Bounds for Preperiodic Points for Rational Maps with Good Reduction

Sebastian Troncoso

Birmingham-Southern College

September 25, 2017.

Notation

For simplicity, during this talk we will work with $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$.

Notation

For simplicity, during this talk we will work with $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$.

However, the theory is more general than that and $\mathbb{Q}$ and $\mathbb{Z}$ may be replace by a number field and $\mathcal{O}_K$ its ring of algebraic integers.

Notation

For simplicity, during this talk we will work with $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$.

However, the theory is more general than that and $\mathbb{Q}$ and $\mathbb{Z}$ may be replace by a number field and $\mathcal{O}_K$ its ring of algebraic integers.

Let $\mathbb{P}^1(\mathbb{Q}) = \{[x : y] \mid [x : y] \sim [\lambda x : \lambda y] \quad \lambda \in \mathbb{Q}^*\} = \mathbb{Q} \cup \{\infty\}$ be the projective line.

Notation

For simplicity, during this talk we will work with $\mathbb{Q}$ and its ring of integers $\mathbb{Z}$.

However, the theory is more general than that and $\mathbb{Q}$ and $\mathbb{Z}$ may be replace by a number field and $\mathcal{O}_K$ its ring of algebraic integers.

Let $\mathbb{P}^1(\mathbb{Q}) = \{[x : y] \mid [x : y] \sim [\lambda x : \lambda y] \quad \lambda \in \mathbb{Q}^*\} = \mathbb{Q} \cup \{\infty\}$ be the projective line. When we write $\mathbb{P}^1$ is assume to be $\mathbb{P}^1(\mathbb{C})$.

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

$\phi([x : y]) = [F(x, y) : G(x, y)]$ where F and G are polynomials of the same degree with coefficients in $\mathbb{Q}$ and with no common zeros.

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

$\phi([x : y]) = [F(x, y) : G(x, y)]$ where F and G are polynomials of the same degree with coefficients in $\mathbb{Q}$ and with no common zeros.

$\phi(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomials with coefficients in $\mathbb{Q}$.

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

$\phi([x : y]) = [F(x, y) : G(x, y)]$ where F and G are polynomials of the same degree with coefficients in $\mathbb{Q}$ and with no common zeros.

$\phi(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomials with coefficients in $\mathbb{Q}$.

ϕ^n is the nth iterate of ϕ.

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

$\phi([x : y]) = [F(x, y) : G(x, y)]$ where F and G are polynomials of the same degree with coefficients in $\mathbb{Q}$ and with no common zeros.

$\phi(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomials with coefficients in $\mathbb{Q}$.

ϕ^n is the nth iterate of ϕ.
The **orbit** of a point $P \in \mathbb{P}^1$ is the set

$$O_\phi(P) = \{P, \phi(P), \phi^2(P), \phi^3(P), \ldots\}.$$

Notation

$\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism defined over $\mathbb{Q}$.

$\phi([x : y]) = [F(x, y) : G(x, y)]$ where F and G are polynomials of the same degree with coefficients in $\mathbb{Q}$ and with no common zeros.

$\phi(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomials with coefficients in $\mathbb{Q}$.

ϕ^n is the nth iterate of ϕ.
The **orbit** of a point $P \in \mathbb{P}^1$ is the set

$$O_\phi(P) = \{P, \phi(P), \phi^2(P), \phi^3(P), \ldots\}.$$

The **orbit length** of P is the cardinality of the orbit of P (as a set).

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.

Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic

i.e. P has finite orbit.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic
i.e. P has finite orbit.

The set of rational preperiodic points for ϕ is denoted by $\mathrm{PrePer}(\phi, \mathbb{Q})$.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic
i.e. P has finite orbit.

The set of rational preperiodic points for ϕ is denoted by $\mathrm{PrePer}(\phi, \mathbb{Q})$.

Tail point: A point that is preperiodic but not periodic.

Notation

Periodic point: $\phi^n(P) = P$ for some $n \geq 1$.
Minimal n is called the **period** of P.

The set of rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, \mathbb{Q})$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic
i.e. P has finite orbit.

