Intelligent Systems Programming

Lecture 5: **Boolean Expression Representations & Binary Decision Diagrams (BDDs)**

Today's Program

- **[10:00-10:40] Classical representations**
  - Boolean expressions and Boolean functions
  - Desirable properties of representations of Boolean functions
  - Classical representations of Boolean expressions
    - Truth tables
    - CNF and DNF
- **[10:50-11:50] Binary Decision Diagrams**
  - If-then-else normal form (INF)
  - Decision trees
  - Ordered Binary Decision Diagrams (OBDDs)
  - Reduced Ordered Binary Decision Diagrams (ROBDDs / BDDs)

Boolean Expressions

- Boolean Expressions

  $t \ ::= x \mid 0 \mid 1 \mid \neg t \mid t \wedge t \mid t \vee t \mid t \Rightarrow t \mid t \Leftrightarrow t$

- Precedence

  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

- Set of truth values

  $\boldsymbol{B} = \{0,1\}$

- Truth assignments

  e.g. $t\,[0/x,\, 1/y]$

Boolean Functions

- Boolean expression E defines **boolean function** $f : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$, where

$$f(x_1,x_2,\ldots,x_n) = E(x_1,x_2,\ldots,x_n)$$

- Above, f is an n-ary function

- **Example**

$$f(x_1,x_2,x_3) = x_1 \Leftrightarrow \neg x_2$$

Properties of Boolean Functions

- Equality

  $f = g \quad \text{iff} \quad \forall \boldsymbol{x} \,.\, f(\boldsymbol{x}) = g(\boldsymbol{x})$

- Order of arguments matter

  $f(x,y) = x \Rightarrow y \qquad \neq \qquad g(y,x) = x \Rightarrow y$

- Several expressions may represent same function

  $f(x,y) = \; x \Rightarrow y \; = \neg x \vee y = (\neg x \vee y) \wedge (\neg x \vee x) = \ldots$

Number of Boolean Functions

Number of Boolean functions $f : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$

x_1	...	x_n	f
0	...	0	$f(0,\ldots,0)$
0	...	1	$f(0,\ldots,1)$
0	...	0	$f(0,\ldots,0)$
0	...	1	$f(0,\ldots,1)$
...	...	...	...
1	...	0	$f(1,\ldots,0)$
1	...	1	$f(1,\ldots,1)$
1	...	0	$f(1,\ldots,0)$
1	...	1	$f(1,\ldots,1)$
...	...	...	...

$$2^{\left(2^n\right)}$$

Representation of Boolean functions

Desirable properties:

1. Compact
2. Equality check easy
3. Easy to evaluate the truth-value of an assignment
4. Boolean operations efficient
5. SAT check efficient
6. Tautology check efficient
7. Canonicity: exactly one representation of each Boolean function. - Solves 2, 5, and 6, why?

Compact representations are rare

- $2^{\left(2^n\right)}$ boolean functions in *n* variables…
  - How do we find a single compact representation for them all?
- The fraction of Boolean functions of *n* variables with a polynomial size in *n* → 0 for *n* → ∞

Curse of Boolean function representations:

This problem exists for all representations we know!

Classical Representations of Boolean Functions

Truth tables

- Compact ☹ table size 2^n
- Equality check easy ☺ canonical
- Easy to evaluate the truth-value of an assignment ☺ log m or constant
- Boolean operations efficient ☺ linear
- SAT check efficient ☺ linear
- Tautology check efficient ☺ linear

x	y	z	$x \wedge y \vee z$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

DNF and CNF

- Is there a DNF and CNF of every expression?
- Given a truth table representation of a Boolean formula, can we easily define a DNF and CNF of the formula?

x	*y*	*z*	*e*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

DNF from on-tuples

- Example DNF of *e* – use *on-tuples*

x	y	z	e	
0	0	0	0	
0	0	1	1	$\neg x \wedge \neg y \wedge z \ \vee$
0	1	0	0	
0	1	1	1	$\neg x \wedge y \wedge z \ \vee$
1	0	0	0	
1	0	1	1	$x \wedge \neg y \wedge z \ \vee$
1	1	0	1	$x \wedge y \wedge \neg z \ \vee$
1	1	1	1	$x \wedge y \wedge z$

