from the user to the satellite (\vec{e}_u^k) and from the reference station to the satellite (\vec{e}_r^k) are approximately the same since the satellite is far from the receivers (see figure 5.6).

Assuming that the multipath error difference $c\Delta M_{ur}^k$ is either small compared to other error terms or can be mitigated by properly shielding the

antennas from multipath reflections, one can consider $\Delta M_{ur}^k \approx 0$. The equations (5.76)–(5.77) for n_s satellites can be written in a reference frame E attached to the reference position (see figure 5.6) as

$$\Delta\bar{\phi} = H\Delta s + \Delta N\lambda_\phi + \Delta\epsilon, \quad (5.78)$$

where H is the matrix given by (5.17) and

$$\begin{aligned}\Delta s &= [{}^E[\vec{s}_{ur}]^T \ c\Delta b_{ur}]^T, \\ \Delta\epsilon &= [c\Delta\epsilon_{ur}^1 \ \dots \ c\Delta\epsilon_{ur}^{n_s}]^T, \\ \Delta N &= [\Delta N_{ur}^1 \ \dots \ \Delta N_{ur}^{n_s}]^T, \\ \Delta\bar{\phi} &= [(\phi_u^1 - \phi_r^1) \ \dots \ (\phi_u^{n_s} - \phi_r^{n_s})]^T \lambda_\phi.\end{aligned}$$

Once the array of integers ΔN is known and a weighting matrix $L > 0$ is chosen, the least squares solution of equation (5.78) can be found as

$$\Delta\hat{s} = (H^T L H)^{-1} H^T L (\Delta\bar{\phi} - \Delta N\lambda_\phi). \quad (5.79)$$

Equation (5.79) should be compared to (5.26), which is the pseudorange solution. For problems in 2D, the attitude angle θ can also be determined once \vec{s}_{ur} is known through the equation (see figure 5.6)

$$\theta = \arctan\left(\frac{{}^E[\vec{s}_{ur}]_y}{{}^E[\vec{s}_{ur}]_x}\right). \quad (5.80)$$

Determination of the angles in 3D is also possible but will not be addressed.

5.9 Integer Ambiguity

Determining the vector of integers ΔN is called the integer ambiguity resolution. This ambiguity can be solved more easily when the distance of the user to the reference point is known, for example by staying steady for a period of time before the initial time of motion. This period is called the calibration. During the calibration one knows the possible different combinations of integer carrier wavelengths between the user and the reference receivers. In fact, these combinations depend only on the direction of the satellite wavefronts, which can be assumed to be planes separated by the wavelength λ_ϕ because the satellite is located far from the receivers (see figure 5.7).

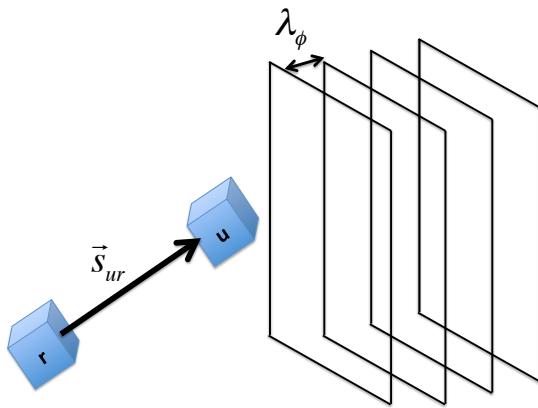


Figure 5.7: Integer ambiguity

The worst case scenario (largest propagation delay between receivers) is the one for which the vector \vec{s}_{ur} is orthogonal to the planar wavelength. The value of the integer for a satellite signal coming from this direction is the largest integer that is smaller than the quotient of $\|\vec{s}_{ur}\|$ and the wavelength of the carrier λ_ϕ . The best case scenario (smallest propagation delay between receivers) is the one for which the vector \vec{s}_{ur} is parallel to the wavefront and the signal arrives at both receivers at the same time. In this case the value of the integer for a satellite signal coming from this direction is zero. The integers for each satellite can thus have any value between zero and the worst case scenario.

For example, if the initial distance between receivers is less than three times the carrier wavelength ($3 \times 19 = 57\text{ cm}$) one determines that the worst case scenario corresponds to an integer of 2. Then each integer can have a value in the set $\{0, 1, 2\}$. For a 2D problem one must have at least three satellites and each satellite adds an extra integer. Therefore, the number of possible integer combinations for three satellites is $3^3 = 9$. For each of the nine possible integer combinations one can then find the least squares solution of equation (5.78) and compute the length of the vector \vec{s}_{ur} , which is assumed to be known during the calibration period. The integer combination

that leads to the best match with the known length $\|\vec{s}_{ur}\|$ (smallest error magnitude) is then the chosen solution for the integers. The integers found by this process can be used for all subsequent calculations because they do not change with time, provided the receivers are maintained locked on the carrier phase.

5.10 Notes and References

For more information on GPS please see references [21], [22], and [19].

5.11 Problems

Problem 5.11.1 For the system in the figure 5.8, satellites 1,2,3 are at 1000 km from the origin and the pseudorange measurements are

$$\begin{aligned}\rho_1 &= 1100 \text{ km}, \\ \rho_2 &= 1000 \text{ km}, \\ \rho_3 &= 1100 \text{ km}.\end{aligned}$$

The 1σ User Equivalent Range Error (UERE) is =10m. The user is assumed to be close to the origin. Answer the following questions:

1. Compute the H matrix as a function of θ_1 and θ_2 .
2. For what values of θ_1 and θ_2 is the matrix H not invertible ? Provide a geometric explanation for each case.
3. For $\theta_1 = \theta_2 = \frac{\pi}{2}$ compute the least squares solution for the position of the user.
4. For $\theta_1 = \theta_2 = \frac{\pi}{2}$ compute XDOP, YDOP, TDOP, PDOP, GDOP.
5. For $\theta_1 = \theta_2 = \frac{\pi}{2}$ compute the standard deviations of the position error σ_x and σ_y .

Problem 5.11.2 Assume that a user is located on a plane with a fixed reference frame with origin O . The user is close to the point with coordinates $[1000 \ 0]^T$ in kilometers. Find the position of the user for the following cases:

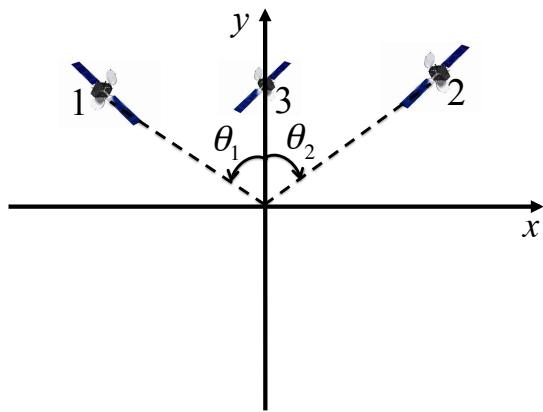


Figure 5.8: GPS geometry for problem 5.11.1

1. The position of the satellites in kilometers is equal to

$$\begin{aligned} {}^I[\vec{s}_{1O}] &= [0 \ 1000]^T, \\ {}^I[\vec{s}_{2O}] &= [1000 \ -1000]^T, \\ {}^I[\vec{s}_{3O}] &= [2000 \ 1000]^T. \end{aligned}$$

and the pseudoranges are equal to

$$\begin{aligned} \rho_1 &= 1534 \text{ km}, \\ \rho_2 &= 1120 \text{ km}, \\ \rho_3 &= 1534 \text{ km}. \end{aligned}$$

2. The same as in item 1 with the following new position for satellite 2,

$${}^I[\vec{s}_{2O}] = [1000 \ 1000]^T.$$

3. The same as in item 1 with the second coordinate of all satellites equal to 5000 km and

$$\begin{aligned} \rho_1 &= 5220 \text{ km}, \\ \rho_2 &= 5120 \text{ km}, \\ \rho_3 &= 5220 \text{ km}. \end{aligned}$$

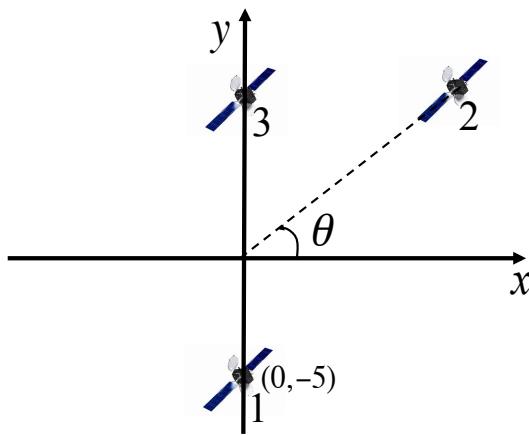


Figure 5.9: GPS geometry for problem 5.11.3

What can you conclude from the results ?

Problem 5.11.3 For the system in figure 5.9 all coordinates are in kilometers. The satellites 1, 2, and 3 provide pseudorange measurements of

$$\begin{aligned}\rho_1 &= 3 \text{ km}, \\ \rho_2 &= 5 \text{ km}, \\ \rho_3 &= 6 \text{ km}.\end{aligned}$$

Satellite 2 is 6 km away from the origin. The position of satellite 3 is symmetric to the position of satellite 1 relative to the x-axis. The user is close to the origin. Using the linearized GPS equations, answer the following questions:

1. Compute the corrected pseudorange vector.
2. Write down the expression for the H matrix for a general angle .
3. What is the position of the user if $\theta = 0^\circ$?

4. *What is the position of the user if $\theta = 45^\circ$? Compared to the answer of question 3 what are the differences in the position of the user and why do these differences exist ?*
5. *For what values of θ will it not be possible to obtain the position of the user ? What is the mathematical reason for not obtaining a solution ? What is the geometric reason ?*
6. *What is the new H matrix if a satellite 4 is located at the same position of satellite 3 ? If satellites 3 and 4 provide the same pseudorange would there now be a solution for the cases without solution in question 5 ? Why ? What is the new solution for $\theta = 0^\circ$?*

Chapter 6

State Space Models and Optimal Estimation

Chapter 4 studied INS and chapter 5 focused on GPS. Kalman filtering and the integration of INS and GPS measurements is the focus of the next chapter. Integration of these measurements is often performed to solve the issue of divergence of the standalone INS error with time. The tool that will be used to integrate INS and GPS measurements is called Kalman Filter. Kalman filter is an optimal estimation framework to combine a mathematical model of a process with real measurements to arrive at a prediction of the state of that process. Therefore, one must define first mathematical models of processes with a random state vector and optimal estimation before addressing Kalman filtering. State space models and optimal estimation are the topics of this chapter.

The chapter starts by discussing linear models in section 6.1. Then the celebrated Cayley-Hamilton theorem is stated in section 6.2 and used to compute one of the parameters of a discrete-time approximation of a continuous-time linear model using a zero-order-hold. The chapter closes with a discussion of several optimal estimation techniques in sections 6.3, 6.4, 6.5, 6.6.

6.1 Linear System Models

The kinematics of a vehicle moving in one direction were addressed in example 4.2.1 and can be written as

$$\dot{r}(t) = v(t), \quad (6.1)$$

$$v(t) = a(t), \quad (6.2)$$

where $r(t)$ is the position, $v(t)$ is the velocity, and $a(t)$ is the acceleration. Defining the state vector $x(t) = [r(t) \ v(t)]^T$, the equations (6.1)–(6.2) can be rewritten in matrix form as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad (6.3)$$

where $u(t) = a(t)$ and

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (6.4)$$

The equation (6.3) is the continuous-time state space representation of a linear model. Any continuous-time linear model can be represented in the general form (6.3). Each specific continuous-time linear model will have its own matrices A_c and B_c as, for example, the kinematics matrices in (6.4).

One can get a discrete-time approximation of the continuous-time model by taking samples every T seconds. The parameter T is therefore called the sampling period. The sampling times will form the infinite sequence $0, T, 2T, 3T, \dots, kT, (k+1)T, \dots$, as shown in figure 6.1. To find a discrete-time approximation of a continuous-time system we start by writing the solution of the differential equation (6.3) with initial condition $x(t_0) = x_0$ as

$$x(t) = e^{A_c(t-t_0)}x_0 + \int_{t_0}^t e^{A_c(t-\tau)}B_c u(\tau)d\tau, \quad (6.5)$$

where the matrix exponential is defined by the Taylor series

$$e^{A_c t} = I + A_c t + \frac{(A_c t)^2}{2!} + \frac{(A_c t)^3}{3!} + \dots = \sum_{m=0}^{\infty} \frac{(A_c t)^m}{m!}, \quad (6.6)$$

with $m! = m(m-1)\dots 1$ for $m > 0$, and $0! = 1$. To verify that (6.5) is the solution of (6.3) one simply replaces (6.5) into (6.3).