The set of rational preperiodic points for ϕ is denoted by $\mathrm{PrePer}(\phi, \mathbb{Q})$.

Tail point: A point that is preperiodic but not periodic.

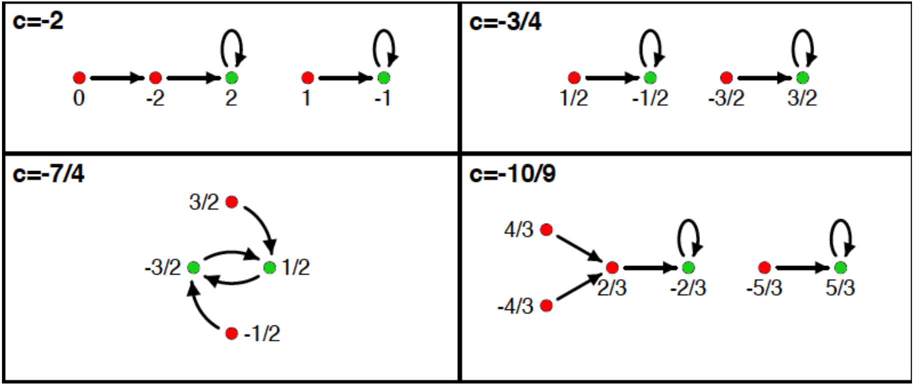
The set of $\mathbb{Q}$-rational tail points for ϕ is denoted by $\mathrm{Tail}(\phi, \mathbb{Q})$.

Examples:

$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ and endomorphism of $\mathbb{P}^1$ as rational functions.
Consider $\phi_c(z) = z^2 + c$.

Examples:

$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ and endomorphism of $\mathbb{P}^1$ as rational functions.
Consider $\phi_c(z) = z^2 + c$.



rational tail points (red) and rational periodic points (green) of $\phi_c(z) = z^2 + c$.

Question:

- Are the sets $\mathrm{Tail}(\phi, \mathbb{Q})$, $\mathrm{Per}(\phi, \mathbb{Q})$ and $\mathrm{PrePer}(\phi, \mathbb{Q})$ finite?

Question:

- Are the sets $\mathrm{Tail}(\phi, \mathbb{Q})$, $\mathrm{Per}(\phi, \mathbb{Q})$ and $\mathrm{PrePer}(\phi, \mathbb{Q})$ finite?

Yes.

Question:

- Are the sets Tail$(\phi, \mathbb{Q})$, Per$(\phi, \mathbb{Q})$ and PrePer$(\phi, \mathbb{Q})$ finite?

Yes.

Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism of degree ≥ 2 defined over $\mathbb{Q}$. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^1(\mathbb{Q})$.

Question:

- Are the sets Tail$(\phi, \mathbb{Q})$, Per$(\phi, \mathbb{Q})$ and PrePer$(\phi, \mathbb{Q})$ finite?

Yes.

Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism of degree ≥ 2 defined over $\mathbb{Q}$. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^1(\mathbb{Q})$.

We can deduce from the original proof of Northcott's theorem a bound for $|\,$PrePer$(\phi, \mathbb{Q})|$ depending on

- the degree of ϕ
- height of the coefficients of ϕ

Goals:

Give explicit bounds for $|\,\mathrm{Tail}(\phi, \mathbb{Q})|$, $|\,\mathrm{Per}(\phi, \mathbb{Q})|$ and $|\,\mathrm{PrePer}(\phi, \mathbb{Q})|$ in terms of:

Goals:

Give explicit bounds for $|\,\mathrm{Tail}(\phi,\mathbb{Q})|$, $|\,\mathrm{Per}(\phi,\mathbb{Q})|$ and $|\,\mathrm{PrePer}(\phi,\mathbb{Q})|$ in terms of:

- The degree d of ϕ.

Goals:

Give explicit bounds for $|\operatorname{Tail}(\phi, \mathbb{Q})|$, $|\operatorname{Per}(\phi, \mathbb{Q})|$ and $|\operatorname{PrePer}(\phi, \mathbb{Q})|$ in terms of:

- The degree d of ϕ.

Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(d)$ such that if $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism of degree $d \geq 2$ defined over $\mathbb{Q}$, then

$$|\operatorname{PrePer}(\phi, \mathbb{Q})| \leq B.$$

Poonen's Conjecture

The simplest case we could consider for the UBC is

Poonen's Conjecture

The simplest case we could consider for the UBC is ϕ a quadratic polynomial in one variable with coefficients in $\mathbb{Q}$.

Poonen's Conjecture

The simplest case we could consider for the UBC is ϕ a quadratic polynomial in one variable with coefficients in $\mathbb{Q}$.

Conjecture (Poonen's Conjecture)

Let ϕ be a quadratic polynomial defined over $\mathbb{Q}$ then

$$|\operatorname{PrePer}(\phi, \mathbb{Q})| \leq 9.$$

Poonen's Conjecture

The simplest case we could consider for the UBC is ϕ a quadratic polynomial in one variable with coefficients in $\mathbb{Q}$.

Conjecture (Poonen's Conjecture)

Let ϕ be a quadratic polynomial defined over $\mathbb{Q}$ then

$$|\operatorname{PrePer}(\phi, \mathbb{Q})| \leq 9.$$

If $\phi = x^2 + d$ then B. Hutz and P. Ingram have shown that Poonen's conjecture holds when the numerator and denominator of d don't exceed 10^8.

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, \mathbb{Q})$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\mathrm{PrePer}(\phi, \mathbb{Q})|$.

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, \mathbb{Q})$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\,\mathrm{PrePer}(\phi, \mathbb{Q})|$.

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, \mathbb{Q})$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\,\mathrm{PrePer}(\phi, \mathbb{Q})|$.

This parameter is the number of primes of bad reduction of ϕ.

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, \mathbb{Q})$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\mathrm{PrePer}(\phi, \mathbb{Q})|$.

This parameter is the number of primes of bad reduction of ϕ.

Give explicit bounds for $|\mathrm{Tail}(\phi, \mathbb{Q})|$, $|\mathrm{Per}(\phi, \mathbb{Q})|$ and $|\mathrm{PrePer}(\phi, \mathbb{Q})|$ in terms of:

- The degree d of ϕ.
- The number of primes of bad reduction of ϕ.

Good/Bad Reduction

- Let $\mathfrak{p}$ be a prime in $\mathbb{Z}$ and $\phi(x) = \frac{F(x)}{G(x)}$ be a rational function $\mathbb{P}^1$ defined over $\mathbb{Q}$.

Good/Bad Reduction

- Let $\mathfrak{p}$ be a prime in $\mathbb{Z}$ and $\phi(x) = \frac{F(x)}{G(x)}$ be a rational function $\mathbb{P}^1$ defined over $\mathbb{Q}$. After multiplying for the common denominator of the coefficients of F and G. We may assume that F and G are polynomials with coefficients in $\mathbb{Z}$.

Good/Bad Reduction

- Let $\mathfrak{p}$ be a prime in $\mathbb{Z}$ and $\phi(x) = \frac{F(x)}{G(x)}$ be a rational function $\mathbb{P}^1$ defined over $\mathbb{Q}$. After multiplying for the common denominator of the coefficients of F and G. We may assume that F and G are polynomials with coefficients in $\mathbb{Z}$.

  Consider the reduction of ϕ modulo $\mathfrak{p}$ given by

$$\tilde{\phi}(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)}$$

  In other words, $\tilde{\phi}$ is obtained by reducing the coefficients of F, G modulo $\mathfrak{p}$.

Good/Bad Reduction

- Let $\mathfrak{p}$ be a prime in $\mathbb{Z}$ and $\phi(x) = \frac{F(x)}{G(x)}$ be a rational function $\mathbb{P}^1$ defined over $\mathbb{Q}$. After multiplying for the common denominator of the coefficients of F and G. We may assume that F and G are polynomials with coefficients in $\mathbb{Z}$.

  Consider the reduction of ϕ modulo $\mathfrak{p}$ given by

  $$\tilde{\phi}(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)}$$

  In other words, $\tilde{\phi}$ is obtained by reducing the coefficients of F, G modulo $\mathfrak{p}$.

