K-rational preperiodic hypersurfaces on $\mathbb{P}^n$

Sebastian Troncoso

Michigan State University
$\downarrow$
Birmingham-Southern College

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Notation

Let K be a number field and $\mathcal{O}_K$ its ring of algebraic integers,

$\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism defined over K
ϕ^m the mth iterate of ϕ.

The **orbit** of a point $P \in \mathbb{P}^n$ is the set

$$O_\phi(P) = \{P, \phi(P), \phi^2(P), \phi^3(P), \ldots\}.$$

Notation

Periodic point: $\phi^m(P) = P$ for some $m \geq 1$.
Minimal m is called the **period** of P.

The set of K-rational periodic points for ϕ is denoted by $\mathrm{Per}(\phi, K)$.

Preperiodic point: $\exists m \geq 0$ such that $\phi^m(P)$ is periodic
i.e. P has finite orbit.

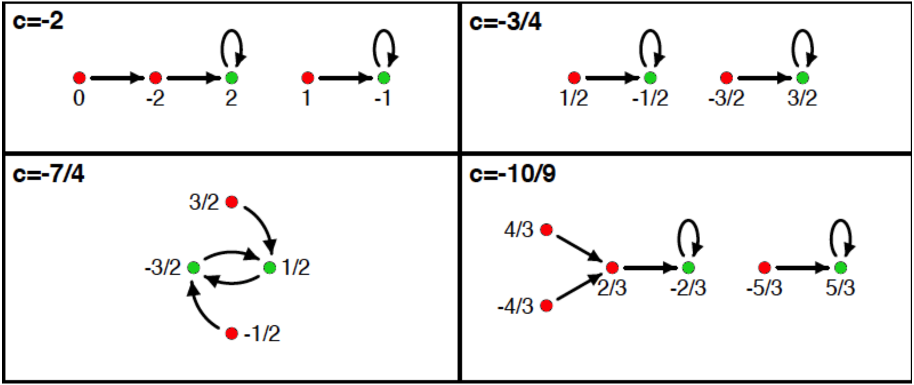
The set of K-rational preperiodic points for ϕ is denoted by $\mathrm{PrePer}(\phi, K)$.

Tail point: A point that is preperiodic but not periodic.

The set of K-rational tail points for ϕ is denoted by $\mathrm{Tail}(\phi, K)$.

Examples:

We can view $\mathbb{P}^1(K)$ as $K \cup \{\infty\}$ and endomorphism of $\mathbb{P}^1$ as rational functions.



$\mathbb{Q}$-rational tail points (red) and $\mathbb{Q}$-rational periodic points (green) of $\phi_c(z) = z^2 + c$.

Question:

- Are the sets $\text{Tail}(\phi, K)$, $\text{Per}(\phi, K)$ and $\text{PrePer}(\phi, K)$ finite? **Yes**.

Theorem (Northcott 1950)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then ϕ has only finitely many preperiodic points in $\mathbb{P}^n(K)$.

We can deduce from the original proof of Northcott's theorem a bound for $|\text{PrePer}(\phi, K)|$ depending on

- n
- $[K : \mathbb{Q}]$
- the degree of ϕ
- height of the coefficients of ϕ

Goals:

Give explicit bounds for $|\operatorname{Tail}(\phi, K)|$, $|\operatorname{Per}(\phi, K)|$ and $|\operatorname{PrePer}(\phi, K)|$ in terms of:

- $D = [K : \mathbb{Q}]$
- The dimension n of the projective space
- The degree d of ϕ.

Conjecture (Uniform Boundedness Conjecture - Morton–Silverman 1994)

There exists a bound $B = B(D, n, d)$ such that if $K/\mathbb{Q}$ is a number field of degree D, and $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism of degree $d \geq 2$ defined over K, then

$$|\operatorname{PrePer}(\phi, K)| \leq B.$$

Goals:

In order to get explicit bounds for the cardinality of the set $\mathrm{PrePer}(\phi, K)$ we need an extra parameter.
Instead of the height of ϕ we can use a weaker and more natural parameter to get bound on $|\mathrm{PrePer}(\phi, K)|$.
This parameter is the number of places of bad reduction of ϕ.
Give explicit bounds for $|\mathrm{Tail}(\phi, K)|$, $|\mathrm{Per}(\phi, K)|$ and $|\mathrm{PrePer}(\phi, K)|$ in terms of:

- $D = [K : \mathbb{Q}]$
- The dimension n of the projective space
- The degree d of ϕ.
- The number of places of bad reduction of ϕ.

Good Reduction

Let S be a finite set of places K, including all archimedean ones.

- We say that ϕ has **good reduction outside** S if ϕ has good reduction for every $\mathfrak{p} \notin S$.

If we allow the number of primes of bad reduction as a parameter, much more is known for the cardinality of the set of K-rational preperiodic points in the case of $\mathbb{P}^1$.

Bounds independent of the degree

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S.

