Scarcity of finite orbits for rational functions over a number fields.

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Spring Southeastern Sectional Meeting
Vanderbilt University, Nashville, TN

April 15, 2018.

Notation

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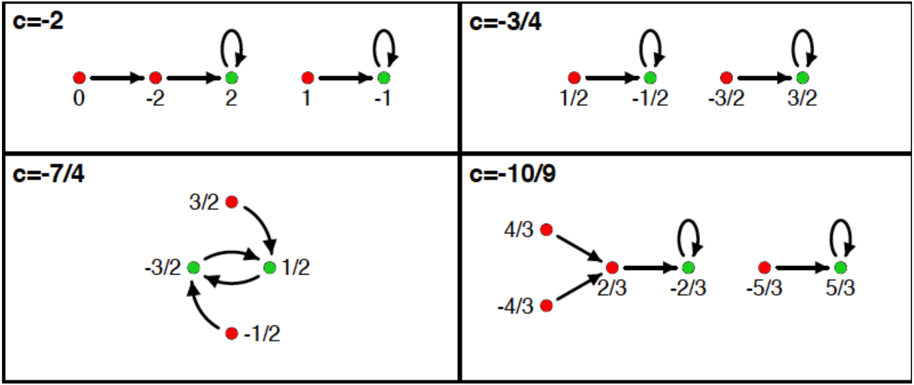
Tail point: A point that is preperiodic but not periodic.

Examples:

We can view $\mathbb{P}^1(K)$ as $K \cup \infty$ and endomorphism of $\mathbb{P}^1$ as rational functions.



$\mathbb{Q}$-rational tail points (red) and $\mathbb{Q}$-rational periodic points (green) of $\phi_c(z) = z^2 + c$.

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Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

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Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

- How large is the set $\mathrm{PrePer}(\phi, K)$?
- Can we give an explicit bound?
- How is the bound depending on ϕ?

We can deduce from the original proof of Northcott's theorem a bound for $|\operatorname{PrePer}(\phi, K)|$ depending on

- $D = [K : \mathbb{Q}]$
- The dimension n of the projective space
- The degree d of ϕ.
- height of the coefficients of ϕ

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Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree D, and $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism of degree $d \geq 2$ defined over K, then

$$|\mathrm{PrePer}(\phi, K)| \leq B.$$

Goal:

- Give an explicit bound for $|\mathrm{PrePer}(\phi, K)|$.

- To do so we need an extra parameter.

- Instead of the height of ϕ we use a weaker and more natural parameter.

- This parameter is the number of places of bad reduction of ϕ

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- Write ϕ in normal form:

$$\phi([x : y]) = [F(x, y) : G(x, y)],$$

where $F(x, y)$ and $G(x, y)$ are coprime homogeneous polynomials of the same degree, with coefficients in $\mathcal{O}_\mathfrak{p}$ and at least one a $\mathfrak{p}$-unit.

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- We say ϕ has **good reduction** at $\mathfrak{p}$ if F and G do not have a common zero module $\mathfrak{p}$ in $\mathbb{P}^1$.

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Theorem

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field K and $[K : \mathbb{Q}] = D$. Suppose ϕ has good reduction outside a finite set of places S, including all archimedean ones. Let $s = |S|$. Then

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- $|PrePer(K, \phi)| \leq \alpha d^2 + \beta d + \gamma$
  where α, β and γ are roughly 2^{78s}.
  J.K. Canci, S. Troncoso and S. Vishkautsan (submitted).

Tools

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- We use big theorems to lift the mild hypothesis and get the theorem. (Riemann-Hurwitz, Baker's Theorem, Kisaka's analysis on Baker's Theorem)

THANK YOU