

Bound for preperiodic points for maps with good reduction

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January 04, 2017.

Notation

Let $\phi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a morphism.

orbit: of a point $P \in \mathbb{P}^n$: $P, \phi(P), \phi^2(P), \phi^3(P), \dots$

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.

Minimal n is called the **period** of P .

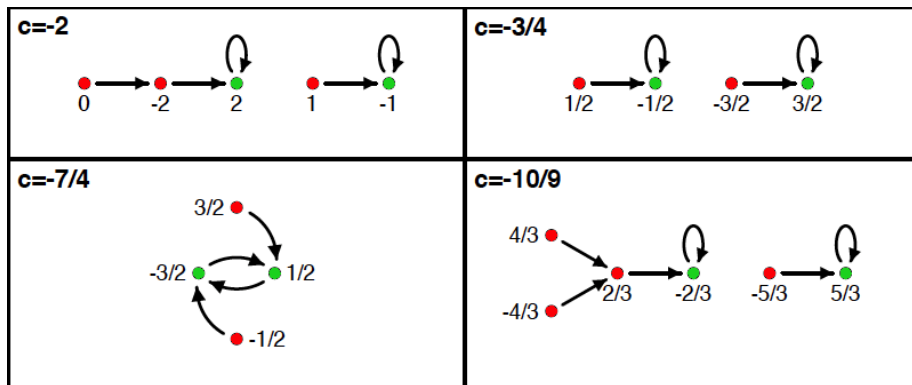
Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic.

The set of K -rational preperiodic points is denoted by $Preper(\phi, K)$.

The **orbit length** of a preperiodic point P is the cardinality of the orbit of P (as a set).

Tail points

Tail point: A point that is preperiodic but not periodic.



Tail points (red) and periodic point (green) of $z^2 + c$.

Uniform boundedness of preperiodic points

Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a morphism of degree ≥ 2 defined over a number field K . Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(D, n, d)$ such that if K/\mathbb{Q} is a number field of degree D , and $\phi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is a morphism of degree $d \geq 2$ defined over K , then

$$\#\text{PrePer}(\phi, K) \leq B.$$

GOAL: Prove good bounds that depend additionally on the number of places of bad reduction of ϕ (and sometimes don't depend on the degree!).

Good reduction

- For simplicity we will from now on deal only with rational maps $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$.
- Let K be a number field, \mathcal{O}_K its ring of algebraic integers, \mathfrak{p} a non zero prime ideal of \mathcal{O}_K and $\mathcal{O}_{\mathfrak{p}}$ the local ring at \mathfrak{p} .
- Write ϕ in normal form:

$$\phi([x : y]) = [F(x, y), G(x, y)],$$

where $F(x, y)$ and $G(x, y)$ are coprime homogeneous polynomials of the same degree, with coefficients in $\mathcal{O}_{\mathfrak{p}}$ and at least one a \mathfrak{p} -unit.

- We say ϕ has **good reduction** at \mathfrak{p} if F and G do not have a common zero module \mathfrak{p} in \mathbb{P}^1 .
- In other words, ϕ does not drop degree mod \mathfrak{p} .
- By convention, we say ϕ has bad reduction at all archimedean places.

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci, L. Paladino 2015)

Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over a number field K and $[K, \mathbb{Q}] = D$. Suppose ϕ has good reduction outside a finite set of places S , including all archimedean ones. Let $s = |S|$. If $P \in \text{PrePer}(\phi, K)$ is of orbit length n , then

$$n \leq \max \left\{ (2^{16s-8} + 3) [12s \log(5s)]^D, [12(s+2) \log(5s+5)]^{4D} \right\}.$$

Bounds independent of the degree

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of \mathbb{P}^1 , defined over K , and $d \geq 2$ the degree of ϕ . Assume ϕ has good reduction outside S .

(a) If there are at least three K -rational tail points of ϕ then

$$|\text{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four K -rational periodic points of ϕ then

$$|\text{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

Bounds for Preperiodic points

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of \mathbb{P}^1 , defined over K , and $d \geq 2$ the degree of ϕ . Assume ϕ has good reduction outside S . Then

- (a) $|\text{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3.$
- (b) $|\text{Tail}(\phi, K)| \leq 4(2^{16|S|d^3}).$
- (c) $|\text{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3.$

Reciprocity of periodic and tail points

Theorem (S. Troncoso 2016)

Let ϕ be an endomorphism of \mathbb{P}^1 , defined over K . Suppose ϕ has good reduction outside S . Let $R \in \mathbb{P}^1(K)$ be a tail point and let n be the period of the periodic part of the orbit of R . Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some m . Then $\delta_p(P, R) = 0$ for every $p \notin S$.

Techniques: S -unit equations

- Almost every pair of K -rational tail point and K -rational periodic point induces S -unit equations.
- That means linear relations of the form

$$au + bv = 1$$

where $(u, v) \in (\mathcal{O}_S^*)^2$ and $a, b \in K$. (\mathcal{O}_S^* is the group of S -units)

- Beukers and Schlickewei's explicit bound on the number of solutions $(u, v) \in (\mathcal{O}_S^*)^2$ to the S -unit equation. For $a, b \in K$ the number of solutions of

$$au + bv = 1$$

is bounded by

$$2^{8(2|S|+2)}$$

- The finitely many solutions to $au + bv = 1$ give us a bound on $Per(\phi, K)$, $Tail(\phi, K)$ and $PrePer(\phi, K)$.

Parallel theorem by Canci and Vishkautsan

Theorem (Canci, Vishkautsan 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let $\phi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be rational map defined over K , where the degree d of ϕ is ≥ 2 . Assume that ϕ has good reduction outside S . Then

$$\#\text{Per}(\phi, K) \leq \kappa d + \lambda,$$

where $\kappa = 2^{2^5 s}$ and $\lambda = 2^{2^{77} s}$.

Future directions: Preliminary results

Joint work with J.K. Canci and S. Vishkautsan

The number of K -rational preperiodic points of rational functions with good reduction outside of S is $O(d^2)$.

Future directions: Preliminary results (2)

- The theorem on the reciprocity of periodic and tail points can be generalize in higher dimensions.
- Reciprocity K -rational tail points and K -rational periodic hypersurface.
- Similarly, we get a reciprocity K -rational tail hypersurface and K -rational periodic points.
- With these theorems we can get similar consequences than the one dimensional case. However Hypotheses of general position are required.
- For instance, if $\text{Per}_\Phi(H) \geq 2N + 1$ and the hypersurfaces on the orbit of H are in general position then $|\text{Tail}(\Phi, K)|$ is bounded.