(a) If there are at least three K-rational tail points of ϕ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four K-rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

Notice that under these hypotheses the bounds are independent of the degree of ϕ. Those hypotheses are sharp, *i.e.* if there are two (three) K-rational tail (periodic) points then $|\operatorname{Per}(\phi, K)|$ $(|\operatorname{Tail}(\phi, K)|)$ must depend on d.

Distance between periodic and tail points

Theorem (S. Troncoso 2016)

Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Suppose ϕ has good reduction outside S. Let $R \in \mathbb{P}^1(K)$ be a tail point and let n be the period of the periodic part of the orbit of R. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some m. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.

For simplicity suppose $\mathcal{O}_S$ is a PID and write $P = [x : y]$ and $Q = [w : t]$ in coprime S-integer coordinates.

Using the theorem we get that there is a S-unit element u such that

$$xt - yw = u$$

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then

(a) $|\operatorname{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3$.

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|d^3})$.

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3$.

Notice that the bounds obtained in the theorem are a significant improvement from the previous bound given by Canci and Paladino which was of the order $d^{2^{16s}(s\log(s))^D}$ for the set $|\operatorname{PrePer}(\phi, K)|$.

Another technique

Using another technique we can get a better result in terms of d.

Theorem (S. Troncoso 2016)

Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then

(a) $|\operatorname{Tail}(\phi, K)| \leq d \max\left\{(5 \cdot 10^6(d^3+1))^{|S|+4}, 4(2^{64(|S|+3)})\right\}.$

(b) In addition, if ϕ has at least one K-rational tail point then then

$$|\operatorname{Per}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d-1))^{|S|+3}, 4(2^{128(|S|+2)})\right\} + 1.$$

Current project: Arithmetic dynamics in $\mathbb{P}^n$.

We are generalizing our results and techniques in $\mathbb{P}^1$ into results and techniques in $\mathbb{P}^n$.

Notation of preperiodic hypersurfaces

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism defined over K and H an irreducible hypersurface defined over K of degree e.

The **orbit** of H is the set

$$O_\phi(H) = \{H, \phi(H), \phi^2(H), \phi^3(H), \ldots\}.$$

Notation of preperiodic hypersurfaces

Periodic hypersurface: $\phi^m(H) = H$ for some $m \geq 1$.
Minimal m is called the **period** of H.

The set of K-rational periodic hypersurface (of degree e) is denoted by $\mathrm{HPer}(\phi, K)$ ($\mathrm{HPer}(\phi, K, e)$).

Preperiodic hypersurface: $\exists m \geq 0$ such that $\phi^m(H)$ is periodic
i.e. H has finite orbit.

The set of K-rational preperiodic hypersurface (of degree e) is denoted by $\mathrm{HPrePer}(\phi, K)$ ($\mathrm{HPrePer}(\phi, K, e)$).

Tail hypersurface: A hypersurface that is preperiodic but not periodic.

The set of K-rational tail hypersurface (of degree e) is denoted by $\mathrm{HTail}(\phi, K)$ ($\mathrm{HTail}(\phi, K, e)$).

Questions:

- Are the sets $\mathrm{HTail}(\phi, K)$, $\mathrm{HPer}(\phi, K)$ and $\mathrm{HPrePer}(\phi, K)$ finite?
  No, an example by J. Bell, D. Ghioca, and T. Tucker have shown that in general these sets are not finite.
- Are the sets $\mathrm{HTail}(\phi, K, e)$, $\mathrm{HPer}(\phi, K, e)$ and $\mathrm{HPrePer}(\phi, K, e)$ finite?
  Yes.

Theorem (B. Hutz 2016)

Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree ≥ 2 defined over a number field K. Then there are only finitely many preperiodic K-rational subvarieties of degree at most e.

His result is based on theory of canonical heights for subvarieties of $\mathbb{P}^N$. From his proof we can give a bound for the cardinality of the set $\mathrm{HPrePer}(\phi, K, e)$ depending on

- n
- $[K : \mathbb{Q}]$
- the degree of ϕ
- e
- height of the coefficients of ϕ

Goals

Just like the one dimensional case.

We would like to give explicit bounds for the cardinality of the sets $\mathrm{HTail}(\phi, K, e)$, $\mathrm{HPer}(\phi, K, e)$ and $\mathrm{HPrePer}(\phi, K, e)$ in terms of:

- $D = [K : \mathbb{Q}]$.
- The dimension n of the projective space.
- The degree d of ϕ.
- The degree e of the hypersurfaces.
- The number of places of bad reduction of ϕ.

- For now, we have just proven the following result for $\mathbb{P}^2$.

If $T \in \mathrm{HTail}(\phi, K, e)$ then n_T is the period of the periodic part of T. Consider N the number of monomials of degree e in three variables.