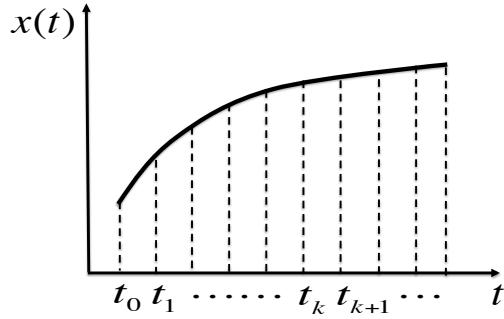


Figure 6.1: Sampling of a continuous-time signal

We consider now the case $t_0 = kT, t = (k + 1)T, x_0 = x(kT), x(t) = x((k + 1)T)$, and we assume a constant value of the input $u(t) = u(kT)$ between samples for $t \in [kT, (k + 1)T]$. Replacing these values in (6.5) and bringing the terms that do not depend on τ out of the integral yields

$$x((k + 1)T) = e^{A_c T} x(kT) + e^{A_c(k+1)T} \int_{kT}^{(k+1)T} e^{-A_c \tau} d\tau B_c u(kT). \quad (6.7)$$

This is called the zero-order hold approximation because one assumes that the input $u(t)$ is held constant between samples (see figure 6.2). The equation (6.7) can be rewritten in shorter notation as

$$x(k + 1) = Ax(k) + Bu(k) \quad (6.8)$$

with $x(k) = x(kT), x(k + 1) = x((k + 1)T), u(k) = u(kT)$, and

$$A = e^{A_c T}, \quad (6.9)$$

$$B = e^{A_c(k+1)T} \int_{kT}^{(k+1)T} e^{-A_c \tau} d\tau B_c = \int_0^T e^{A_c \lambda} d\lambda B_c. \quad (6.10)$$

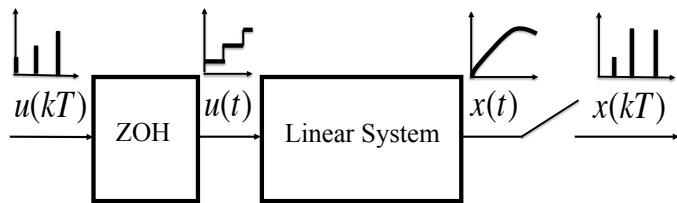


Figure 6.2: Zero-order hold (ZOH) approximation of continuous-time system

Example 6.1.1 Use the formulas (6.9) and (6.10) to compute the zero-order hold discrete-time approximation of the continuous-time kinematics model (6.3) with parameters (6.4).

Solution: We first note that the matrix exponential (6.6) becomes

$$e^{A_c T} = I + A_c T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad (6.11)$$

because $A_c^m = 0$ for $m \geq 2$. Computing the integral in (6.10) gives

$$B = \int_0^T e^{A_c \lambda} d\lambda B_c = \begin{bmatrix} \int_0^T 1 d\lambda & \int_0^T \lambda d\lambda \\ \int_0^T 0 d\lambda & \int_0^T 1 d\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}.$$

6.2 Cayley-Hamilton Theorem

In example 6.1.1 one could use the Taylor series (6.6) to compute the matrix exponential because all terms of order higher or equal to two were zero. However, for a general case this will not be true and one cannot resort to the infinite series (6.6) to compute the matrix exponential. For such cases one can use the Cayley-Hamilton Theorem.

Theorem 6.2.1 (Cayley-Hamilton) *Every real $n \times n$ matrix A_c satisfies its characteristic equation. In other words, if the characteristic polynomial is written as*

$$p(\lambda) = |\lambda I - A_c| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0, \quad (6.12)$$

then $p(\lambda_{A_c}) = 0$ and

$$p(A_c) = A_c^n + c_{n-1}A_c^{n-1} + \dots + c_0 = 0, \quad (6.13)$$

where λ_{A_c} is any eigenvalue of the matrix A_c .

The Cayley-Hamilton theorem relates a polynomial function of a matrix A_c with a polynomial function of a scalar, which is zero when the scalar is replaced by an eigenvalue λ_{A_c} of the matrix A_c .

We note that the infinite Taylor series becomes a finite series using the Cayley-Hamilton Theorem. This is possible because all terms of the form A_c^m , with $m \geq n$, can be written as a linear combination of terms of the form A_c^l , with $l < n$, using (6.13), as

$$e^{A_c t} = \sum_{m=0}^{\infty} \frac{(A_c t)^m}{m!} = \sum_{l=0}^{n-1} \alpha_l (A_c)^l = f(A_c) \quad (6.14)$$

for some coefficients $\alpha_l, l = 0, \dots, n-1$. To be able to compute $e^{A_c t}$ one must find a process to compute these coefficients. To achieve that goal, one can write the function corresponding to the polynomial (6.14) as a function of a scalar λ using the same coefficients $\alpha_l, l = 0, \dots, n-1$, as

$$f(\lambda) = \sum_{l=0}^{n-1} \alpha_l (\lambda)^l. \quad (6.15)$$

This polynomial function can be rewritten as

$$f(\lambda) = q(\lambda)p(\lambda) + r(\lambda) \quad (6.16)$$

by performing long division, where q is called the quotient, r is called the remainder, and p is the characteristic polynomial of the matrix A_c . Evaluating the function $f(\lambda)$ in (6.16) at the eigenvalues λ_{A_c} of the matrix A_c yields

$$f(\lambda_{A_c}) = r(\lambda_{A_c}) \quad (6.17)$$

because $p(\lambda_{A_c}) = 0$ by definition of characteristic polynomial. Writing the matrix polynomial $f(A_c)$ corresponding to (6.14) using (6.16) yields

$$e^{A_c t} = \sum_{l=0}^{n-1} \alpha_l (A_c)^l = f(A_c) = q(A_c)p(A_c) + r(A_c). \quad (6.18)$$

The Cayley-Hamilton theorem states that

$$f(A_c) = r(A_c) \quad (6.19)$$

because $p(A_c) = 0$. Therefore, from (6.18) and (6.19) we can conclude that

$$e^{A_c t} = \sum_{l=0}^{n-1} \alpha_l (A_c)^l = r(A_c). \quad (6.20)$$

From (6.15)–(6.17) the corresponding scalar polynomial is

$$e^{\lambda t} = \sum_{l=0}^{n-1} \alpha_l (\lambda)^l = r(\lambda), \quad (6.21)$$

where $\lambda = \lambda_{A_c}$. This is a system of n equations in n unknowns that can be solved to obtain the coefficients $\alpha_l, l = 0, \dots, n-1$, if the equations are independent.

Example 6.2.1 Using Cayley-Hamilton's theorem compute $e^{A_c T}$ for

$$A_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution: We first compute the eigenvalues of A_c by solving

$$|\lambda I - A_c| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = -1. \quad (6.22)$$

The eigenvalues are $\lambda_1 = i, \lambda_2 = -i$, where i is the imaginary unit. According to equation (6.21) with $t = T, \lambda = \lambda_1 = i, n = 2$, one can write

$$e^{iT} = \alpha_0 + \alpha_1 i. \quad (6.23)$$

Equation (6.21) for $\lambda = \lambda_2 = -i$ yields

$$e^{-iT} = \alpha_0 - \alpha_1 i. \quad (6.24)$$

Adding and subtracting (6.23) and (6.24) yields the solution

$$\begin{aligned}\alpha_0 &= \cos(T), \\ \alpha_1 &= \sin(T),\end{aligned}$$

using Euler's equations

$$\begin{aligned}\cos(T) &= \frac{e^{iT} + e^{-iT}}{2}, \\ \sin(T) &= \frac{e^{iT} - e^{-iT}}{2i}.\end{aligned}$$

Replacing $\alpha_0 = \cos(T)$, $\alpha_1 = \sin(T)$, $t = T$, and $n = 2$ in (6.20) yields

$$e^{A_c T} = \alpha_0 I + \alpha_1 A_c = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}.$$

When there are repeated eigenvalues equations (6.21) are not independent. One can take derivatives of both sides of (6.21) to write

$$te^{\lambda_{A_c}^i t} = \left. \frac{dr(\lambda)}{d\lambda} \right|_{\lambda_{A_c}^i}, \quad (6.25)$$

$$\vdots \quad \vdots \quad (6.26)$$

$$t^{\bar{k}-1} e^{\lambda_{A_c}^i t} = \left. \frac{d^{\bar{k}-1} r(\lambda)}{d\lambda^{\bar{k}-1}} \right|_{\lambda_{A_c}^i}, \quad (6.27)$$

The equations (6.21)–(6.27) form a set of n independent equations in the n unknowns α_l , $l = 0, \dots, n-1$.

Example 6.2.2 Compute $e^{A_c T}$ for A_c given in (6.4) using Cayley-Hamilton's theorem and confirm that the result is the same as in example 6.1.1.

Solution: We first compute the eigenvalues of A_c by solving

$$|\lambda I - A_c| = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2 = 0. \quad (6.28)$$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$. Therefore, there is one eigenvalue with multiplicity $\bar{k} = 2$. According to equation (6.21) with $t = T$, $\lambda = \lambda_1 = 0$, $n = 2$, one can write

$$1 = e^{\lambda_1 T} = \alpha_0 + \alpha_1 \lambda_1 = \alpha_0. \quad (6.29)$$

We can also use equation (6.25) to write

$$T = T e^{\lambda_1 T} = \alpha_1. \quad (6.30)$$

Replacing $\alpha_0 = 1$, $\alpha_1 = T$, $t = T$, and $n = 2$ in (6.20) yields

$$e^{A_c T} = \alpha_0 I + \alpha_1 A_c = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix},$$

which is the same result as the one in equation (6.11).

6.3 Minimum Mean Square Error Estimation

We will now start our study of state estimation from measurements so that a measurement equation can be added later to the state space models described in section 7.1. Measurements add information that can be taken into account to improve the estimate of the state. Before proceeding the reader may find it useful to review section 2.9. Given measurements y and a state x to be estimated, the minimum mean square error estimation is the solution of

$$\min_{\hat{x}(y)} E [\|x - \hat{x}(y)\|^2 \mid y] \quad (6.31)$$

Taking the derivative of the objective function $f(x, \hat{x}(y)) = E [\|x - \hat{x}(y)\|^2 \mid y]$ with respect to $\hat{x}(y)$ and equating to zero yields the necessary condition

$$\partial_{\hat{x}} f = -2E [x - \hat{x}(y) \mid y] = 0. \quad (6.32)$$

Since $\hat{x}(y)$ is a constant parameter once y is measured, the necessary condition yields the optimal solution

$$\hat{x}(y) = E [x \mid y]. \quad (6.33)$$

Taking the second derivative of the objective function yields the matrix $2I$, which is positive definite and confirms that the optimal solution (6.33) is a

minimizer. Replacing the minimizer (6.33) in the objective function yields the optimal value

$$E [\|x - E[x|y]\|^2 | y] = \sigma_{x|y}^2, \quad (6.34)$$

which is the minimum mean square error and corresponds to the trace of the conditional covariance matrix of x given y .

Example 6.3.1 Let the value of the random variable x be the number on the face of a dice that is rolled on a casino table. Let a measurement variable y have the value zero if the result is even and one if the result is an odd number. Compute the minimum mean square error estimate of the variable x given that $y = 0$ and compute the corresponding conditional minimum variance.

