Bounds for Preperiodic Points for Maps with Good Reduction

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Let $\mathbb{P}^1(K) = \{[x : y] \mid [x : y] \sim [\lambda x : \lambda y] \quad \lambda \in K^*\} = K \cup \{\infty\}$ be the projective line. When we write $\mathbb{P}^1$ is assume to be $\mathbb{P}^1(\bar{K})$.

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The **orbit length** of P is the cardinality of the orbit of P (as a set).

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The set of K-rational tail points for ϕ is denoted by $\mathrm{Tail}(\phi, K)$.

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We can view $\mathbb{P}^1(K)$ as $K \cup \{\infty\}$ and endomorphism of $\mathbb{P}^1$ as rational functions. Consider $\phi_c(z) = z^2 + c$.

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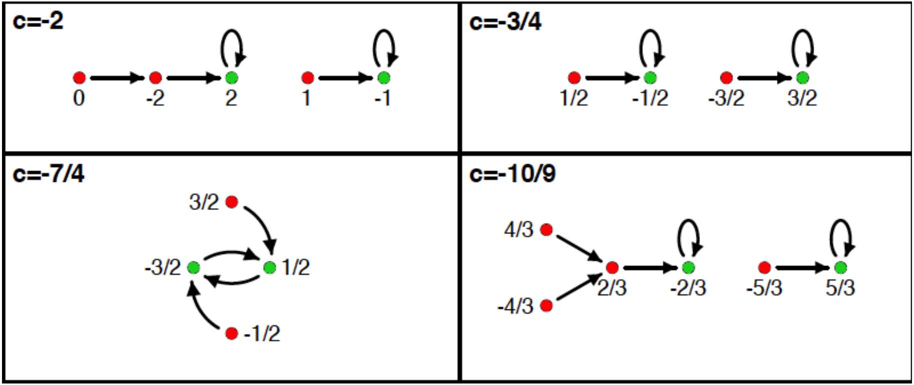

$\mathbb{Q}$-rational tail points (red) and $\mathbb{Q}$-rational periodic points (green) of $\phi_c(z) = z^2 + c$.

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Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

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We can deduce from the original proof of Northcott's theorem a bound for $|\text{PrePer}(\phi, K)|$ depending on

- n
- $[K : \mathbb{Q}]$
- the degree of ϕ
- height of the coefficients of ϕ

Goals:

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Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree D, and $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism of degree $d \geq 2$ defined over K, then

$$|\operatorname{PrePer}(\phi, K)| \leq B.$$

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- Lattes maps are the only nontrivial family of rational maps for which the UBC is currently known.

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Let ϕ be a quadratic polynomial defined over $\mathbb{Q}$ then

$$|\operatorname{PrePer}(\phi, \mathbb{Q})| \leq 9.$$

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If $\phi = x^2 + d$ then B. Hutz and P. Ingram have shown that Poonen's conjecture holds when the numerator and denominator of d don't exceed 10^8.

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, K)$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\mathrm{PrePer}(\phi, K)|$.

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Give explicit bounds for $|\mathrm{Tail}(\phi, K)|$, $|\mathrm{Per}(\phi, K)|$ and $|\mathrm{PrePer}(\phi, K)|$ in terms of:

- $D = [K : \mathbb{Q}]$
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Normalized Form and Good Reduction

Let ϕ be an endomorphism of $\mathbb{P}^1$ defined over K, $\mathfrak{p}$ be a non zero prime ideal of $\mathcal{O}_K$, $\mathcal{O}_\mathfrak{p}$ the local ring at $\mathfrak{p}$ and $k = \mathcal{O}_\mathfrak{p}/\mathfrak{p}$ the residue field of $\mathcal{O}_\mathfrak{p}$. Let $F, G \in K[X, Y]$ be homogeneous polynomials of the same degree with no common zero on $\mathbb{P}^1$

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- For an any representation $\phi = [F, G]$ we can find a $c \in K^*$ such that $[cF, cG]$ is in normalized form with respect to $\mathfrak{p}$.

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- Write $\phi = [F, G]$ in normalized form with respect to $\mathfrak{p}$. Consider the reduction of ϕ modulo $\mathfrak{p}$ given by

$$\tilde{\phi} = [\tilde{F}, \tilde{G}]$$

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- ϕ has **bad reduction** at $\mathfrak{p}$ if it does not have good reduction at $\mathfrak{p}$.