Theorem (S. Troncoso 2017)

Let ϕ be an endomorphism of $\mathbb{P}^2$, defined over K and suppose ϕ has good reduction outside S. Let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^2$. Assume that no $N+1$ of them lie in a curve of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K) \colon \forall 1 \leq i \leq 2N+1 \quad P_i \notin supp\ H'\}$ and $\mathcal{A} = \{T \in \mathrm{HTail}(\phi, K, e) \colon \text{there is } l \geq 0 \quad \phi^{l n_T}(T) \in \mathcal{B}\}$. Then

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3 (s+2N+1)}$$

- There is strong result from dynamical system that states that the set of periodic points is Zariski dense.
- On $\mathbb{P}^2$. We can give an alternative proof than the one given by B. Hutz for the finiteness of the set $\text{HPrePer}(\phi, K, e)$,
  the idea is to use the previous theorem and the Zariski density of periodic points.

Tools

The theorem is based on the following three tools:

- Logarithmic v-adic distance between a point and a hypersurface.
- Study the distance between tail hypersurfaces and periodic points.
- Finiteness of integral points of $\mathbb{P}^n$ - $\{2n+1$ hyperplanes in general position$\}$ due to Ru and Wong.
- Bounds for the number of solutions to decomposable form equations (Evertse).

logarithmic v-adic distance

Let H be a hypersurface of $\mathbb{P}^n$ defined over K of degree e.
Further, suppose H is defined by

$$f = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \in K[\mathbf{X}]$$

an homogeneous polynomial of degree e.
Let v be a nonarchimedean place of K and $P = [x_0 : \cdots : x_n]$ a point in $\mathbb{P}^n(K)$ such that $P \notin supp(H)$.
logarithmic v-adic distance between P and H with respect to v is given by

$$\delta_v(P;H) = v(f(x_0, \ldots, x_n)) - e \min_{0 \leq i \leq n}\{v(x_i)\} - \min_{|\mathbf{i}|=e}\{v(a_{\mathbf{i}})\}$$

Distance between tail hypersurfaces and periodic points

Theorem (S. Troncoso 2017)

Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Suppose ϕ has good reduction outside S. Let H be a K-rational tail hypersurface , m the period of the periodic part of the orbit of H and H' the periodic hypersurface such that $H' = \phi^{m_0 m}(H)$ for some $m_0 > 0$. Let $P \in \mathbb{P}^n(K)$ be any periodic point such that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.

For simplicity suppose $\mathcal{O}_S$ is a PID. Assume H is defined by

$$f = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \in \mathcal{O}_S[\mathbf{X}]$$

an homogeneous polynomial of degree e with at least one coefficient in $\mathcal{O}_S^*$ and $P = [x_0 : \cdots : x_n]$ in coprime S-integer coordinates.
Using the theorem we get that there is a S-unit element u such that

$$f(x_0, \ldots, x_n) = u$$

If $T \in \mathrm{HTail}(\phi, K, e)$ then n_T is the period of the periodic part of T. Consider N the number of monomials of degree e in three variables.

Theorem (S. Troncoso 2017)

Let ϕ be an endomorphism of $\mathbb{P}^2$, defined over K and suppose ϕ has good reduction outside S. Let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^2$. Assume that no $N+1$ of them lie in a curve of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K) \colon \forall 1 \leq i \leq 2N+1 \quad P_i \notin supp\ H'\}$ and $\mathcal{A} = \{T \in \mathrm{HTail}(\phi, K, e) \colon \text{there is } l \geq 0 \quad \phi^{l n_T}(T) \in \mathcal{B}\}$. Then

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3(s+2N+1)}$$

Idea of the proof:

1. Use the d-veronese map.

$$\{P_i\}_{i=1}^{2N+1} \to \{H_i\}_{i=1}^{2N+1}$$
$$T \to P_T$$
$$T(P_i) = H_i(P_T)$$

2. Use the arithmetic relation given by the v-adic distance between a periodic point and a tail curve.
   $T(P_i)$ is a S-unit $\Rightarrow H_i(P_T)$ is a S-unit.
3. Use the Finiteness of integral points of $\mathbb{P}^n$ - $\{2n+1$ hyperplanes in general position$\}$ to prove that our system of equation has finitely many solutions.
   Take $F = H_1 \cdot \ldots \cdot H_{2N+1} \in O_S^*$ has finitely many solutions.
4. Use the known bound for the number of solutions to the decomposable form equation (Evertse).
   We get an explicit bound previous descomposable equation.

Future work:

1. Generalize the explicit result that we have in $\mathbb{P}^2$ to a result in $\mathbb{P}^n$.

2. Assume that we have enough K-rational periodic points $(2N + 1 + e^2)$ to get a bound for the cardinality of $\mathrm{HTail}(\phi, K, e)$ in $\mathbb{P}^2$ in terms of the number of places of bad reduction, $[K : \mathbb{Q}]$ and e (independent on the degree of ϕ).

3. Get a bound for $\mathrm{HPrePer}(\phi, K, e)$ in $\mathbb{P}^2$ in terms of the number of places of bad reduction, $[K : \mathbb{Q}]$, e and the degree of ϕ.

4. Do the previous results in $\mathbb{P}^n$.

THANK YOU