Solution: The possible values taken by the random variable x when $y = 0$ are the even numbers $x_1 = 2, x_2 = 4, x_3 = 6$. Since all three possible outcomes have the same probability, then $P(x = x_n) = \frac{1}{3}, n = 1, 2, 3$. The conditional expected value (6.33) is then

$$E[x|y=0] = \sum_{n=1}^3 x_n P(x = x_n) = \frac{1}{3}(2 + 4 + 6) = 4. \quad (6.35)$$

The conditional minimum variance is

$$\sigma_{x|y}^2 = E[x^2|y=0] - E^2[x|y=0] = \frac{2^2 + 4^2 + 6^2}{6} - 4^2 = \frac{8}{3}. \quad (6.36)$$

6.4 Affine Minimum Variance Estimation

In this section we study affine minimum variance estimation. This study is important because the Kalman filter is the minimum variance affine estimator, provided the noise signals $w(k)$ and $\nu(k)$ are zero-mean, uncorrelated, and white. In other words, the Kalman filter is the one that provides the minimum variance of the estimation error among all possible estimators that are an affine (linear with offset) function of the measurements. Given the measurement y , assuming that $E[(y - \mu_y)(y - \mu_y)^T] > 0$ is finite, where $\mu_y = E[y]$, we are interested in solving the optimization problem

$$\begin{aligned} \min_{W,b} \quad & E[\|x - \hat{x}\|^2] \\ \text{s.t.} \quad & \hat{x} = Wy + b \end{aligned} \quad (6.37)$$

where the unknowns are the matrix W and the vector b . The solution of this problem yields the best affine estimator in the least squares sense. Using the trace operator (see example 2.37) the problem (6.37) can be rewritten as

$$\begin{aligned} \min_{W,b} \quad & \text{Tr} \left[E \left((x - \hat{x}) (x - \hat{x})^T \right) \right] \\ \text{s.t.} \quad & \hat{x} = Wy + b \end{aligned} \quad (6.38)$$

where we have used the fact that the operators E and Tr can be interchanged because they are both linear. Although the objective function is the sum of the variances of the coordinates of the error vector, one can show that the optimal solution corresponds to the minimum variance of each coordinate because the problem can be decoupled (see [23]). The optimization problem (6.38) is therefore called minimum variance estimation. Using the constraint $\hat{x} = Wy + b$, to replace \hat{x} in the objective function, (6.38) can be rewritten

$$\min_{W,b} \quad \text{Tr} \left[E \left((x - Wy - b) (x - Wy - b)^T \right) \right]. \quad (6.39)$$

The argument of the trace and expected value operators can be expanded as

$$f_{W,b} = xx^T - xy^T W^T - xb^T - Wyx^T + Wyy^T W^T + Wyb^T - bx^T + by^T W^T + bb^T. \quad (6.40)$$

Solving the necessary condition

$$\partial_W \text{Tr} [E (f_{W,b})] = 0 \quad (6.41)$$

taking into account the partial derivatives

$$\partial_W [\text{Tr} (WB^T)] = B, \quad \partial_W [\text{Tr} (WBW^T)] = 2WB, \quad (6.42)$$

which are valid for any matrix B , as well as the two properties of the trace operator in equations (2.36) and (2.37), yields

$$E[xy^T] = WE[yy^T] + bE[y^T]. \quad (6.43)$$

Solving the necessary condition

$$\partial_b \text{Tr} [E (f_{W,b})] = 0 \quad (6.44)$$

using the same approach gives

$$E[x] = WE[y] + b. \quad (6.45)$$

Solving equations (6.43) and (6.45) with $E[(y - \mu_y)(y - \mu_y)^T] > 0$ yields

$$W = \mathcal{K}_{xy}\mathcal{K}_{yy}^{-1}, \quad (6.46)$$

$$b = E[x] - WE[y], \quad (6.47)$$

where

$$\mathcal{K}_{xy} = E[xy^T] - E[x]E[y^T], \quad (6.48)$$

$$\mathcal{K}_{yy} = E[yy^T] - E[y]E[y^T]. \quad (6.49)$$

From (6.46), (6.47), and the constraint $\hat{x} = Wy + b$, the solution \hat{x} is

$$\hat{x} = E[x] + \mathcal{K}_{xy}\mathcal{K}_{yy}^{-1}(y - E[y]). \quad (6.50)$$

Note that from (6.50) and linearity of the expected value operator we have $E[\hat{x}] = E[x]$. Therefore, the affine minimum variance estimator is an *unbiased estimator*. The minimum mean square error is obtained by first computing the estimation error

$$\delta x = (x - E[x]) - \mathcal{K}_{xy}\mathcal{K}_{yy}^{-1}(y - E[y]), \quad (6.51)$$

and then replacing (6.51) in the objective function of the optimization (6.38), which yields

$$\Sigma\sigma_{opt}^2(\delta x) = \text{Tr}(E[\delta x \delta x^T]).$$

Expanding the terms inside the expected value operator and using the rules (2.34)–(2.35) for the transpose of the sum and product of matrices yields

$$\Sigma\sigma_{opt}^2(\delta x) = \text{Tr}(P - K_{xy}K_{yy}^{-1}K_{xy}^T - K_{xy}K_{yy}^{-1}K_{xy}^T + K_{xy}K_{yy}^{-1}K_{yy}K_{yy}^{-1}K_{xy}^T), \quad (6.52)$$

where $P = E[(x - E[x])(x - E[x])^T]$ is the *a-priori* covariance matrix and $E[x]$ is the *a-priori* expected value of x . After canceling terms are removed from (6.52) the final result for the minimum mean square error is

$$\Sigma\sigma_{opt}^2(\delta x) = \text{Tr}(P - K_{xy}K_{yy}^{-1}K_{xy}^T). \quad (6.53)$$

Note that the *a-posteriori* or corrected covariance matrix $P - K_{xy}K_{yy}^{-1}K_{xy}^T$ has smaller eigenvalues then P because $K_{xy}K_{yy}^{-1}K_{xy}^T \geq 0$. Therefore, the trace of the corrected covariance martix is also smaller than the trace of P .

Example 6.4.1 Go back to example 6.3.1 and compute the affine minimum variance estimator and the corresponding minimum mean square error.

Solution: Since $y = 0$ if the dice shows an even number and $y = 1$ if it shows an odd number then the possible pair of values (x, y) are

$$(1, 1), (2, 0), (3, 1), (4, 0), (5, 1), (6, 0),$$

each happening with equal probability. Therefore,

$$\begin{aligned} E[xy] &= \frac{1+3+5}{6} = \frac{3}{2} \\ E[y^2] &= 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Since $E[x] = 3.5$ (see example 2.9.2) and $E[y] = E[y^2]$, replacing all values in equation (6.50) yields

$$\hat{x} = \frac{7}{2} + \left(\frac{3}{2} - \frac{7}{2} \right) \left(\frac{1}{2} - \frac{1}{2^2} \right)^{-1} \left(y - \frac{1}{2} \right) = 4 - y.$$

Therefore, if the result of rolling a dice is even then $y = 0$ and $\hat{x} = 4$, which is the same result we obtained in example 6.3.1. If the result is odd then $y = 1$ and $\hat{x} = 3$, which is the same value as $E[x|y = 1]$. The minimum mean square error is obtained from equation (6.53) and the value of σ^2 in example (2.9.2) as

$$\Sigma \sigma_{opt}^2 = \sigma^2 - \left(\frac{3}{2} - \frac{7}{2} \right)^2 \left(\frac{1}{2} - \frac{1}{2^2} \right)^{-1} = \frac{35}{12} - \frac{1}{4} = \frac{8}{3},$$

which is also the same result as the one in example 6.3.1. Note however that the minimum mean square error of the estimator calculated in example 6.3.1 must always be less or equal to the one of the affine minimum variance estimator because there is no guarantee that the optimal estimator is affine.

Example 6.4.2 Use the GPS linearized equation (5.13) repeated here for convenience

$$y = Hx + \nu, \quad (6.54)$$

where $x = \delta r$ and $y = \delta \rho$. Assume that the unknown position δr is a random vector with $E[\delta r] = 0$, $E[\delta r \delta r^T] = P > 0$, the measurement noise is a zero

mean random vector with covariance $E[\nu\nu^T] = R > 0$, and the that the two random vectors are uncorrelated, i.e., $E[\nu\delta r^T] = 0$. Compute the minimum variance estimation $\hat{\delta}r$ of the position.

Solution: Using (6.54) and (2.35) we start by computing the matrices

$$\begin{aligned} E[\delta r \delta \rho^T] &= E[\delta r (H \delta r + \nu)^T] = E[\delta r \delta r^T] H^T + [\delta r \nu^T] = PH^T, \\ E[\delta \rho \delta \rho^T] &= HE[\delta r \delta r^T] H^T + 0 + 0 + E[\nu \nu^T] = HPH^T + R. \end{aligned} \quad (6.55)$$

Using (6.50) and (6.55) with $E[\delta r] = 0, E[\delta \rho] = 0$, the minimum variance estimation is

$$\hat{\delta}r = PH^T(HPH^T + R)^{-1}\delta\rho. \quad (6.56)$$

Equation (6.56) is equivalent to

$$\hat{\delta}r = (H^T R^{-1} H + P^{-1})^{-1} H^T R^{-1} \delta\rho, \quad (6.57)$$

which can be proved by multiplying the equation

$$H^T R^{-1} (HPH^T + R) = (H^T R^{-1} H + P^{-1}) PH^T$$

by $(H^T R^{-1} H + P^{-1})^{-1}$ on the left and by $(HPH^T + R)^{-1}$ on the right.

6.5 Regularized Least Squares

In section 5.4 we computed the least squares solution of the position for the linearized GPS equations (6.54). However, the linearized GPS equations are only valid if the magnitude of the position deviation vector is not very large. To make sure that this is the case one can introduce an extra term into the objective function that will weight the magnitude of the deviation vector. This is called regularization and leads to the regularized least squares

$$\begin{aligned} \min_{x, \nu} \quad & \nu^T L \nu + x^T M x \\ \text{s.t.} \quad & y = Hx + \nu \end{aligned} \quad (6.58)$$

for given positive definite symmetric matrices L and M . Comparing the structure of the optimization problems (6.58) and (5.19) we see that there is an extra quadratic term $x^T M x$ in the objective function of (6.58), which

penalizes large values of the entries of x . If $M = I$ then the extra term $x^T x$ is the squared norm of x . For other positive definite matrices M the extra term is a weighted sum of squares. Solving the equality constraint of the optimization problem (6.58) for ν yields

$$\nu = y - Hx. \quad (6.59)$$

Replacing (6.59) into the objective function and rearranging using the rules (2.34)–(2.35) for the transpose of the sum and product of matrices, as well as the fact that $y^T L H x = x^T H^T L y$ for a scalar, yields the equivalent unconstrained optimization problem

$$\min_x f(x), \quad (6.60)$$

where

$$f(x) = x^T (H^T L H + M) x - 2x^T H^T L y + y^T L y. \quad (6.61)$$

As in chapter 5 to find the solution of the minimization (6.60) we solve the necessary condition

$$\partial_x f = 2 (H^T L H + M) x - 2H^T L y = 0. \quad (6.62)$$

The matrix $(H^T L H + M)$ is always invertible for a positive definite M , and the solution of (6.62) is the regularized least squares estimate

$$\hat{x} = (H^T L H + M)^{-1} H^T L y. \quad (6.63)$$

If M was allowed to be positive semi-definite (instead of positive definite) then one would recover the weighted least squares solution (5.26) for $M = 0$ in (6.63).

Example 6.5.1 *For the geometry of example 5.3.1 assume that the user is located somewhere close to the origin and that the three GPS satellites have the following pseudorange measurements:*

$$\begin{aligned} \rho_1 &= 1500 \text{ km}, \\ \rho_2 &= 2000 \text{ km}, \\ \rho_3 &= 2000 \text{ km} \end{aligned}$$

The magnitude of the x coordinate of the position of satellites 2 and 3 is 1000 km. The y coordinate of the position of satellite 1 is also 1000 km.

Assuming a 1σ User Equivalent Range Error (UERE) of 10 m compute the regularized least squares solution for $L = I$ and $M = \mathcal{K}^{-1}$, where \mathcal{K} is given by equation (5.41).

Solution: Recall that the matrix H of example 5.3.1 is

$$H = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \\ 0 & -1 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \end{bmatrix}. \quad (6.64)$$

With $L = I$ we have

$$H^T H = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The covariance matrix of the position error given by the equation (5.41) for $L = I$ is equal to

$$\mathcal{K} = (H^T H)^{-1} \sigma^2 = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} 100. \quad (6.65)$$

Therefore,

$$M = \mathcal{K}^{-1} = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} 0.01,$$

and

$$(H^T L H + M)^{-1} = \begin{bmatrix} \frac{200}{303} & 0 & 0 \\ 0 & \frac{200}{303} & 0 \\ 0 & 0 & \frac{100}{303} \end{bmatrix}. \quad (6.66)$$

From (5.27), (6.63), (6.64), (6.89), the regularized least squares solution is obtained as

$$\delta \hat{r} = \begin{bmatrix} \frac{200}{303} (750\sqrt{3} - 1000) \\ -\frac{200}{303\sqrt{3}} (750\sqrt{3} - 1000) \\ \frac{100}{303\sqrt{3}} (4500\sqrt{3} - 4000) \end{bmatrix}.$$

6.6 Special Cases of Regularized Least Squares

In this section we analyze two special cases of the regularized least squares estimation: maximum likelihood and minimum variance estimation.

