Formally Real Fields

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Classical examples are:

- $\mathbb{Q}$
- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{Q}(\sqrt{2}) = \{x \in \mathbb{R} \mid x = a + b\sqrt{2} \quad a, b \in \mathbb{Q}\}$

Definition

Informally, a formally real field (also called ordered field) is a field with a linear order such that the operations of the fields are preserved *i.e.*

$$x \leq y \implies x + z \leq y + z$$

$$x \leq y \text{ and } 0 \leq z \implies xz \leq yz.$$

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Definition

Let F be a field. An **ordering** $\leq$ of F is a binary relation satisfying

1. $a \leq a$
2. $a \leq b, b \leq c \Longrightarrow a \leq c$
3. $a \leq b, b \leq a \Longrightarrow a = b$
4. $a \leq b$ or $b \leq a$
5. $a \leq b \Longrightarrow a + c \leq b + c$
6. $0 \leq a, 0 \leq b \Longrightarrow 0 \leq ab$

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For instance, $\mathbb{Q}(\sqrt{2})$

Positive Numbers

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3. $P \cap -P = \{0\}$
4. $P \cup -P = F$

where $-P = \{-a \in F \mid a \in P\} = \{b \in F \mid 0 \geq b\}$.

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$$a + (-a) \geq 0 + (-a)$$

$$0 \geq -a$$

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Lemma: Squares are positive

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Let F be a ordered field. Then

1. $0 \leq a \implies -a \leq 0$ for every $a \in F$.
2. $0 \leq 1$ and $-1 \leq 0$.
3. $0 \leq a^2$ for every $a \in F$.

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2. If $1 \leq 0$ then $0 \leq -1$. Hence $0 \leq (-1)^2 = 1$ which is a contradiction since the only element which is positive and negative at the same time is 0.
3. If $0 \leq a$ then $0 \leq a^2$ by definition of order. If $a \leq 0$ then $0 \leq -a$. Hence, $0 \leq (-a)^2 = a^2$ by definition of order.

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$\mathbb{C}$ is **NOT** an ordered field because $i^2 = -1$.

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From a previous talk we know that $\mathbb{Z}_p$ is a field when p is a prime number. $\mathbb{Z}_p$ is **NOT** an ordered field because

$$\begin{aligned} p - 1 &= 1 + 1 + \ldots + 1 \geq 0 \\ &= -1 \leq 0 \end{aligned}$$

THANK YOU