Bound for preperiodic hypersurfaces and preperiodic points

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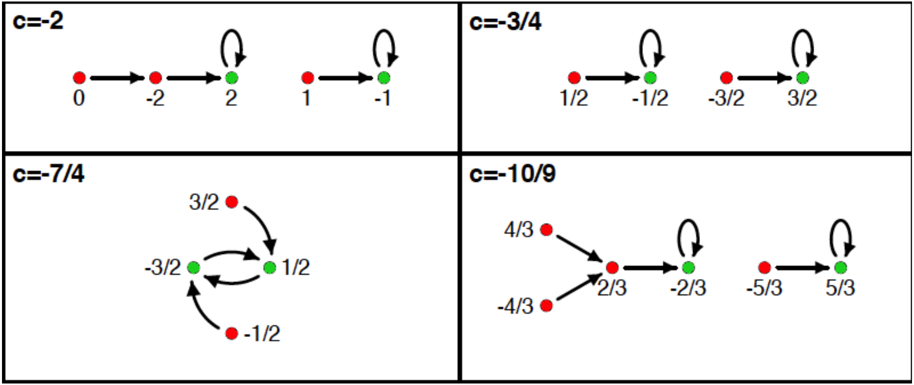
Tail point: A point that is preperiodic but not periodic.

Examples:

We can view $\mathbb{P}^1(K)$ as $K \cup \infty$ and endomorphism of $\mathbb{P}^1$ as rational functions.



$\mathbb{Q}$-rational tail points (red) and $\mathbb{Q}$-rational periodic points (green) of $\phi_c(z) = z^2 + c$.

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Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

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We can deduce from the original proof of Northcott's theorem a bound for $|\mathrm{PrePer}(\phi, K)|$ depending on

- $D = [K : \mathbb{Q}]$
- The dimension n of the projective space
- The degree d of ϕ.
- height of the coefficients of ϕ

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Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree D, and $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism of degree $d \geq 2$ defined over K, then

$$|\mathrm{PrePer}(\phi, K)| \leq B.$$

Goal:

- Give an explicit bound for $|\mathrm{PrePer}(\phi, K)|$.
- To do so we need an extra parameter.
- Instead of the height of ϕ we use a weaker and more natural parameter.
- This parameter is the number of places of bad reduction of ϕ

Good reduction

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- Write ϕ in normal form:

$$\phi([x:y]) = [F(x,y) : G(x,y)],$$

where $F(x,y)$ and $G(x,y)$ are coprime homogeneous polynomials of the same degree, with coefficients in $\mathcal{O}_\mathfrak{p}$ and at least one a $\mathfrak{p}$-unit.

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  where $F(x, y)$ and $G(x, y)$ are coprime homogeneous polynomials of the same degree, with coefficients in $\mathcal{O}_\mathfrak{p}$ and at least one a $\mathfrak{p}$-unit.
- We say ϕ has **good reduction** at $\mathfrak{p}$ if F and G do not have a common zero module $\mathfrak{p}$ in $\mathbb{P}^1$.

Bound on the set of preperiodic point for a rational map

Theorem

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field K and $[K : \mathbb{Q}] = D$. Suppose ϕ has good reduction outside a finite set of places S, including all archimedean ones. Let $s = |S|$. Then

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- $|PrePer(K, \phi)| \leq d^{2^{16s}(s\log(s))^D}$
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- $|PrePer(K, \phi)| \leq \alpha d^2 + \beta d + \gamma$
  where α, β and γ are roughly 2^{78s}.
  J.K. Canci, S. Troncoso and S. Vishkautsan (submitted).

Tools

- Logarithmic v-adic distance between points in $\mathbb{P}^1(K)$.
- Study the distance between tail point and periodic.

$$\downarrow$$

- Logarithmic v-adic distance between a point and a hypersurface in $\mathbb{P}^n$.
- Study the distance between tail hypersurfaces and periodic points.

Notation of preperiodic hypersurfaces

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism defined over K and H an irreducible hypersurface defined over K of degree e.

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Theorem (B. Hutz 2016)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then there are only finitely many preperiodic K-rational subvarieties of degree at most e.

Goal

Just like the one dimensional case, we would like to give explicit bounds for the cardinality of the set $HPrePer(\phi, K, e)$. In terms of:

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- The degree of the endomorphism.
- The degree of the number field.
- The dimension of the projective space.
- The degree e of the hypersurface.
- The number of places of bad reduction of ϕ.

Partial result in $\mathbb{P}^2$

Theorem (S. Troncoso 2017)

Let ϕ be an endomorphism of $\mathbb{P}^2$, defined over K and suppose ϕ has good reduction outside S. Let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^2$. Assume that no $N+1$ of them lie in a curve of degree e.

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Then there is a set *large subset* of K-rational tail hypersurfaces such that its cardinality is bounded by

$$\left(2^{33}\cdot(2N+1)^2\right)^{(N+1)^3(s+2N+1)}$$

where $N=\binom{e+2}{e}-1$.