6.6.1 Maximum Likelihood Estimation

We now assume that the noise ν is Gaussian with zero mean and covariance matrix R . From (6.54) the random vector $(y|x = \bar{x})$ is also Gaussian because it is equal to the sum of ν with the constant vector $H\bar{x}$. Assuming that y has m coordinates its joint probability density function is given by (2.87), i.e.,

$$p(y|x = \bar{x}) = \frac{1}{(2\pi)^{\frac{m}{2}} \sqrt{|\mathcal{K}_{y|x=\bar{x}}|}} e^{-\frac{1}{2}(y - \mu_{y|x=\bar{x}})^T \mathcal{K}_{y|x=\bar{x}}^{-1} (y - \mu_{y|x=\bar{x}})}. \quad (6.67)$$

The expected value of this random vector can be computed using (6.54) and the linearity property (2.80) as

$$\mu_{y|\bar{x}} = E[y|x = \bar{x}] = H\bar{x} + E[\nu|x = \bar{x}] = H\bar{x}. \quad (6.68)$$

The covariance matrix is

$$\mathcal{K}_{y|\bar{x}} = E[(y - \mu_{y|\bar{x}})(y - \mu_{y|\bar{x}})^T | x = \bar{x}]. \quad (6.69)$$

Given (6.54), (6.68), and the fact that $x = \bar{x}$, equation (6.69) leads to

$$\mathcal{K}_{y|\bar{x}} = E[\nu\nu^T] = R. \quad (6.70)$$

Replacing (6.68) and (6.70) into (6.67) yields

$$p(y|x = \bar{x}) = \frac{1}{(2\pi)^{\frac{m}{2}} \sqrt{|R|}} e^{-\frac{1}{2}(y - H\bar{x})^T R^{-1} (y - H\bar{x})}. \quad (6.71)$$

The maximum likelihood estimator corresponds to the maximization of

$$P(y|x = \bar{x}) = \int_{s \in S} p(y(s)|x(s) = \bar{x}) ds, \quad (6.72)$$

where the probability P is computed based on the set S of possible outcomes of the random experiment generating x . The estimator is the solution of the following optimization problem

$$\min_{\bar{x}} (y - H\bar{x})^T R^{-1} (y - H\bar{x}) \quad (6.73)$$

Table 6.1: Least Squares Solutions

Solution	Equation
Least Squares	$\hat{x} = (H^T H)^{-1} H^T y$
Weighted Least Squares	$\hat{x} = (H^T L H)^{-1} H^T L y$
Regularized Least Squares	$\hat{x} = (H^T L H + M)^{-1} H^T L y$
Maximum Likelihood	$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$
Affine Minimum Variance	$\delta \hat{x} = (H^T R^{-1} H + P^{-1})^{-1} H^T R^{-1} \delta y$

because the maximum probability occurs for the minimum value of the exponent of $p(y|x = \bar{x})$ from (6.71) since the integral in (6.72) is non-negative. Comparing (6.73) to (6.58) we see that the maximum likelihood estimator is the regularized least squares solution with $L = R^{-1}$ and $M = 0$. In other words, it is the weighted least squares solution when the weight matrix is chosen as the inverse of the measurement error covariance. For GPS measurements the matrix R is

$$R = \sigma^2 I, \quad (6.74)$$

where σ is the variance of the measurement. All least squares solutions are summarized in table 6.1 where $\delta \hat{x} = \hat{x} - E[x]$ and $\delta y = y - E[y]$.

6.7 Recursive Least Squares

So far it was assumed that all measurements are available at once and therefore we can use all information to obtain one of the least squares solutions of table 6.1. This is called a batch least squares solution. However, there may be cases where not all the measurements are available at the same time. In such situations one can compute a recursive least squares solution by updating a previous solution as new data arrives. To do this one must use the following result called the matrix inversion lemma.

Lemma 6.7.1 *Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times m}$ be two invertible matrices. Let $B \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{m \times n}$ be such that the matrices $(A + BCD)$ and $(C^{-1} + DA^{-1}B)$ are invertible. Then,*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \quad (6.75)$$

Proof: To prove this result we right multiply $(A + BCD)$ by the right hand side of equation (6.75) and show that it yields the identity matrix. The detailed steps are as follows:

$$\begin{aligned} & (A + BCD) \left(A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \right) = \\ & I - B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1} \left(I - B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \right) \\ & = I + BCDA^{-1} - BC(C^{-1} + DA^{-1}B)(C^{-1} + DA^{-1}B)^{-1}DA^{-1} = I \end{aligned}$$

If $(A + BCD)$ is left multiplied by the right hand side of equation (6.75) the result is also the identity matrix. This can be shown using very similar steps as the ones used above for the right multiplication. \square

The matrix inversion lemma will now be applied to the regularized least squares solution of the affine minimum variance estimator. The idea is to transform this regularized least squares solution into a recursive least squares solution. We start by noting that the set of measurement equations (6.54) can be analyzed one by one if we consider a single row of the matrix H at a time. The equation corresponding to the row i of matrix H , denoted by h_i^T , is written as

$$y(i) = h_i^T x + \nu(i), \quad (6.76)$$

where $y(i)$ and $\nu(i)$ denote the element i of the column vectors y and ν , respectively. When the standard deviation of the random variable $\nu(i)$ is $\sigma_i \neq 0$, equation (6.76) can be divided by σ_i to yield

$$\bar{y}(i) = \frac{y(i)}{\sigma_i} = \bar{h}_i^T x + \bar{\nu}(i), \quad (6.77)$$

where $\bar{\nu}(i)$ is a random variable with standard deviation equal to one and $\bar{h}_i = \sigma_i^{-1}h_i$. Assuming that the vector ν has uncorrelated entries we can therefore without loss of generality (upon a change of coordinates) assume that its covariance matrix is the identity. Therefore, defining $y_i = [y(1) \dots y(i)]^T$ and $\nu_i = [\nu(1) \dots \nu(i)]^T$, we can write the first i equations of (6.54) as

$$y_i = H_i x + \nu_i, \quad (6.78)$$

where

$$H_i = [h_1 \dots h_i]^T. \quad (6.79)$$

It is assumed that

$$R_i = E [\nu_i \nu_i^T] = I, \quad (6.80)$$

$$S_i = E [\nu_i x^T] = 0. \quad (6.81)$$

Moreover, we assume that x is a random vector with mean \bar{x} and covariance

$$P = E [(x - \bar{x})(x - \bar{x})^T] > 0. \quad (6.82)$$

Let the matrix P_i be defined as

$$P_i = (P^{-1} + H_i^T H_i)^{-1} = (P_{i-1}^{-1} + h_i h_i^T)^{-1}. \quad (6.83)$$

Applying the matrix inversion lemma to P_i with the choice of matrices $A = P_{i-1}^{-1}, B = h_i, C = I, D = h_i^T$ yields

$$P_i = P_{i-1} - \frac{P_{i-1} h_i h_i^T P_{i-1}}{1 + h_i^T P_{i-1} h_i}, \quad P_0 = P, \quad (6.84)$$

which is a recursive formula to compute the covariance matrix P_i . We will now derive a recursive formula for the regularized least squares solution (6.63) with the choice of matrices for the affine minimum variance estimator, i.e., $L = R_i^{-1} = I$ and $M = P^{-1}$. Under this choice, using the definition of P_i in (6.83), the state estimate is linear in y_i and can be written in the form

$$\hat{x}_i = P_i H_i^T y_i. \quad (6.85)$$

Using the recursive formula (6.84) and the relation

$$H_i^T y_i = H_{i-1}^T y_{i-1} + h_i y(i), \quad (6.86)$$

equation (6.85) can be expanded and rewritten as the recursive formula

$$\hat{x}_i = \hat{x}_{i-1} + \frac{P_{i-1} h_i}{1 + h_i^T P_{i-1} h_i} (y(i) - h_i^T \hat{x}_{i-1}), \quad \hat{x}_0 = 0. \quad (6.87)$$

Example 6.7.1 For the GPS geometry in figure 6.3 assume that all satellites are at 1000 kilometers from the origin. The user is assumed to be located close to the origin. The pseudorange measurements are given by

$$\begin{aligned} \rho_1 &= 1100 \text{ km}, \\ \rho_2 &= 1150 \text{ km}, \\ \rho_3 &= 880 \text{ km}, \\ \rho_4 &= 1120 \text{ km}. \end{aligned}$$

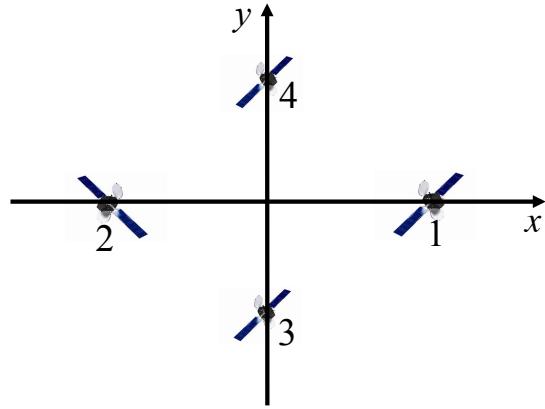


Figure 6.3: GPS geometry for the example 6.7.1

Assume that the measurement from satellite number 4 comes later than the measurements from the other three satellites. Compute the regularized least squares solution using satellites 1, 2, 3, and then update that solution using the pseudorange of satellite 4. Assume that $L = I$ and $M = P^{-1} = I$.

Solution: The matrices H_3 and y_3 for measurements 1, 2, 3 are

$$H_3 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 1100 - 1000 \\ 1150 - 1000 \\ 880 - 1000 \end{bmatrix}. \quad (6.88)$$

The matrix P_3 is given by

$$P_3 = (I + H_3^T H_3)^{-1} = \frac{1}{7} \begin{bmatrix} \frac{7}{3} & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \quad (6.89)$$

The regularized least squares solution is

$$\hat{x}_3 = P_3 H_3^T y_3 = \begin{bmatrix} 16.7 \\ -87.1 \\ 54.3 \end{bmatrix}.$$

For the measurement of satellite 4 the matrices are

$$h_4^T = [0 \ -1 \ 1], \quad y(4) = 1120 - 1000 = 120.$$

Using the recursive least squares formula (6.87) the updated solution is

$$\hat{x}_4 = \hat{x}_3 + \frac{P_3 h_4}{1 + h_4^T P_3 h_4} (y(4) - h_4^T \hat{x}_3) = \begin{bmatrix} 16.7 \\ -80.0 \\ 50.0 \end{bmatrix}.$$

Note that this is the same solution one would get if the measurements of all satellites were considered at once, which is called batch least squares.

To be able to interpret the recursive least squares procedure from a geometrical point of view, we observe that one can rewrite equation (6.84) as

$$P_i = P_{i-1} - \frac{(P_{i-1} h_i) (P_{i-1} h_i)^T}{1 + h_i^T P_{i-1} h_i} \quad (6.90)$$

since $P_{i-1} = P_{i-1}^T$. The formula (6.90) needs the matrix P_0 to start the recursive algorithm to compute P_i given P_{i-1} . It is often assumed that the initial covariance matrix P_0 of the state estimate \hat{x}_0 is diagonal with large diagonal elements. This is done because of two reasons. First, it makes sense to assume that the entries of \hat{x}_0 are uncorrelated. Second, the selection $\hat{x}_0 = 0$ in the recursive least squares algorithm is quite arbitrary. This uncertainty in the value of \hat{x}_0 can be expressed in the covariance matrix P_0 by selecting large positive diagonal elements, which are the variances of the entries of \hat{x}_0 . Therefore, we will assume that $P_0 = \sigma_x^2 I$ with a large $\sigma_x >> 1$. This implies that $P_0 h_1 = \sigma_x^2 h_1$. Using $\hat{x}_0 = 0$ and equation (6.87) the updated state estimate is then equal to

$$\hat{x}_1 = 0 + \frac{\sigma_x^2 h_1}{1 + \sigma_x^2 \|h_1\|^2} (y(1) - 0) \approx \frac{h_1}{\|h_1\|^2} y(1) \quad (6.91)$$

given that $\sigma_x^2 >> 1$. Using equation (6.90) the updated covariance matrix is

$$P_1 = \sigma_x^2 I - \frac{\sigma_x^4 h_1 h_1^T}{1 + \sigma_x^2 h_1^T h_1} \approx \sigma_x^2 (I - e_1 e_1^T) = \sigma_x^2 P_{h_1}^\perp, \quad (6.92)$$

where

$$e_1 = \frac{h_1}{\|h_1\|} \quad (6.93)$$

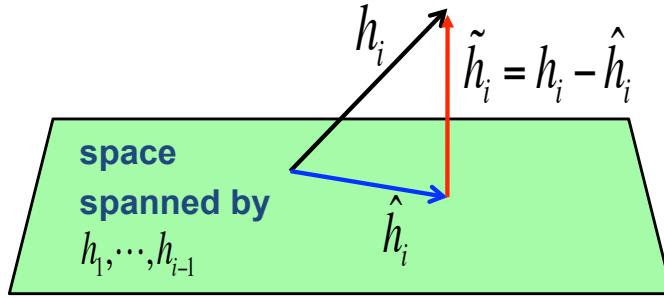


Figure 6.4: Geometry of recursive least squares for $P_0 = \sigma_x^2 I$, $\sigma_x \gg 1$

are the coordinates of the unit vector in the direction of h_1 , and $P_{h_1}^\perp$ is the matrix that projects any vector onto the space orthogonal to¹ h_1 . Let the component of h_2 orthogonal to h_1 be denoted by \tilde{h}_2 , which from (6.92) is

$$\tilde{h}_2 = P_{h_1}^\perp h_2 = h_2 - \hat{h}_2, \quad (6.94)$$

where

$$\hat{h}_2 = e_1 (e_1^T h_2) \quad (6.95)$$

is the projection of h_2 onto h_1 (see figure 6.4 for the case $i = 2$). Using equations (6.87) and (6.91) the updated state estimate is

$$\hat{x}_2 = \frac{h_1}{\|h_1\|^2} y(1) + \frac{\sigma_x^2 \tilde{h}_2}{1 + \sigma_x^2 h_2^T \tilde{h}_2} \tilde{y}(2) \approx \frac{h_1}{\|h_1\|^2} y(1) + \frac{\tilde{h}_2}{\|\tilde{h}_2\|^2} \tilde{y}(2), \quad (6.96)$$

where

$$\tilde{y}(2) = y(2) - h_2^T \hat{x}_1 \quad (6.97)$$

is the estimation error of the measurement $y(2)$ and

$$h_2^T \tilde{h}_2 = (h_2 - \hat{h}_2)^T \tilde{h}_2 = \|\tilde{h}_2\|^2 \quad (6.98)$$