CNF from off-tuples

- Example CNF of *e* - use *off-tuples*

x	y	z	e	
0	0	0	0	$\neg(\neg x \wedge \neg y \wedge \neg z)\ \wedge$
0	0	1	1	
0	1	0	0	$\neg(\neg x \wedge y \wedge \neg z)\ \wedge$
0	1	1	1	
1	0	0	0	$\neg(x \wedge \neg y \wedge \neg z)\ \wedge$
1	0	1	1	
1	1	0	1	
1	1	1	1	

CNF from off-tuples

- Example CNF of *e* - use *off-tuples*

x	y	z	e	
0	0	0	0	$(x \vee y \vee z) \wedge$
0	0	1	1	
0	1	0	0	$(x \vee \neg y \vee z) \wedge$
0	1	1	1	
1	0	0	0	$(\neg x \vee y \vee z)$
1	0	1	1	
1	1	0	1	
1	1	1	1	

DNF and CNF

Every Boolean formula has a DNF and CNF representation

- The special version DNF and CNF representations produced from *on* and *off*-tuples are canonical and called cDNF and cCNF
- Are cDNF and cCNF minimum size DNF and CNF representations?

Checking DNF and CNF-formulas

- Symmetry properties of DNF and CNF

	SAT	Tautology
CNF	NP complete	Polynomial (exercise)
DNF	Polynomial (exercise)	Co-NP complete

- Idea: Solve CNF-SAT by conversion to DNF-SAT
  – Problem: conversion between CNF and DNF may be exponential

Converting CNF to DNF

- Example
  - CNF $\left(x_0^1 \vee x_1^1\right) \wedge \left(x_0^2 \vee x_1^2\right) \wedge \cdots \wedge \left(x_0^n \vee x_1^n\right)$
  - Corresponding DNF blows up

$$
\begin{gathered}
\left(x_0^1 \wedge x_0^2 \wedge \cdots \wedge x_0^{n-1} \wedge x_0^n\right) \vee \\
\left(x_0^1 \wedge x_0^2 \wedge \cdots \wedge x_0^{n-1} \wedge x_1^n\right) \vee \\
\vdots \\
\left(x_1^1 \wedge x_1^2 \wedge \cdots \wedge x_1^{n-1} \wedge x_0^n\right) \vee \\
\left(x_1^1 \wedge x_1^2 \wedge \cdots \wedge x_1^{n-1} \wedge x_1^n\right)
\end{gathered}
$$

Binary Decision Diagrams

If-then-else operator

- The *if-then-else* Boolean operator is defined by

$$x \to y_1 , y_0 \;\equiv\; (x \wedge y_1) \vee (\neg x \wedge y_0)$$

- We have

$$(x \to y_1 , y_0)\,[1/x] \;\equiv\; (1 \wedge y_1) \vee (0 \wedge y_0) \equiv y_1$$

$$(x \to y_1 , y_0)\,[0/x] \;\equiv\; (0 \wedge y_1) \vee (1 \wedge y_0) \equiv y_0$$

- What is $x \to 1 , 0$ equivalent to? And $x \to 0 , 1$?

If-then-else operator

- All operators in propositional logic can be expressed using only $\rightarrow$ operators with:
  - $\rightarrow$ expressions and 0 and 1 for y_1 and y_0
  - tests on un-negated variables
  - Variables only as tests
- What are *if-then-else* expressions for
  - x, $\neg x$
  - $x \wedge y$
  - $x \vee y$
  - $x \Rightarrow y$

If-then-else Normal Form (INF)

An *if-then-else* Normal Form (INF) is a Boolean expression build entirely from the if-then-else operator and the constants 0 and 1 such that all test are performed only on un-negated variables

- **Proposition**: any Boolean expression t is equivalent to an expression in INF

  Proof:

  $t \equiv x \rightarrow t[1/x], t[0/x]$ (Shannon expansion of t)

  Apply the Shannon expansion recursively on t. The recursion must terminate in 0 or 1, since the number of variables is finite