  Then ϕ has **good reduction** at $\mathfrak{p}$ if the degree of ϕ is equal to the degree of $\tilde{\phi}$.

Good/Bad Reduction

- ϕ has **bad reduction** at $\mathfrak{p}$ if it does not have good reduction at $\mathfrak{p}$.

Good/Bad Reduction

- ϕ has **bad reduction** at $\mathfrak{p}$ if it does not have good reduction at $\mathfrak{p}$.
- We say that ϕ has **good reduction outside** S if ϕ has good reduction for every $\mathfrak{p} \notin S$.

Good/Bad Reduction

- ϕ has **bad reduction** at $\mathfrak{p}$ if it does not have good reduction at $\mathfrak{p}$.
- We say that ϕ has **good reduction outside** S if ϕ has good reduction for every $\mathfrak{p} \notin S$.

If we allow the number of primes of bad reduction as a parameter, much more is known for the cardinality of the set of rational preperiodic points.

Bound on maximal period

Theorem (W. Narkiewicz 1988)

Let $\phi \in \mathbb{Q}[z]$ be a polynomial of degree ≥ 2 defined over $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S.

If P is a rational periodic point of period n, then

$$n \leq (6 \cdot 7^{D+2|S|})^{\alpha},$$

where $\alpha = O(|S| \log |S|)$.

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci 2006)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree at least two defined $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S.

If $P \in \mathrm{PrePer}(\phi, \mathbb{Q})$ is of orbit length n, then

$$n \leq \left[e^{10^{12}} (|S| + 1)^8 (log(5(|S| + 1)))^8 \right]^{|S|}.$$

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci, L. Paladino 2015)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S of $\mathbb{Z}$. If $P \in \mathrm{PrePer}(\phi, \mathbb{Q})$ is of orbit length n, then

$$n \leq \max\left\{ (2^{16|S|-8} + 3)\,[12|S| \log(5|S|)]^D, [12(|S|+2)\log(5|S|+5)]^{4D} \right\}.$$

Bound on maximal orbit length of a preperiodic point

Theorem (J.K. Canci, L. Paladino 2015)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S of $\mathbb{Z}$. If $P \in \mathrm{PrePer}(\phi, \mathbb{Q})$ is of orbit length n, then

$$n \leq \max\left\{(2^{16|S|-8}+3)\,[12|S|\log(5|S|)]^D\,,[12(|S|+2)\log(5|S|+5)]^{4D}\right\}.$$

From here we can deduce a bound for $|\mathrm{PrePer}(\phi, \mathbb{Q})|$ that is roughly of the order $d^{2^{16|S|}(|S|\log(|S|))^D}$.

Theorem (S. Troncoso 2016)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S of $\mathbb{Z}$. Then

(a) $|\operatorname{Per}(\phi, \mathbb{Q})| \leq 2^{16|S|d^3} + 3$.

(b) $|\operatorname{Tail}(\phi, \mathbb{Q})| \leq 4(2^{16|S|d^3})$.

(c) $|\operatorname{PrePer}(\phi, \mathbb{Q})| \leq 5(2^{16|S|d^3}) + 3$.

Notice that the bounds obtained in the theorem are a significant improvement from the previous bound given by Canci and Paladino which was of the order $d^{2^{16s}(s\log(s))^D}$ for the set $|\operatorname{PrePer}(\phi, \mathbb{Q})|$.

Bounds for Preperiodic points

The previous theorem use some important results like:

Bounds for Preperiodic points

The previous theorem use some important results like:

Bounds for Preperiodic points

The previous theorem use some important results like:

- Riemann-Hurwitz formula

Bounds for Preperiodic points

The previous theorem use some important results like:

- Riemann-Hurwitz formula
- Baker's Theorem on existence of periodic points

The previous theorem use some important results like:

- Riemann-Hurwitz formula
- Baker's Theorem on existence of periodic points
- Kisaka's analysis on Baker's Theorem

Bounds for Preperiodic points

The previous theorem use some important results like:

- Riemann-Hurwitz formula
- Baker's Theorem on existence of periodic points
- Kisaka's analysis on Baker's Theorem
- Logarithmic p-adic chordal distance.