¹see equation (2.55)

because $\hat{h}_2^T \tilde{h}_2 = 0$ given that \hat{h}_2 has the same direction of h_1 and \tilde{h}_2 is orthogonal to h_1 (see figure 6.4 for the case $i = 2$). Continuing this process we can see that at each iteration i the recursive formula is approximated by

$$\hat{x}_i = \hat{x}_{i-1} + \frac{\tilde{h}_i}{\|\tilde{h}_i\|^2} \tilde{y}(i). \quad (6.99)$$

After k iterations the recursive least squares estimate is approximated by

$$\hat{x}_k \approx \sum_{i=1}^k \frac{\tilde{h}_i}{\|\tilde{h}_i\|^2} \tilde{y}(i), \quad (6.100)$$

where $\tilde{h}_1 = h_1$ and $\tilde{y}(1) = y(1)$. The geometric interpretation of the recursive least squares estimation when $P_0 = \sigma_x^2 I$ with $\sigma_x >> 1$ is depicted in figure 6.4. We will get back to this geometric idea in the next chapter on Kalman filtering.

Chapter 7

Kalman Filtering

This chapter is about Kalman filtering and extended Kalman filtering. The chapter starts by defining linear random process models in section 7.1, which will be used as a prediction model for Kalman filtering. Section 7.2 will focus on both the prediction and correction stages of the Kalman filter, whereas the geometry of Kalman filtering is discussed in section 7.3. Extended Kalman filtering is then addressed in section 7.4.

7.1 Linear Random Process Models

In this section we will add an output measurement equation to the linear model developed previously in section 6.1. Measurement and process noise will also be added. We assume a sampling time $T > 0$ and a measurement $y(k) = y(kT)$ that is related to the state $x(k) = x(kT)$ and measurement noise $\nu(k) = \nu(kT)$ as

$$y(k) = H(k)x(k) + \nu(k), \quad (7.1)$$

where $H(k)$ indicates that the matrix H may be time-varying. The measurement noise is assumed to be a random process with mean and covariance given respectively by

$$E[\nu(k)] = 0, \quad (7.2)$$

$$E[\nu(k)\nu^T(l)] = R(k)\delta_{kl}, \quad (7.3)$$

where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise. The measurement noise $\nu(k)$ is a vector and $R(k)$ is its covariance matrix. Equations (7.2)–(7.3) imply that

the measurement noise is a zero mean white noise process. The value of a white noise process at a sampling time kT is uncorrelated with its value at any other sampling time lT if $k \neq l$, as stated in equation (7.3).

Example 7.1.1 Write down all equations of the linear random process model for the kinematics (6.1)–(6.2) assuming that only the velocity is measured, the initial state is known to be x_0 , and assuming that the variance of the measurement noise is $\sigma^2 = 2 \text{ m}^2$.

Solution: The linear random process model is described by the equations (6.3), (7.1), (7.2), and (7.3), where A_c and B_c are given by (6.4), the matrix H is $[0 \ 1]$, and $R(k) = \sigma^2 = 2 \text{ m}^2$. The initial state is assumed to be $x(0) = x_0$, where x_0 is known.

A more general linear random process model includes both process and measurement noise and a random initial condition. It is of the form

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + B_c u(t) + \eta(t) \\ y(k) &= H(k)x(k) + \nu(k)\end{aligned}\tag{7.4}$$

where the process noise $\eta(t)$ and measurement noise $\nu(kT)$ have zero mean and are uncorrelated with each other, i.e.,

$$E[\eta(t)\nu^T(kT)] = 0,\tag{7.5}$$

where $t \geq 0, kT \geq 0$. The process noise is assumed to be a random process with mean and covariance given respectively by

$$E[\eta(t)] = 0,\tag{7.6}$$

$$E[\eta(t)\eta^T(\tau)] = Q(t)\delta(t - \tau),\tag{7.7}$$

where $t \geq 0, \tau \geq 0$ and $\delta(t - \tau)$ is the Dirac delta, which is equal to zero for $t \neq \tau$ and has unit integral. The measurement noise is assumed to be a random process with mean and covariance given by (7.2) and (7.3), respectively. It is also assumed that the initial condition $x(0)$ is a random vector with mean and covariance given respectively by

$$E[x(0)] = x_0,\tag{7.8}$$

$$E[(x(0) - x_0)(x(0) - x_0)^T] = P(0),\tag{7.9}$$

and is independent of the processes $\eta(t), \nu(kT)$.

In Kalman filtering we will be working with a discrete-time approximation of the state $x(t)$ using a zero-order-hold model. Therefore, the matrices A and B will be obtained by equations (6.9) and (6.10), respectively. Using the same approach that led to equation (6.7), the zero-order-hold discrete-time random process model, with possibly time-varying $A(k), B(k)$, is written as

$$x(k+1) = A(k)x(k) + B(k)u(k) + w(k), \quad (7.10)$$

$$y(k) = H(k)x(k) + \nu(k), \quad (7.11)$$

with $x(k+1) = x((k+1)T), x(k) = x(kT), y(k) = y(kT), \nu(k) = \nu(kT)$, and¹

$$w(k) = w(kT) = e^{A_c(k+1)T} \int_{kT}^{(k+1)T} e^{-A_c\tau} \eta(\tau) d\tau \quad (7.12)$$

Note that $E[w(k)] = 0$ and

$$E[w(k)w^T(l)] = 0 \quad (7.13)$$

for $l \neq k$ because of equations (7.7) and (7.12). The covariance of $w(k)$ is computed using (7.7) as

$$Q(k) = E[w(k)w(k)^T] = e^{A_c(k+1)T} \int_{kT}^{(k+1)T} e^{-A_c\tau} Q(\tau) e^{-A_c^T\tau} d\tau e^{A_c^T(k+1)T}. \quad (7.14)$$

With the change of variables $\lambda = (k+1)T - \tau$, the integration (7.14) can be rewritten as

$$Q(k) = \int_0^T e^{A_c\lambda} Q((k+1)T - \lambda) e^{A_c^T\lambda} d\lambda. \quad (7.15)$$

Example 7.1.2 Compute the covariance matrix for the process noise w_k using the matrix exponential given by (6.11) and

$$Q(\tau) = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}. \quad (7.16)$$

¹The integral in (7.12) is a stochastic integral and is formally written using a Brownian motion process instead of white noise, which would however not change the formula (7.14).

Solution: Computing the integral in (7.14) with $T = -\tau$ in (6.11) gives

$$\begin{aligned} & \int_{kT}^{(k+1)T} \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} d\tau = \\ & = \sigma_v^2 \begin{bmatrix} \frac{\sigma_r^2}{\sigma_v^2} T + (k^2 + k + \frac{1}{3}) T^3 & - (k + \frac{1}{2}) T^2 \\ - (k + \frac{1}{2}) T^2 & T \end{bmatrix}. \end{aligned} \quad (7.17)$$

Using (6.11) with T replaced by $(k+1)T$ and (7.17), $Q(k)$ is the product

$$\sigma_v^2 \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sigma_r^2}{\sigma_v^2} T + (k^2 + k + \frac{1}{3}) T^3 & - (k + \frac{1}{2}) T^2 \\ - (k + \frac{1}{2}) T^2 & T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (k+1)T & 1 \end{bmatrix}.$$

Therefore, the final answer is

$$Q(k) = \sigma_v^2 \begin{bmatrix} \frac{\sigma_r^2}{\sigma_v^2} T + \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}.$$

7.2 Kalman Filtering

A Kalman filter is a predictor-corrector algorithm. It uses a mathematical model to predict the system's state at the next sampling time given the state estimate at the current sampling time, and then corrects this prediction after the next sample of the measurements is obtained. The prediction and correction phases will be the focus of sections 7.2.1 and 7.2.2, respectively.

7.2.1 Prediction

The objective of the Kalman filter is to have estimates of the state and the covariance of the estimation error at each sampling time $t = kT$, where $T > 0$ is the sampling period. As before, we will write $x(k), y(k)$ for the state and output at $t = kT$ as a shorter notation for $x(kT)$ and $y(kT)$, respectively. After the k -th measurement is received at $t = kT$, the estimate of the state is updated and is denoted by $\hat{x}(k|k)$. This is the state estimate at $t = kT$ given all measurements collected for $t = nT, n = 0, \dots, k$, and is the topic of section 7.2.2. In this section we will derive the estimate of the state at $t = (k+1)T$ given the measurements collected for $t = nT, n = 0, \dots, k$, which will be denoted as $\hat{x}(k+1|k)$. Since in the time interval $t \in (kT, (k+1)T)$ there is

no measurement, the estimate of the state at $t = (k+1)T$ will be a prediction of the state evolution based on the linear model. This prediction is computed applying the conditional estimation operator to equation (7.10). Recall from equation (6.33) that the conditional estimation yields the minimum mean square error estimator.

In what follows we will use the notation " $|k$ " to denote the sentence "given $y(0), \dots, y(kT)$ ". Applying the conditional estimation operator to equation (7.10) and using the linearity properties (2.80)–(2.81) with $A_k = A(k)$ and $B_k = B(k)$ yields

$$\hat{x}(k+1|k) = E[x(k+1)|k] = A_k E[x(k)|k] + B_k E[u(k)|k] + E[w(k)|k]. \quad (7.18)$$

Note that from (7.5) and (7.12) we have

$$E[w(k)\nu^T(l)] = 0. \quad (7.19)$$

Moreover, $w(k)$ is independent of the initial condition $x(0)$. Therefore, $w(k)$ is zero mean white noise uncorrelated with the measurements $y(nT)$, $n = 0, \dots, k$. Additionally, $u(k)$ is a non-random input. Therefore, equation (7.18) can be rewritten as

$$\hat{x}(k+1|k) = A_k \hat{x}(k|k) + B_k u(k). \quad (7.20)$$

After the k -th measurement is received at $t = kT$ the covariance of the state estimation error is denoted by (see section 7.2.2)

$$P(k|k) = E \left[(x(k) - \hat{x}(k|k)) (x(k) - \hat{x}(k|k))^T \right]. \quad (7.21)$$

Since in the time interval $t \in (kT, (k+1)T)$ there is no measurement, the covariance of the state estimation error

$$P(k+1|k) = E \left[(x(k+1) - \hat{x}(k+1|k)) (x(k+1) - \hat{x}(k+1|k))^T \right] \quad (7.22)$$

can be predicted at $t = (k+1)T$ based on the state evolution given by the linear model. Using equations (7.10) and (7.20), one can write

$$\tilde{x}(k+1|k) = x(k+1) - \hat{x}(k+1|k) = A_k (x(k) - \hat{x}(k|k)) + w(k). \quad (7.23)$$