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If we allow the number of primes of bad reduction as a parameter, much more is known for the cardinality of the set of K-rational preperiodic points.

Bound on maximal period

Theorem (W. Narkiewicz 1988)

Let $\phi \in K[z]$ be a polynomial of degree ≥ 2 defined over a number field K of degree $D = [K : \mathbb{Q}]$. Suppose ϕ has good reduction outside a finite set of places S, including all archimedean ones. Let $s = |S|$.

If P is a K-rational periodic point of period n, then

$$n \leq (6 \cdot 7^{D+2s})^{\alpha},$$

where $\alpha = O(s \log s)$.

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci 2006)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree at least two defined over a number field K. Suppose ϕ has good reduction outside a finite set of places S, including all archimedean ones. Let $s = |S|$.

If $P \in \mathrm{PrePer}(\phi, K)$ is of orbit length n, then

$$n \leq \left[e^{10^{12}} (s+1)^8 (log(5(s+1)))^8 \right]^s .$$

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci, L. Paladino 2015)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over a number field K and $[K : \mathbb{Q}] = D$. Suppose ϕ has good reduction outside a finite set of places S, including all archimedean ones. Let $s = |S|$. If $P \in \mathrm{PrePer}(\phi, K)$ is of orbit length n, then

$$n \leq \max\left\{(2^{16s-8} + 3)\left[12s\log(5s)\right]^D, \left[12(s+2)\log(5s+5)\right]^{4D}\right\}.$$

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From here we can deduce a bound for $|\,\mathrm{PrePer}(\phi, K)|$ that is roughly of the order $d^{2^{16s}(s\log(s))^D}$.

Bounds independent of the degree

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S.

(a) If there are at least three K-rational tail points of ϕ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four K-rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

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Notice that under these hypotheses the bounds are independent of the degree of ϕ. Those hypotheses are sharp, *i.e.* if there are two (three) K-rational tail (periodic) points then $|\operatorname{Per}(\phi, K)|$ ($|\operatorname{Tail}(\phi, K)|$) must depend on d.

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We can prove an explicit bound for the desire sets.

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(a) $|\operatorname{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3$.

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|d^3})$.

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$.

Notice that the bounds obtained in the theorem are a significant improvement from the previous bound given by Canci and Paladino which was of the order $d^{2^{16s}(s\log(s))^D}$ for the set $|\operatorname{PrePer}(\phi, K)|$.

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- Logarithmic p-adic chordal distance.
- S-unit equation.
- Study the distance between periodic and tail points.

logarithmic p-adic chordal distance

Let K be a number field, and $\nu_\mathfrak{p}$ a valuation associated to a prime $\mathfrak{p}$ of K. Let $P = [X_1 : Y_1], Q = [X_2 : Y_2] \in \mathbb{P}^1(K)$. Then

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logarithmic p-adic chordal distance:

$$\delta_{\mathfrak{p}}(P, Q) = v_{\mathfrak{p}}(X_1 Y_2 - X_2 Y_1) - \min\{v_{\mathfrak{p}}(X_1), v_{\mathfrak{p}}(Y_1)\} - \min\{v_{\mathfrak{p}}(X_2), v_{\mathfrak{p}}(Y_2)\}$$

S-unit equations

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$$au + bv = 1$$
where $(u, v) \in (\mathcal{O}_S^*)^2$ and $a, b \in K^*$ is called a S-unit equation.
- Beukers and Schlickewei give an explicit bound for the S-unit equation. The number of solutions $(u, v) \in (\mathcal{O}_S^*)^2$ to
$$au + bv = 1$$
is bounded by
$$2^{8(2|S|+2)}.$$

Distance between periodic and tail points

Theorem (S. Troncoso 2016)

Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Suppose ϕ has good reduction outside S. Let $R \in \mathbb{P}^1(K)$ be a tail point and let n be the period of the periodic part of the orbit of R. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some m. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.

For simplicity suppose $\mathcal{O}_S$ is a PID and write $P = [x : y]$ and $Q = [w : t]$ in coprime S-integer coordinates.

Distance between periodic and tail points

Theorem (S. Troncoso 2016)

Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Suppose ϕ has good reduction outside S. Let $R \in \mathbb{P}^1(K)$ be a tail point and let n be the period of the periodic part of the orbit of R. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some m. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.