Bounds for Preperiodic points

The previous theorem use some important results like:

- Riemann-Hurwitz formula
- Baker's Theorem on existence of periodic points
- Kisaka's analysis on Baker's Theorem
- Logarithmic p-adic chordal distance.
- Study the distance between periodic and tail points.

The previous theorem use some important results like:

- Riemann-Hurwitz formula
- Baker's Theorem on existence of periodic points
- Kisaka's analysis on Baker's Theorem
- Logarithmic p-adic chordal distance.
- Study the distance between periodic and tail points.
- Number of solution of the S-unit equation.

Bounds independent of the degree

Theorem (S. Troncoso 2016)

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree ≥ 2 defined over $\mathbb{Q}$. Suppose ϕ has good reduction outside a finite set of primes S of $\mathbb{Z}$.

(a) If there are at least three rational tail points of ϕ then

$$|\operatorname{Per}(\phi, \mathbb{Q})| \leq 2^{16|S|} + 3.$$

(b) If there are at least four rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, \mathbb{Q})| \leq 4(2^{16|S|}).$$

S-unit equations

Let S be a finite set of primes of $\mathbb{Z}$ and $\mathbb{Z}_S^*$ be the group of S-units *i.e.* $\mathbb{Z}_S^*$ is the set of fraction of the form $\dfrac{a}{b}$ where a and b are not divisable by primes in S.

S-unit equations

Let S be a finite set of primes of $\mathbb{Z}$ and $\mathbb{Z}_S^*$ be the group of S-units *i.e.* $\mathbb{Z}_S^*$ is the set of fraction of the form $\dfrac{a}{b}$ where a and b are not divisable by primes in S.

- A linear relation of the form

$$au + bv = 1$$

where $(u, v) \in (\mathbb{Z}_S^*)^2$ and $a, b \in \mathbb{Q}^*$ is called a S-unit equation.

S-unit equations

Let S be a finite set of primes of $\mathbb{Z}$ and $\mathbb{Z}_S^*$ be the group of S-units *i.e.* $\mathbb{Z}_S^*$ is the set of fraction of the form $\dfrac{a}{b}$ where a and b are not divisable by primes in S.

- A linear relation of the form
$$au + bv = 1$$
where $(u, v) \in (\mathbb{Z}_S^*)^2$ and $a, b \in \mathbb{Q}^*$ is called a S-unit equation.
- Beukers and Schlickewei give an explicit bound for the S-unit equation. The number of solutions $(u, v) \in (\mathbb{Z}_S^*)^2$ to
$$au + bv = 1$$
is bounded by
$$2^{8(2|S|+2)}.$$

Almost ready project

Joint work with J.K. Canci and S. Vishkautsan

Almost ready project

Joint work with J.K. Canci and S. Vishkautsan

Theorem (S. Troncoso 2016)

Let S be a finite set of primes of $\mathbb{Z}$. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over $\mathbb{Q}$, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then

$$|\mathrm{PrePer}(\phi, \mathbb{Q})| \leq \kappa_1 d^2 + \lambda_1$$

If we assume that ϕ has a rational periodic point of minimal period at least two then

$$|\mathrm{PrePer}(\phi, \mathbb{Q})| \leq \kappa_2 d + \lambda_2.$$

Almost ready project

Joint work with J.K. Canci and S. Vishkautsan

Theorem (S. Troncoso 2016)

Let S be a finite set of primes of $\mathbb{Z}$. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over $\mathbb{Q}$, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then

$$|\mathrm{PrePer}(\phi, \mathbb{Q})| \leq \kappa_1 d^2 + \lambda_1$$

If we assume that ϕ has a rational periodic point of minimal period at least two then

$$|\mathrm{PrePer}(\phi, \mathbb{Q})| \leq \kappa_2 d + \lambda_2.$$

We emphasize that the constants $\kappa_1, \kappa_2, \lambda_1$ and λ_2 in the theorem depend only on the cardinality of S. An explicit definition of the constants $\kappa_1, \kappa_2, \lambda_1$ and λ_2 can be given.

Arithmetic dynamics in $\mathbb{P}^n$

THANK YOU