Note that the estimation error $\tilde{x}(k+1|k)$ depends on $\nu(l), w(m)$, for $l = 0, \dots, k, m = 0, \dots, k-1$, as well as on the initial condition $x(0)$. Therefore, $w(k)$ is uncorrelated with the estimation error $\tilde{x}(k+1|k)$ due to its

independence of the initial condition and due to equations (7.13) and (7.19). Using this fact and equations (7.23), (7.21), (7.14), the covariance (7.22) can be expanded and rewritten as

$$P(k+1|k) = A_k P(k|k) A_k^T + Q(k). \quad (7.24)$$

7.2.2 Correction

In the previous section we obtained predictions for the state $\hat{x}(k+1|k)$ and the covariance matrix of the estimation error $P(k+1|k)$ at $t = (k+1)T$ given the measurements collected for $t = nT, n = 0, \dots, k$. These predictions are used as *a-priori* estimates that will be corrected upon the arrival of a new measurement at $t = (k+1)T$ using equation (6.50). Using (6.50) yields

$$E[x(k+1)|k+1] = \hat{x}(k+1|k) + \mathcal{K}_{xy} \mathcal{K}_{yy}^{-1} \tilde{y}(k+1|k), \quad (7.25)$$

where

$$\tilde{y}(k+1|k) = y(k+1) - E[y(k+1)|k] = H\tilde{x}(k+1|k) + \nu(k+1), \quad (7.26)$$

using (7.1) and (7.23) for the last equality. Equation (7.25) is rewritten as

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \mathcal{K}_{xy} \mathcal{K}_{yy}^{-1} \tilde{y}(k+1|k). \quad (7.27)$$

We recall that the estimation error $\tilde{x}(k+1|k)$ depends on $\nu(l), w(m)$, for $l = 0, \dots, k, m = 0, \dots, k-1$, as well as on the initial condition $x(0)$. Therefore, $\nu(k+1)$ is uncorrelated with the estimation error $\tilde{x}(k+1|k)$ due to its independence of the initial condition and due to equations (7.3) and (7.19). Computing the covariance matrices \mathcal{K}_{xy} and \mathcal{K}_{yy} using (7.3), (7.22), (7.23), (7.26), and the linearity of the expected value yields

$$\begin{aligned} \mathcal{K}_{xy} &= E[\tilde{x}(k+1|k)\tilde{y}^T(k+1|k)] = P(k+1|k)H_k^T, \\ \mathcal{K}_{yy} &= E[\tilde{y}(k+1|k)\tilde{y}^T(k+1|k)] = H_k P(k+1|k)H_k^T + R(k+1). \end{aligned} \quad (7.28)$$

The correction of the covariance matrix estimate is defined as

$$P(k+1|k+1) = E[\tilde{x}(k+1|k+1)\tilde{x}^T(k+1|k+1)], \quad (7.29)$$

where

$$\tilde{x}(k+1|k+1) = \tilde{x}(k+1|k) - \mathcal{K}_{xy} \mathcal{K}_{yy}^{-1} \tilde{y}(k+1|k), \quad (7.30)$$

Table 7.1: Kalman filter equations

Step	Equation
Initialization	$\hat{x}(0 1) = \hat{x}_0 = E[x(0)], P_0 = E[(x(0)-\hat{x}_0)(x(0)-\hat{x}_0)^T]$
State Prediction	$\hat{x}(k+1 k) = A_k \hat{x}(k k) + B_k u(k)$
State Covariance	$P(k+1 k) = A_k P(k k) A_k^T + Q(k)$
Output Covariance	$\mathcal{K}_{yy} = H_k P(k+1 k) H_k^T + R(k)$
State Correction	$\hat{x}(k+1 k+1) = \hat{x}(k+1 k) + P(k+1 k) H_k^T \mathcal{K}_{yy}^{-1} \tilde{y}(k+1 k)$
Covariance Correction	$P(k+1 k+1) = P(k+1 k) (I - H_k^T \mathcal{K}_{yy}^{-1} H_k P(k+1 k))$

using equation (7.27). Expanding the expected value on the right hand side of (7.29), using (7.22) and the linearity of the expected value yields the following four terms,

$$P(k+1|k) - Z - Z^T + \mathcal{K}_{xy} \mathcal{K}_{yy}^{-1} E[\tilde{y}(k+1|k) \tilde{y}^T(k+1|k)] \mathcal{K}_{yy}^{-1} \mathcal{K}_{xy}^T, \quad (7.31)$$

where $Z = E[\tilde{x}(k+1|k) \tilde{y}^T(k+1|k)] \mathcal{K}_{yy}^{-1} \mathcal{K}_{xy}^T$. From (7.28), (7.29), and (7.31),

$$P(k+1|k+1) = P(k+1|k) (I - H_k^T \mathcal{K}_{yy}^{-1} H_k P(k+1|k)), \quad (7.32)$$

where \mathcal{K}_{yy} is defined in (7.28) and $P(k+1|k) = P^T(k+1|k)$.

It can be shown that the affine minimum variance estimation (7.25) used for the correction is the minimum mean square error estimate $E[x(k+1)|y(0), \dots, y(k+1), x(0)]$ when the $x(k)$ and $y(k)$ are jointly Gaussian. This is the case when the noise processes $w(k), v(k)$, and the initial condition process $x(0)$ are all Gaussian. In this case, the Kalman filter is the minimum mean square error filter. Moreover, having estimates of the mean and covariance implies also having an estimate of the probability density function when the process and measurement noise are Gaussian random vectors. Note that even when $x(k)$ and $y(k)$ are not jointly Gaussian the Kalman filter is still the affine estimator with the least minimum mean square error among all possible affine estimators. The equations of all steps of the Kalman filtering procedure are summarized in table 7.1.

Example 7.2.1 An aircraft is climbing at a speed of 100 meters per second under the effect of turbulent wind, which is modelled as additive zero mean

white noise horizontal and vertical velocity with a covariance matrix of $10I$ meters squared per second squared. The flight path angle is 20 degrees. The measurement of altitude is contaminated with additive white noise with zero mean and one meter standard deviation. The initial position of the aircraft is unknown and modeled by white noise with zero mean and identity covariance matrix. All white noise processes are uncorrelated with each other. Write down the Kalman filter equations and show the results for one iteration of prediction and correction.

Solution: The state of the system is $x(t) = [r(t) \ h(t)]^T$ where $r(t)$ is the range and $h(t)$ is the altitude. The state space model of the system is

$$\begin{aligned}\dot{x}(t) &= u(t) + \eta(t), \\ y(kT) &= Hx(kT) + \nu(kT),\end{aligned}$$

where $H = [0 \ 1]$ and $u(t) = 100[\cos(20^\circ) \ \sin(20^\circ)]^T$. The process and measurement noise are characterized by $E[\eta(t)] = 0 \text{ ms}^{-1}$, $Q = E[\eta(t)\eta^T(t)] = 10I \text{ m}^2\text{s}^{-2}$ and $E[\nu(kT)] = 0 \text{ ms}^{-1}$, $R = E[\nu^2(kT)] = 1\text{m}^2$. The initial condition has mean $\hat{x}_0 = E[x(0)] = 0$ meters and covariance $P(0) = E[x(0)x(0)^T] = I \text{ m}^2$. Using equations (6.9) and (6.10) the zero-order-hold discrete-time model matrices are

$$\begin{aligned}A &= e^{0T} = I, \\ B &= TI.\end{aligned}$$

From equations (7.12) and (7.14) the discrete-time process noise and covariance matrix are

$$w(k) = w(kT) = \int_{kT}^{(k+1)T} \eta(\tau) d\tau \text{ ms}^{-1}$$

and

$$Q(k) = E[w(k)w(k)^T] = \int_{kT}^{(k+1)T} 10Id\tau = 10TI \text{ m}^2\text{s}^{-2}.$$

The Kalman filter equations are written in table 7.2. For $k = 0$ the output covariance is

$$\mathcal{K}_{yy} = P_{22}(0) - 1 + 1 = P_{22}(0) + 1 = 2.$$

Table 7.2: Kalman filter equations for example 7.2.1

Step	Equation
Initialization	$\hat{x}(0 0) = \hat{x}_0 = 0, P_0 = I$
State Prediction	$\hat{x}(k+1 k) = \hat{x}(k k) + Tu(k)$
State Covariance	$P(k+1 k) = P(k k) + 10TI$
Output Covariance	$\mathcal{K}_{yy} = P_{22}(k+1 k) + 1$
State Correction	$\hat{x}(k+1 k+1) = \hat{x}(k+1 k) + P(k+1 k) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathcal{K}_{yy}^{-1} \tilde{y}(k+1 k)$
Covariance Correction	$P(k+1 k+1) = P(k+1 k) (I - H^T \mathcal{K}_{yy}^{-1} H P(k+1 k))$

The first iteration of correction then yields

$$\begin{aligned}\hat{x}(0|0) &= 0 + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (y(0) - 0), \\ P(0|0) &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.\end{aligned}$$

The first iteration of prediction gives

$$\hat{x}(1|0) = \hat{x}(0|0) + Tu(0) = \begin{bmatrix} 100T \cos(20^\circ) \\ 0.5y(0) + 100T \sin(20^\circ) \end{bmatrix}$$

and

$$P(1|0) = P(0|0) + 10TI = \begin{bmatrix} 1 + 10T & 0 \\ 0 & \frac{1}{2} + 10T \end{bmatrix}.$$

7.3 Kalman Filter as a Recursive Projection

In Kalman filtering the corrected estimate (7.27) is the orthogonal projection onto the space of all available measurements. For a vector space of non-random vectors a projection is determined based on the dot product. Therefore, in order to generalize projections to random vectors, we define a product \langle , \rangle between random vectors with the same properties as the dot product, which are the following:

1. **Commutativity:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$,
2. **Linearity:** $\langle \alpha\vec{x}_1 + \beta\vec{x}_2, \vec{y} \rangle = \alpha \langle \vec{x}_1, \vec{y} \rangle + \beta \langle \vec{x}_2, \vec{y} \rangle$,
3. **Non-degeneracy:** $\langle \vec{x}, \vec{x} \rangle = 0$ only if $\vec{x} = 0$.

We call a product between real vectors that satisfies these properties an inner product. Recall from section 2.8 that the dot product of non-random vectors is commutative. It is also linear and non-degenerate. Therefore, the dot product is an example of an inner product.

In a vector space generated by n random vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ with zero mean and finite second moments, which are the entries of the covariance matrix, any vector \vec{v} has coordinates given by

$$[\vec{v}] = V_1 [\vec{v}_1] + \dots + V_n [\vec{v}_n], \quad (7.33)$$

where $V_i, i = 1, \dots, n$, are real coefficient square matrices of appropriate dimensions. Note that in the case of non-random vectors the equation (7.33) is written using real scalar coefficients instead of matrices. This is an important difference between spaces of random and non-random vectors. For random vectors with zero mean and finite second moments the inner product is the expected value of the dot product and it is written as

$$\langle \vec{x}, \vec{y} \rangle = E \left[[\vec{x}]^T [\vec{y}] \right] = E[x^T y]. \quad (7.34)$$

As an easy exercise the reader can show that the product (7.34) satisfies the properties of commutativity, linearity, and non-degeneracy. The space of random vectors with zero mean and finite second moments equipped with an inner product is called a Hilbert space². One can now define the notion of orthogonal vectors.

Definition 7.3.1 *Two vectors \vec{x} and \vec{y} are orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$.*

With a notion of orthogonality one can rewrite the projection formulas (2.24) and (2.54), respectively, as

$$\vec{p}_{\vec{v}_1, \vec{v}_2} = \langle \vec{v}_1, \vec{v}_2 \rangle \langle \vec{v}_2, \vec{v}_2 \rangle^{-1} \vec{v}_2, \quad (7.35)$$

$$\vec{p}_{\vec{v}_1, \vec{v}_2^\perp} = \vec{v}_1 - p_{\vec{v}_1, \vec{v}_2}, \quad (7.36)$$

²given that all Cauchy sequences converge in this space

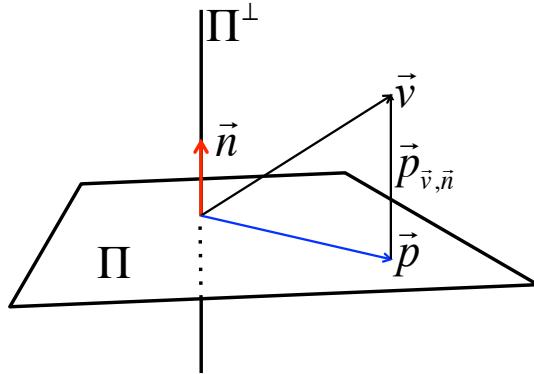


Figure 7.1: Orthogonal complement of a plane and projection on the plane

where \vec{v}_2^\perp is used to denote the orthogonal complement of the set spanned³ by \vec{v}_2 . The orthogonal complement of a set Π is the set of all vectors that are orthogonal to Π . For example, if the set Π is a plane then the set Π^\perp is a vertical line (see figure 7.1). Note that the projection operator (7.35) is linear, i.e., it satisfies the property

$$[\vec{p}_{V_1 \vec{v}_1 + V_2 \vec{v}_2, \vec{v}}] = V_1 [\vec{p}_{v_1, \vec{v}}] + V_2 [\vec{p}_{v_2, \vec{v}}] \quad (7.37)$$

for any real square matrices V_1 and V_2 of appropriate dimensions. We now define the direct sum of a subspace and its orthogonal complement and state one of the most important theorems of linear algebra.

