# Bound for preperiodic points for maps with good reduction

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January 04, 2017.

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The **orbit length** of a preperiodic point $P$ is the cardinality of the orbit of $P$ (as a set).

# Tail points

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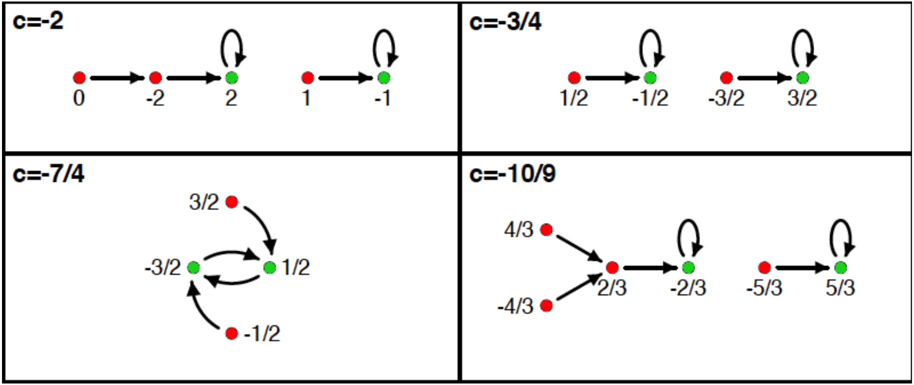


Tail points (red) and periodic point (green) of $z^2 + c$.

# Uniform boundedness of preperiodic points

**Theorem (Northcott 1950)**

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**Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)**

There exists a bound $B = B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree $D$, and $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is a morphism of degree $d \geq 2$ defined over $K$, then

$$\#\mathrm{PrePer}(\phi, K) \leq B.$$

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- Let $K$ be a number field, $\mathcal{O}_K$ its ring of algebraic integers, $\mathfrak{p}$ a non zero prime ideal of $\mathcal{O}_K$ and $\mathcal{O}_{\mathfrak{p}}$ the local ring at $\mathfrak{p}$.
- Write $\phi$ in normal form:

$$\phi([x : y]) = [F(x, y), G(x, y)],$$

where $F(x, y)$ and $G(x, y)$ are coprime homogeneous polynomials of the same degree, with coefficients in $\mathcal{O}_{\mathfrak{p}}$ and at least one a $\mathfrak{p}$-unit.

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- We say $\phi$ has **good reduction** at $\mathfrak{p}$ if $F$ and $G$ do not have a common zero module $\mathfrak{p}$ in $\mathbb{P}^1$.
- In other words, $\phi$ does not drop degree mod $\mathfrak{p}$.
- By convention, we say $\phi$ has bad reduction at all archimedean places.

# Bound on maximal orbit length of a preperiodic point

**Theorem (J.K. Canci, L. Paladino 2015)**

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\geq 2$ defined over a number field $K$ and $[K, \mathbb{Q}] = D$. Suppose $\phi$ has good reduction outside a finite set of places $S$, including all archimedean ones. Let $s = |S|$. If $P \in \mathrm{PrePer}(\phi, K)$ is of orbit length $n$, then

$$n \leq \max\left\{(2^{16s-8} + 3)\,[12s\log(5s)]^D\,,\,[12(s+2)\log(5s+5)]^{4D}\right\}.$$

# Bounds independent of the degree

**Theorem (S. Troncoso 2016)**

Let $K$ be a number field and $S$ a finite set of places of $K$ containing all the archimedean ones. Let $\phi$ be an endomorphism of $\mathbb{P}^1$, defined over $K$, and $d \geq 2$ the degree of $\phi$. Assume $\phi$ has good reduction outside $S$.

(a) If there are at least three $K$-rational tail points of $\phi$ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four $K$-rational periodic points of $\phi$ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

# Bounds for Preperiodic points

**Theorem (S. Troncoso 2016)**

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(a) $|\operatorname{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3$.

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|d^3})$.

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$.

# Reciprocity of periodic and tail points

**Theorem (S. Troncoso 2016)**

Let $\phi$ be an endomorphism of $\mathbb{P}^1$, defined over $K$. Suppose $\phi$ has good reduction outside $S$. Let $R \in \mathbb{P}^1(K)$ be a tail point and let $n$ be the period of the periodic part of the orbit of $R$. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some $m$. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.

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- The finitely many solutions to $au + bv = 1$ give us a bound on $Per(\phi, K)$, $Tail(\phi, K)$ and $PrePer(\phi, K)$.

# Parallel theorem by Canci and Vishkautsan

**Theorem (Canci, Vishkautsan 2016)**

Let $K$ be a number field and $S$ a finite set of places of $K$ containing all the archimedean ones. Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be rational map defined over $K$, where the degree $d$ of $\phi$ is $\geq 2$. Assume that $\phi$ has good reduction outside $S$. Then

$$\#\mathrm{Per}(\phi, K) \leq \kappa d + \lambda,$$

where $\kappa = 2^{2^5 s}$ and $\lambda = 2^{2^{77} s}$.

# Future directions: Preliminary results

Joint work with J.K. Canci and S. Vishkautsan

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The number of $K$-rational preperiodic points of rational functions with good reduction outside of $S$ is $O(d^2)$.

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- With these theorems we can get similar consequences than the one dimensional case. However Hypotheses of general position are required.
- For instance, if $\mathrm{Per}_\Phi(H) \geq 2N + 1$ and the hypersurfaces on the orbit of $H$ are in general position then $|Tail(\Phi, K)|$ is bounded.