For simplicity suppose $\mathcal{O}_S$ is a PID and write $P = [x : y]$ and $Q = [w : t]$ in coprime S-integer coordinates.
Using the theorem we get that there is a S-unit element u such that

$$xt - yw = u$$

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Theorem (S. Troncoso 2017)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then

(a) $|\operatorname{Tail}(\phi, K)| \leq d \max\left\{(5 \cdot 10^6(d^3+1))^{|S|+4}, 4(2^{64(|S|+3)})\right\}.$

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The proof of this theorem uses all the techniques mentions before. But it uses Thue Mahler equations instead of using S-unit equations.

Another technique: Thue-Mahler equations

Let $F(X, Y)$ be a binary form of degree $r \geq 3$ with coefficients in $\mathcal{O}_S$ which is irreducible over K.

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An $\mathcal{O}_S^*$-coset of solutions of

$$F(x, y) \in \mathcal{O}_S^* \qquad \text{in} \qquad (x, y) \in \mathcal{O}_S^2 \tag{1}$$

is a set $\{\epsilon(x, y) : \epsilon \in \mathcal{O}_S^*\}$, where (x, y) is a fixed solution of (1).

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Evertse proved in 1997 that: the set of solutions of (1) is the union of at most

$$(5 \cdot 10^6 r)^s$$

$\mathcal{O}_S^*$-cosets of solutions.

Almost ready project

Joint work with J.K. Canci and S. Vishkautsan

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$$|\mathrm{PrePer}(\phi, K)| \leq \kappa_1 d^2 + \lambda_1$$

If we assume that ϕ has a K-rational periodic point of minimal period at least two then

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We emphasize that the constants $\kappa_1, \kappa_2, \lambda_1$ and λ_2 in the theorem depend only on the cardinality of S and thus implicitly on the degree $[K : \mathbb{Q}]$ but not on the field K itself. An explicit definition of the constants $\kappa_1, \kappa_2, \lambda_1$ and λ_2 can be given.

Arithmetic dynamics in $\mathbb{P}^n$

Notation of preperiodic hypersurfaces

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The **orbit** of H is the set

$$O_\phi(H) = \{H, \phi(H), \phi^2(H), \phi^3(H), \ldots\}.$$

The **orbit length** of H is the cardinality of the orbit of H (as a set).

Notation of preperiodic hypersurfaces

Periodic hypersurface: $\phi^n(H) = H$ for some $n \geq 1$.
Minimal n is called the **period** of H.

The set of K-rational periodic hypersurface (of degree e) is denoted by $\mathrm{HPer}(\phi, K)$ ($\mathrm{HPer}(\phi, K, e)$).

Preperiodic hypersurface: $\exists m \geq 0$ such that $\phi^m(H)$ is periodic
i.e. H has finite orbit.

The set of K-rational preperiodic hypersurface (of degree e) is denoted by $\mathrm{HPrePer}(\phi, K)$ ($\mathrm{HPrePer}(\phi, K, e)$).

Tail hypersurface: A hypersurface that is preperiodic but not periodic.

The set of K-rational tail hypersurface (of degree e) is denoted by $\mathrm{HTail}(\phi, K)$ ($\mathrm{HTail}(\phi, K, e)$).

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His result is based on theory of canonical heights for subvarieties of $\mathbb{P}^N$. From his proof we can give a bound for the set $\mathrm{HPrePer}(\phi, K)$ depending on n, $[K : \mathbb{Q}]$, the degree of ϕ and height of ϕ.

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- The dimension of the projective space.
- The number of places of bad reduction of ϕ.
- The degree e of the hypersurface.

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Theorem (S. Troncoso 2017)

Let ϕ be an endomorphism of $\mathbb{P}^2$, defined over K and suppose ϕ has good reduction outside S. Let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^2$. Assume that no $N+1$ of them lie in a curve of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K) \colon \forall 1 \leq i \leq 2N+1 \quad P_i \notin supp\ H'\}$ and $\mathcal{A} = \{T \in \mathrm{HTail}(\phi, K, e) \colon \text{there is } l \geq 0 \quad \phi^{l n_T}(T) \in \mathcal{B}\}$. Then

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3(s+2N+1)}$$

THANK YOU