Definition 7.3.2 *The direct sum $\Pi \oplus \Pi^\perp$ is the set of vectors of the form $\vec{h} = \vec{v} + \vec{v}^\perp$, $\vec{v} \in \Pi$, $\vec{v}^\perp \in \Pi^\perp$, with $\Pi \cap \Pi^\perp = \vec{0}$.*

Theorem 7.3.3 *Let $\Pi \subset H$ be a closed linear subspace of a Hilbert vector space H , and let Π^\perp be the orthogonal complement of Π . Then $H = \Pi \oplus \Pi^\perp$ and the decomposition $\vec{h} = \vec{v} + \vec{v}^\perp$, $\vec{v} \in \Pi$, $\vec{v}^\perp \in \Pi^\perp$ is unique for any $\vec{h} \in H$.*

Perhaps one of the strongest results in Hilbert space theory is the projection theorem that is used in Kalman filtering and is stated next.

³The set spanned by \vec{v}_2 are all vectors with coordinates $V_2 [\vec{v}_2]$ for any real matrix V_2

Theorem 7.3.4 Let H be a Hilbert space and let $\Pi \subset H$ be a closed linear subspace of H . Then for all $\vec{v} \in H$ there exists a unique $\vec{p} \in \Pi$ such that

$$\|\vec{v} - \vec{p}\| \leq \|\vec{v} - \vec{w}\|, \quad \forall \vec{w} \in \Pi. \quad (7.38)$$

Moreover, the vector $\vec{v} - \vec{p}$ belongs to Π^\perp , i.e.,

$$\langle \vec{v} - \vec{p}, \vec{w} \rangle = 0, \quad \forall \vec{w} \in \Pi. \quad (7.39)$$

Equation (7.39) is called the normal equation and the vector \vec{p} is called the projection of the vector \vec{v} onto Π .

Example 7.3.1 Let H be the space of n dimensional random vectors with zero mean and finite second moments and let $w = [\vec{w}] = [w_1 \ w_2 \ \dots \ w_m]^T$ be the coordinates of an m -dimensional random vector with zero mean and finite second moments. Find an expression for the projection of a vector $\vec{v} \in H$ onto the subspace Π spanned by \vec{w} given by

$$\Pi = \{Ww \mid \forall W \in \mathbb{R}^{n \times m}\}. \quad (7.40)$$

Solution: Invoking the Projection Theorem 7.3.4, there exists a unique projection $p = W_0 w$ that satisfies (7.39) so that one must find W_0 such that

$$E \left[(v - W_0 w)^T W w \right] = 0, \quad \forall W \in \mathbb{R}^{n \times m}.$$

This can equivalently be written as

$$E \left[\text{Tr} \left((v - W_0 w) w^T W^T \right) \right] = 0, \quad \forall W \in \mathbb{R}^{n \times m}.$$

Interchanging the trace and expected value operators, which are both linear,

$$\text{Tr} \left(E \left[(v - W_0 w) w^T \right] W^T \right) = 0, \quad \forall W \in \mathbb{R}^{n \times m}.$$

This condition must be true for all matrices W and it is thus equivalent to

$$E \left[(v - W_0 w) w^T \right] = 0.$$

Assuming $E \left[w w^T \right]$ is nonsingular and solving for W_0 yields

$$p = W_0 w = E \left[v w^T \right] (E \left[w w^T \right])^{-1} w. \quad (7.41)$$

Comparing (7.41) with (7.35) we see that

$$\langle \vec{v}, \vec{w} \rangle = E \left[[\vec{v}] [\vec{w}]^T \right] = E [vw^T] \quad (7.42)$$

can also be interpreted as an inner product. Note however that this inner product yields a matrix value instead of a scalar and therefore is it is not commutative. Although it fails to be commutative it is reflexive because it satisfies $E [vw^T] = (E [wv^T])^T$. It is thus a perfectly valid generalization of an inner product (see [24] for more details).

We can see an application of Theorems 7.3.3 and 7.3.4 in figure 7.1 (see also figure 6.4) where any vector $\vec{v} \in H$ can be expressed as $\vec{v} = \vec{p} + \vec{p}_{\vec{v}, \vec{n}}$, $\vec{p} \in \Pi$, $\vec{p}_{\vec{v}, \vec{n}} \in \Pi^\perp$, where \vec{n} is a unit vector normal to the plane and (7.39) is satisfied. Another application is the correction update in Kalman filtering. One can compute the corrected estimate $\hat{x}(k+1|k+1)$ using the prediction $\hat{x}(k+1|k)$ by representing the space of measurements as a direct sum of two orthogonal subspaces:

1. a subspace $\Pi = \mathcal{Y}_k$ corresponding to measurements for $t = nT, n = 0, \dots, k$,
2. a subspace $\Pi^\perp = \tilde{\mathcal{Y}}_{k+1}$ that is obtained based on the space spanned by the new measurement at $t = (k+1)T$.

This process is done recursively for each value of the index k . To show how it works let us start with $k = 0$. Note that $E[x(0)] = 0$ if it is assumed that $x(0)$ has either a uniform distribution in the interval $[-\bar{x}, \bar{x}]$, or if it has a zero-mean Gaussian distribution. In the case where $E[x(0)] = 0$, which from (7.1) implies $E[y(0)] = 0$, using (7.35), (7.42), (6.48), (6.49), the solution (7.27) can be interpreted as

$$\vec{x}(0|0) = \langle \vec{x}(0), \vec{y}(0) \rangle \langle \vec{y}(0), \vec{y}(0) \rangle^{-1} \vec{y} = \vec{p}_{\vec{x}(0), \vec{y}(0)}, \quad (7.43)$$

which is the projection of the vector $\vec{x}(0)$, with coordinates $x(0) = [\vec{x}(0)]$, onto the vector $\vec{y}(0)$. Furthermore, note that the solution for the correction $\hat{x}(k+1|k+1) - \hat{x}(k+1|k)$ and $k > 0$ proceeds in the same manner as for $k = 0$ but with x, y replaced by \tilde{x}, \tilde{y} , respectively. Comparing this with the solution (6.50) we conclude that the Kalman filter correction is an affine minimum variance estimator.

Suppose now that all measurements $y(nT)$ for $n = 0, \dots, k$ have been collected for a given $k > 0$. These measurements span the space \mathcal{Y}_k . We will use the notation $\hat{x}(k+1|k)$ to denote the projection of $x(k+1)$ onto the space \mathcal{Y}_k and we will show later in (7.51) that $\hat{x}(k+1|k)$ is given by (7.20). From $\hat{x}(k+1|k)$ one can compute the corresponding covariance matrix $P(k+1|k)$ leading to (7.24). For $k = 0$, $\hat{x}(0|-1) = \hat{x}_0$ and $P(0|-1) = P(0)$ (table 7.1).

Then at time $t = (k+1)T$ a new measurement arrives that satisfies the equation $y(k+1) = H(k+1)x(k+1) + \nu(k+1)$. As shown in (7.37) the projection operator is linear. Therefore, the projection of $y(k+1)$ onto the space \mathcal{Y}_k is

$$\hat{y}(k+1|k) = H(k+1)\hat{x}(k+1|k) + \hat{\nu}(k+1|k) = H(k+1)\hat{x}(k+1|k), \quad (7.44)$$

given that the noise is uncorrelated with all past measurements, i.e.,

$$E[\nu(k+1)y^T(n)] = 0, \forall n = 0, \dots, k, \quad (7.45)$$

which from the generalized inner product (7.42) implies that $\nu(k+1)$ is orthogonal to the space \mathcal{Y}_k . Note that (7.45) also implies that

$$E[\nu(k+1)^T y(n)] = 0, \forall n \leq k, \quad (7.46)$$

which in accordance with (7.34) again confirms that $\nu(k+1)$ is orthogonal to the space \mathcal{Y}_k .

We now find the vector that spans the space $\tilde{\mathcal{Y}}_{k+1} = \mathcal{Y}_k^\perp$. Using (7.36) the space $\tilde{\mathcal{Y}}_{k+1}$ is spanned by

$$\tilde{y}(k+1|k) = y(k+1) - \hat{y}(k+1|k). \quad (7.47)$$

Given that one must have $\hat{x}(k+1|k+1) \in \mathcal{Y}_k \oplus \tilde{\mathcal{Y}}_k$, from Theorem 7.3.3,

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + \vec{p}_{x(k+1), \tilde{y}(k+1|k)}. \quad (7.48)$$

This is equivalent to (7.27) using (7.35), (7.42), and the fact that

$$\langle x(k+1), \tilde{y}(k+1|k) \rangle = \langle \tilde{x}(k+1|k), \tilde{y}(k+1|k) \rangle \quad (7.49)$$

where

$$\tilde{x}(k+1|k) = x(k+1) - \hat{x}(k+1|k). \quad (7.50)$$

Equation (7.49) holds because $\hat{x}(k+1|k) \in \mathcal{Y}_k$ is orthogonal to $\tilde{y}(k+1|k) \in \tilde{\mathcal{Y}}_{k+1}$. Computing the covariance of $\hat{x}(k+1|k+1)$ from (7.48) yields (7.32).

From (7.48), using (7.10) and linearity of the projection operator,

$$\hat{x}(k+2|k+1) = A_k \hat{x}(k+1|k+1) + B_k \hat{u}(k+1|k+1) + \hat{w}(k+1|k+1). \quad (7.51)$$

Since $u(k)$ does not depend on the measurements and

$$E[w(k+1)y^T(n)] = 0, n = 0, \dots, k+1, \quad (7.52)$$

then (7.51) implies (7.20). The prediction-correction then repeats itself.

7.4 Extended Kalman Filtering

To obtain the Kalman filter equations we have assumed that the mathematical model was linear. However, there are quite often nonlinearities in the model of a dynamic system. Let us assume a nonlinear model of the form

$$\dot{x}(t) = f(x(t), u(t)) + \eta(t), \quad (7.53)$$

$$y(k) = h(x(k)) + \nu(k), \quad (7.54)$$

where the functions f and h are continuously differentiable. Given a reference trajectory $\bar{x}(t)$ satisfying the equations

$$\dot{\bar{x}}(t) = f(\bar{x}(t), u(t)), \quad (7.55)$$

$$y(k) = h(\bar{x}(k)), \quad (7.56)$$

let

$$A(k) = [\partial_x f]_{\bar{x}} \quad (7.57)$$

$$H(k) = [\partial_x h]_{\bar{x}}. \quad (7.58)$$

Defining the errors $\delta x(t) = x(t) - \bar{x}(t)$ and $\delta y(k) = y(k) - \bar{y}(k)$, one can write

$$\delta \dot{x}(t) = f(x(t), u(t)) - f(\bar{x}(t), u(t)) + \eta(t), \quad (7.59)$$

$$\delta y(k) = h(x(k)) - h(\bar{x}(k)) + \nu(k). \quad (7.60)$$

Using a first order Taylor series approximation taking into account (7.57) and (7.58) one can write the linearized model as

$$\delta \dot{x}(t) = A(k) \delta x(t) + \eta(t), \quad (7.61)$$

$$\delta y(k) = H(k) \delta x(k) + \nu(k). \quad (7.62)$$

Kalman filtering can now be applied to this linearized model. This process is called *extended Kalman filtering* and is illustrated in the next example.

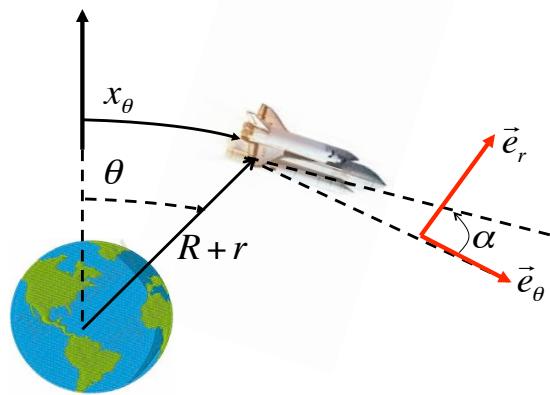


Figure 7.2: Constant altitude flight over spherical Earth

Example 7.4.1 Consider a flying vehicle on a constant altitude planar trajectory around the Earth, assumed to be a perfect sphere. The vehicle has one accelerometer to measure accelerations along the x body axis and one gyro to measure the angular velocity around the y body axis pointing out of the plane (see figure 7.2). Assume that both the accelerometer and the gyro have known biases b_a, b_g , respectively, and are subject to additive zero mean white measurement noise. The additive noise signals ν_A, ν_g for the accelerometer and the gyro, respectively, are uncorrelated with each other. Write down the linearized INS error equations in the matrix form (6.3) that can then be used in Kalman filtering. Assume a small angle α and $(R + r) \gg v^2$, where v is the speed of the vehicle, r is the flight altitude, and R is the Earth's radius.

Solution: Since the altitude is constant the velocity in the radial direction is zero, i.e., $v_r = 0$ and, consequently, $\dot{v}_r = 0$. Therefore, the kinematic

equations (4.28), (4.29), and (4.30) can be rewritten as

$$\dot{\theta} = \frac{v_\theta}{R+r}, \quad (7.63)$$

$$0 = a_r + \frac{v_\theta^2}{R+r}, \quad (7.64)$$

$$\dot{v}_\theta = a_\theta, \quad (7.65)$$

where r is the constant altitude. The distance travelled x satisfies the equation

$$\dot{x} = v_\theta, \quad (7.66)$$

and the angular velocity relative to inertial space is

$$\omega = \dot{\alpha} - \dot{\theta}, \quad (7.67)$$

where $\dot{\alpha}$ is the pitch rate of the vehicle and $\dot{\theta}$ is the rate of change of the angular displacement along the trajectory (see figure 7.2). The accelerometer and gyro measurements are

$$a_m = a_\theta \cos \alpha + (a_r + g) \sin \alpha + b_a + \nu_a, \quad (7.68)$$

$$\omega_m = \omega + b_g + \nu_g, \quad (7.69)$$

where b_a, b_g are the known biases and ν_a, ν_g are the additive measurement noise signals. Following the strategy outlined in section 4.3, the mechanization equations are

$$\dot{\hat{x}} = \hat{v}_\theta, \quad (7.70)$$

$$\dot{\hat{\theta}} = \frac{\hat{v}_\theta(t)}{R+r}, \quad (7.71)$$

$$\dot{\hat{v}}_\theta = a_m - b_a - (\hat{a}_r + g) \sin \hat{\alpha}, \quad (7.72)$$

$$\dot{\hat{\alpha}} = \omega_m - b_g + \dot{\hat{\theta}}, \quad (7.73)$$

where

$$\hat{a}_r = -\frac{\hat{v}_\theta^2}{R+r}, \quad (7.74)$$

from (7.64). Let the $\delta x = x - \hat{x}$, $\delta v_\theta = v_\theta - \hat{v}_\theta$, and $\delta \alpha = \alpha - \hat{\alpha}$. The INS error equations are then written as

$$\delta \dot{x} = \delta v_\theta, \quad (7.75)$$

$$\delta \dot{\alpha} = \frac{\delta v_\theta}{R+r} - \nu_g, \quad (7.76)$$

$$\delta \dot{v}_\theta = a_\theta (1 - \cos \alpha) - (a_r \sin \alpha - \hat{a}_r \sin \hat{\alpha}) - g (\sin \alpha - \sin \hat{\alpha}) - \nu_a. \quad (7.77)$$

Assuming a small angle α we have $\sin \alpha \approx \alpha$, $\cos \alpha \approx 1$. With the additional assumption of small $\hat{\alpha}$ equation (7.77) can be approximated by

$$\delta\dot{v}_\theta = \left(\frac{v_\theta^2}{R+r} \alpha - \frac{\hat{v}_\theta^2}{R+r} \hat{\alpha} \right) - g\delta\alpha - \nu_a \quad (7.78)$$

using (7.64) and (7.74). Assuming small δv_θ and small $\delta\alpha$, the term inside brackets can be approximated by a first order Taylor series as

$$\delta\dot{v}_\theta = \left(2\frac{\hat{v}_\theta}{R+r} \hat{\alpha} \delta v_\theta + \frac{\hat{v}_\theta^2}{R+r} \delta\alpha \right) - g\delta\alpha - \nu_a \quad (7.79)$$

The first term inside brackets is the product of three small terms because δv_θ and $\hat{\alpha}$ are assumed to be small and $\hat{v}_\theta \ll R+r$, given that R is the radius of the Earth. Neglecting this term, equation (7.79) can be approximated by

$$\delta\dot{v}_\theta = \left(\frac{\hat{v}_\theta^2}{R+r} - g \right) \delta\alpha - \nu_a. \quad (7.80)$$

Finally, since $g(R+r) \gg \hat{v}_\theta^2$ equation (7.80) can be approximated by

$$\delta\dot{v}_\theta = -g\delta\alpha - \nu_a. \quad (7.81)$$

Therefore, the linearized INS error equations are

$$\delta\dot{x} = \delta v_\theta, \quad (7.82)$$

$$\delta\dot{v}_\theta = -g\delta\alpha - \nu_a, \quad (7.83)$$

$$\delta\dot{\alpha} = \frac{\delta v_\theta}{R+r} - \nu_g, \quad (7.84)$$

which are valid for small angles $\alpha, \hat{\alpha}$, small errors $\delta\alpha, \delta v_\theta$, and $(R+r) \gg \hat{v}_\theta^2$. Equations (7.82)–(7.84) can be written in the matrix form (6.3) with $u = [\nu_a \ \nu_b]^T$ and

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & \frac{1}{R+r} & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (7.85)$$

The INS error equations can be combined with GPS measurements using a Kalman filter. Since the INS sensor provides measurements at a higher rate than the GPS sensor, the integration of the two measurements will occur in a multi-rate scenario.

Chapter 8

INS and GPS Integration

This chapter will use Kalman filtering to integrate measurements from INS and GPS. These two sensors have complementary characteristics. One of the most important differences between the two sensors is that INS errors will get quite large in long term whereas GPS errors are long term stable. In terms of the measurement frequency rate, INS is a high rate sensor while GPS is a self-initializing sensor that provides measurements at a lower rate. The possible outages of GPS measurements in urban centers due to loss of satellite tracking or limited coverage can be compensated for by the INS measurements, which experience no outages. In terms of attitude measurements, INS supplies angular velocity whereas a GPS receiver with a single antenna does not. Finally, unlike the mechanically driven INS, the GPS signal can be spoofed or jammed because it is a propagating electrical signal. For all of these complementary characteristics INS and GPS are perfect candidates for sensor integration.

8.1 Integration Approaches

There are essentially four different approaches for integration of INS and GPS measurements:

- Uncoupled
- Loosely coupled
- Tightly coupled

- Deep or ultra-tight integration

In uncoupled integration both signals are independently available and an algorithm chooses either one or the other at any given time (see figure ??). Typically GPS is more accurate and is chosen unless it is not available. GPS is also used to reset INS errors without estimating drift.

Example 8.1.1 Assume motion in a 1D flat Earth with an accelerometer and a GPS sensor. The accelerometer measurement has a bias b for which an estimate \hat{b} is known. The GPS measurement of position has an error e and a sampling period of T seconds. Compute the INS position error and reset it at time $t = T$ seconds using the first GPS measurement as it would be done in a decoupled integration.

Solution: The kinematic equations are

$$\dot{x} = v, \quad (8.1)$$

$$\dot{v} = a. \quad (8.2)$$

and the INS mechanization equations are

$$\dot{\hat{x}} = \hat{v}, \quad (8.3)$$

$$\dot{\hat{v}} = a + \delta b, \quad (8.4)$$

where $\delta b = b - \hat{b}$. Defining $\delta x = x - \hat{x}$ and $\delta v = v - \hat{v}$ the error equations are

$$\delta \dot{x} = \delta v, \quad (8.5)$$

$$\delta \dot{v} = -\delta b, \quad (8.6)$$

with solutions

$$\delta x(t) = \delta x(0) + \delta v(0)t - \frac{\delta b}{2}t^2, \quad (8.7)$$

$$\delta v(t) = \delta v(0) - \delta bt. \quad (8.8)$$

At time $t = T$ the GPS position measurement is $y(T) = x(T) + e$. Therefore, one can reset the estimate of position to be $\hat{x}(T) = y(T) = x(T) + e$, which leads to a new error $\delta x(T) = e$. Usually the GPS measurement error e is much smaller than the value of $\delta x(T)$ obtained from equation (8.7) replacing t by T .

Both loosely coupled and tightly coupled integration use a Kalman filter to fuse the measurements of INS and GPS. One of the main differences is that GPS position, velocity and time (PVT) solutions are used in loosely coupled integration, whereas tightly coupled integration uses GPS raw measurements: pseudo range, delta pseudo range, and carrier phase. The other important difference is that in tightly coupled integration the position and velocity of the INS are used in the tracking loops of the GPS receiver to reduce the tracking loop bandwidths.

In deep or ultra-tight integration the INS and GPS sensors no longer work as independent systems. To implement this type of integration one needs firmware access or software receivers. The deeply integrated approach corresponds to the formulation and solution of a state estimation problem for the entire multidimensional navigation state vector. The optimal solution of this problem is a nonlinear filter that departs from the more classical Kalman or extended Kalman filter solutions used in other integration approaches.

In the rest of this chapter we will only focus on loosely coupled integration.

8.2 Loosely Coupled Integration

The loosely coupled integration approach is shown in figure ???. Recall from example 8.1.1 that the state estimation error is defined to be $\delta x(t) = x(t) - \hat{x}(t)$, where $x(t)$ is the state and $\hat{x}(t)$ is the state estimate at time t . Equivalently, one can write $\hat{x}(t) = x(t) - \delta x(t)$. We will consider that the estimates are done every sampling time $T > 0$ and therefore we will write $\hat{x}(KT) = x(kT) - \delta x(kT)$, or simply $\hat{x}(k) = x(k) - \delta x(k)$. Therefore, the idea of INS and GPS loosely coupled integration is that the INS measurement will lead to the estimate $\hat{x}(t)$ and the GPS measurement will provide an estimate of the term $\delta x(t)$ that can be added to $\hat{x}(t)$ to yield the true value of the state $x(t)$. However, only in the ideal case can one obtain the true value of the state. In practice, the GPS measurement allows to correct the state estimate from the INS to make it closer to the true value. To understand how this correction is performed, recall that in chapter 7.2 we saw that there will be two estimates of the state: $\hat{x}^-(k)$ is the estimate based on the model prediction and $\hat{x}^+(k)$ is the estimate based on the correction of the prediction after a measurement arrives. In figure ?? the INS mechanization equations lead to the estimate $\hat{x}^-(k) = x(k) - \delta x^-(k)$. Assume as usual that

the output measurement satisfies the equation

$$y(k) = H_k x(k) + \nu(k) \quad (8.9)$$

where $\nu(k)$ is zero mean white measurement noise. The difference between the output measurement $y(k)$ coming from the GPS and the predicted value of the measurement $\hat{y}^-(k)$ based on the INS mechanization equations is

$$\delta y(k) = y(k) - \hat{y}^-(k) = y(k) - H_k \hat{x}^-(k). \quad (8.10)$$

This output error will be used in a tightly coupled integration as the measurement equation of a Kalman filter. The filter then provides the correction of the state estimation error as

$$\delta \hat{x}^+(k) = \delta \hat{x}^-(k) + K_k (\delta y(k) - H_k \delta \hat{x}^-(k)) \quad (8.11)$$

where the Kalman gain is given by

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \quad (8.12)$$

with $R_k = E [\nu(k)\nu^T(k)]$ and $P_k^- = E [\delta x^-(k) (\delta x^-(k))^T]$, and the *a-priori* estimation error satisfies the equation

$$\delta \hat{x}^-(k+1) = A \delta \hat{x}^-(k) + B u(k) \quad (8.13)$$

where $u(k)$ is the control input and the matrices A, B are obtained from the INS mechanization state space model. Based on this model a similar derivation as the one for Kalaman filtering in chapter 7.2 leads to the update of the covariance matrix of the state estimation written as

$$P_{k+1}^- = A P_k^- (A^T + Q_k) \quad (8.14)$$

where $Q_k = E [w(k)w(k)^T]$ and $w(k)$ is zero mean white process noise. The correction of the covariance matrix of the state estimation after a measurement arrives is written as

$$P_k^+ = [I - K_k H_k] P_k^- . \quad (8.15)$$

In the next two sections we apply the loosely coupled integration to two case studies: flat Earth and curved Earth.

8.3 Loosely Coupled Integration for Flat Earth

8.4 Loosely Coupled Integration for Curved Earth

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