Shannon Expansion

Expression t with 4 variables: t x_1,x_2,x_3,x_4

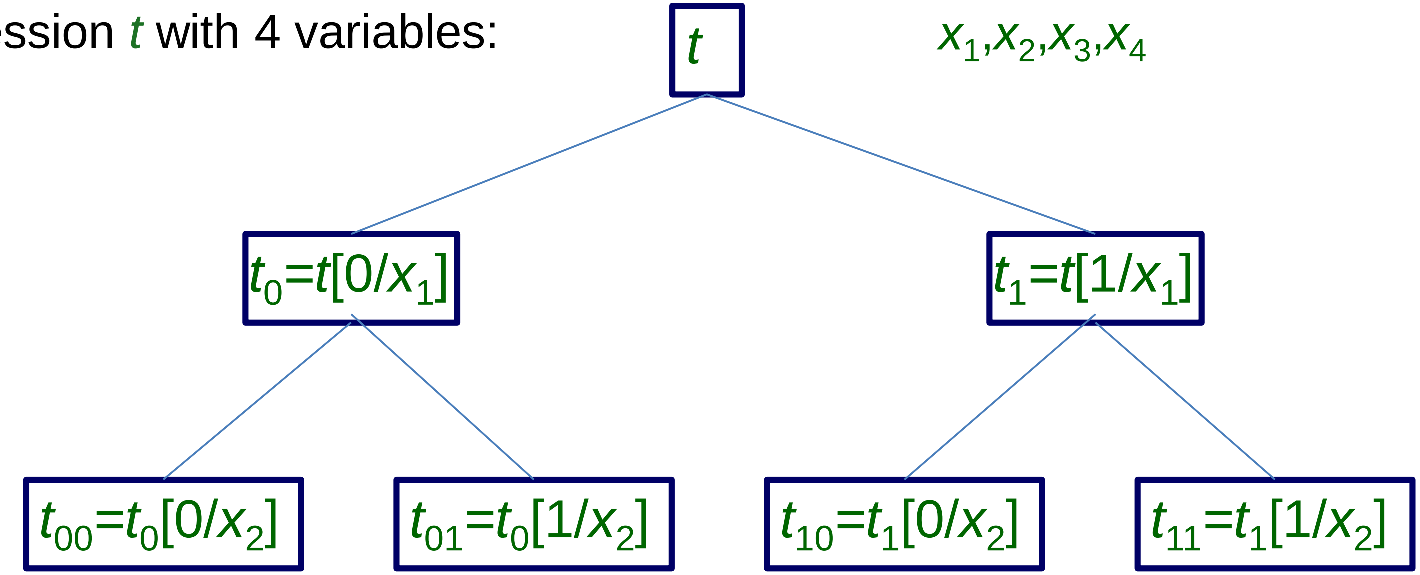
$t_0=t[0/x_1]$ $t_1=t[1/x_1]$

$$t \equiv x_1 \rightarrow t_1,\ t_0$$

Shannon Expansion

Expression t with 4 variables:

x_1, x_2, x_3, x_4

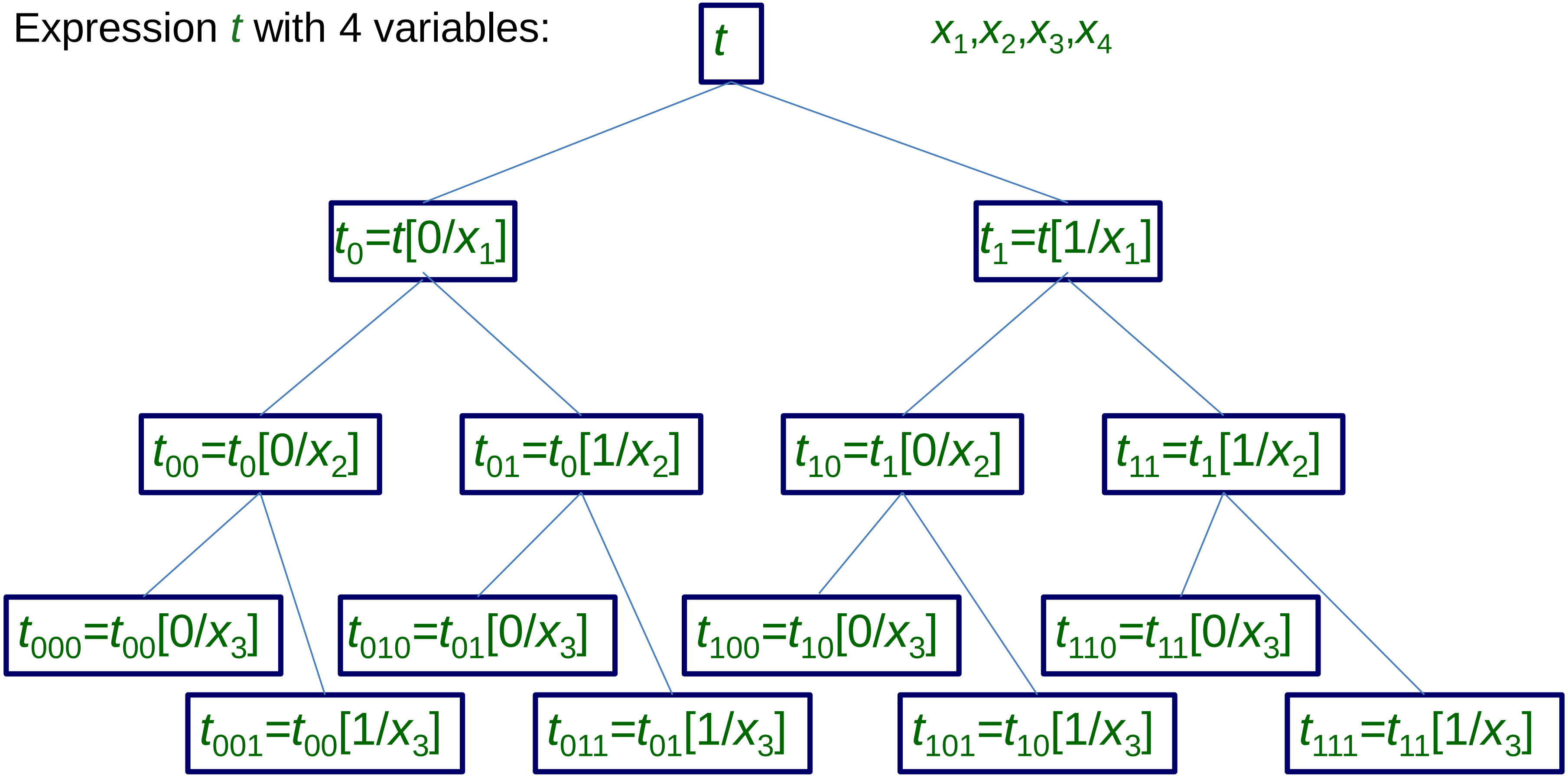


$$t \equiv x_1 \rightarrow t_1,\ t_0$$
$$t_0 \equiv x_2 \rightarrow t_{01},\ t_{00}$$
$$t_1 \equiv x_2 \rightarrow t_{11},\ t_{10}$$

Shannon Expansion

Expression t with 4 variables:

x_1, x_2, x_3, x_4



$t \equiv x_1 \to t_1, t_0$

$t_0 \equiv x_2 \to t_{01}, t_{00}$

$t_1 \equiv x_2 \to t_{11}, t_{10}$

$t_{00} \equiv x_3 \to t_{001}, t_{000}$

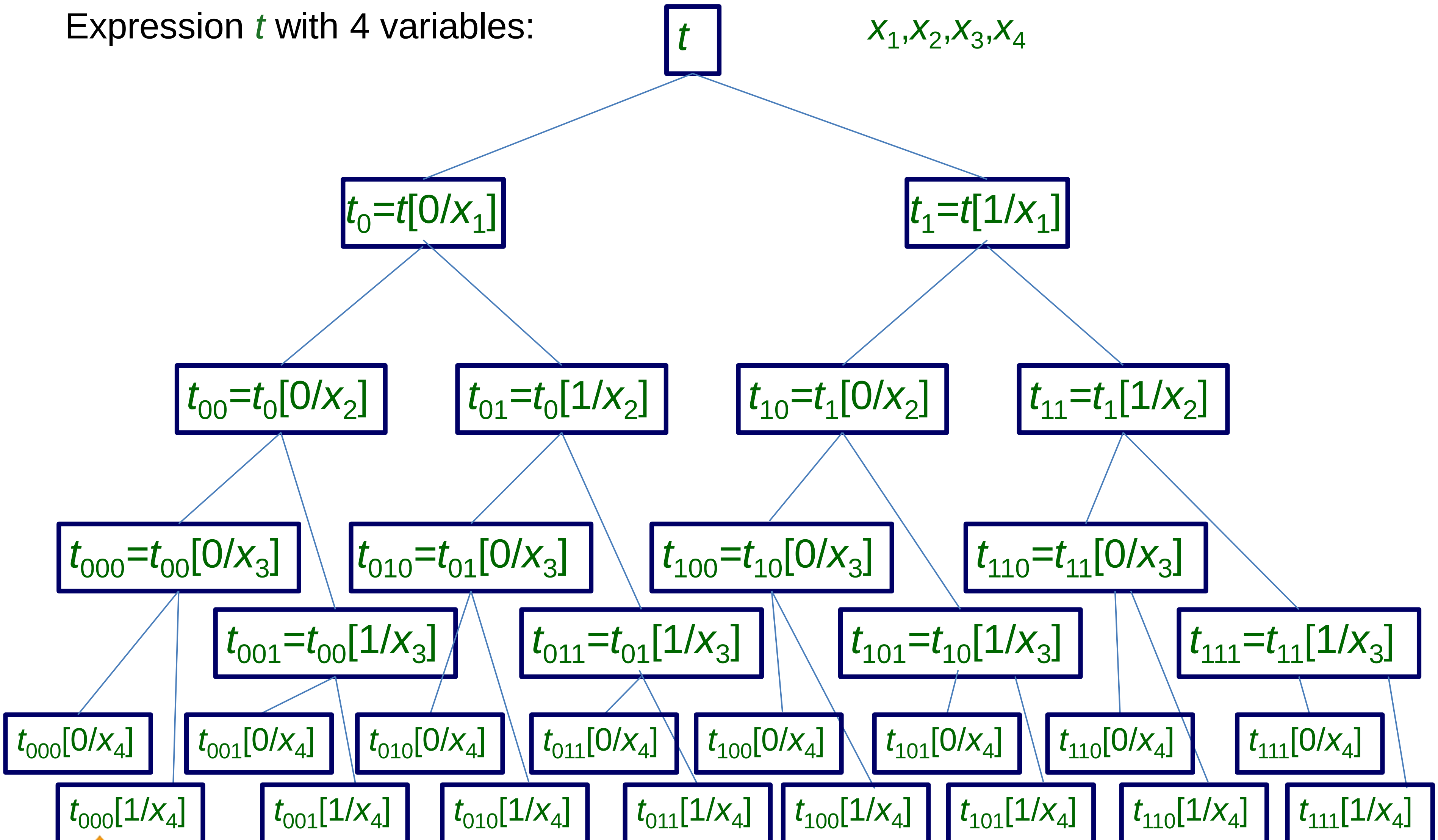
$t_{01} \equiv x_3 \to t_{011}, t_{010}$

$t_{10} \equiv x_3 \to t_{101}, t_{110}$

$t_{11} \equiv x_3 \to t_{111}, t_{110}$

Shannon Expansion

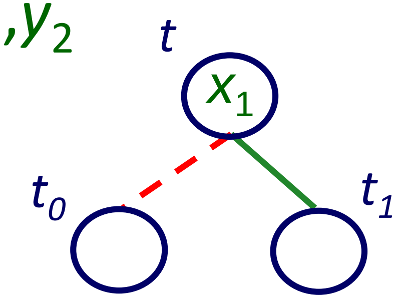
Expression t with 4 variables:



Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1,y_1,x_2,y_2

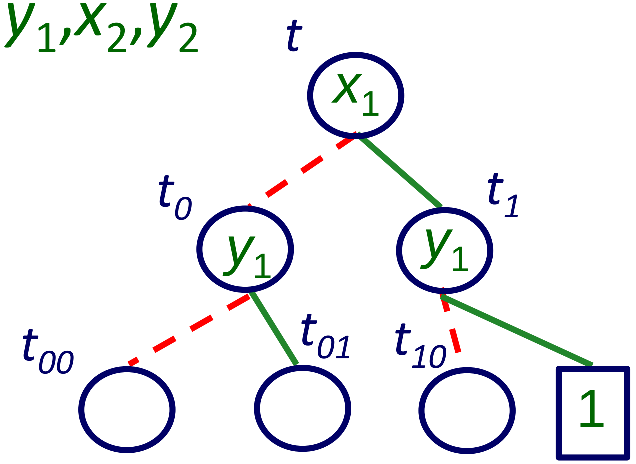
$t \quad \equiv x_1 \rightarrow t_1, t_0$



Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1, y_1, x_2, y_2

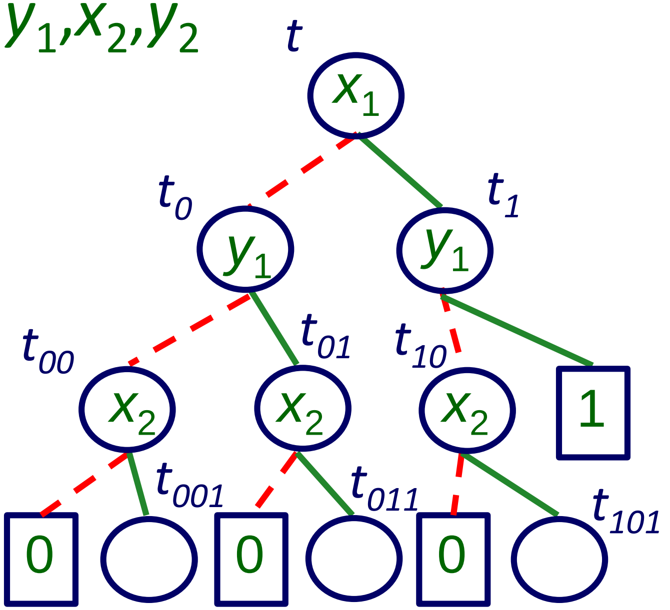
$$t \equiv x_1 \rightarrow t_1, t_0$$
$$t_0 \equiv y_1 \rightarrow t_{01}, t_{00}$$
$$t_1 \equiv y_1 \rightarrow 1, t_{10}$$



Example

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1, y_1, x_2, y_2

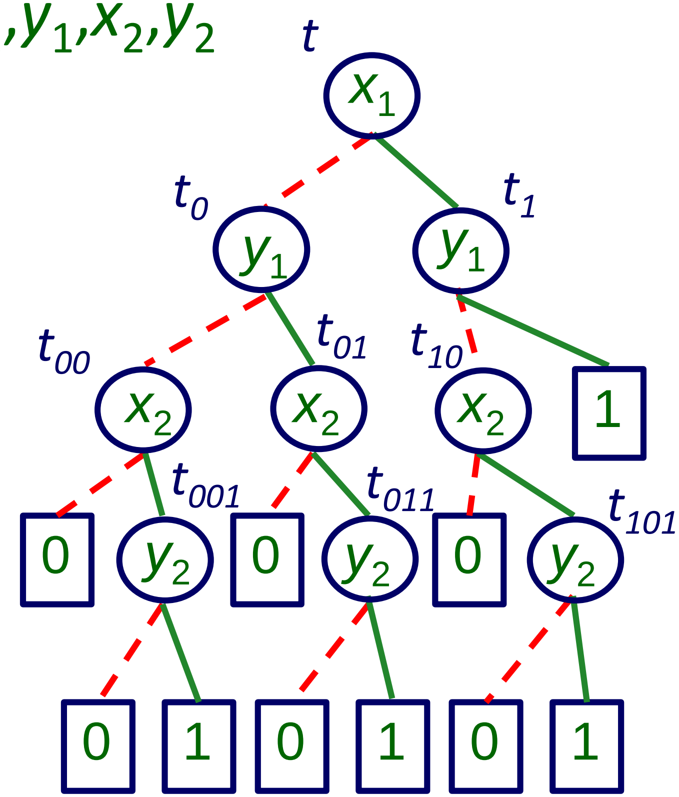
$$
\begin{aligned}
t &\equiv x_1 \to t_1, t_0 \\
t_0 &\equiv y_1 \to t_{01}, t_{00} \\
t_1 &\equiv y_1 \to 1, t_{10} \\
t_{01} &\equiv x_2 \to t_{011}, 0 \\
t_{00} &\equiv x_2 \to t_{001}, 0 \\
t_{10} &\equiv x_2 \to t_{101}, 0
\end{aligned}
$$



Decision Tree

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1, y_1, x_2, y_2

$$
\begin{aligned}
t &\equiv x_1 \to t_1, t_0 \\
t_0 &\equiv y_1 \to t_{01}, t_{00} \\
t_1 &\equiv y_1 \to 1, t_{10} \\
t_{01} &\equiv x_2 \to t_{011}, 0 \\
t_{00} &\equiv x_2 \to t_{001}, 0 \\
t_{10} &\equiv x_2 \to t_{101}, 0 \\
t_{011} &\equiv y_2 \to 1,0 \\
t_{001} &\equiv y_2 \to 1,0 \\
t_{101} &\equiv y_2 \to 1,0
\end{aligned}
$$

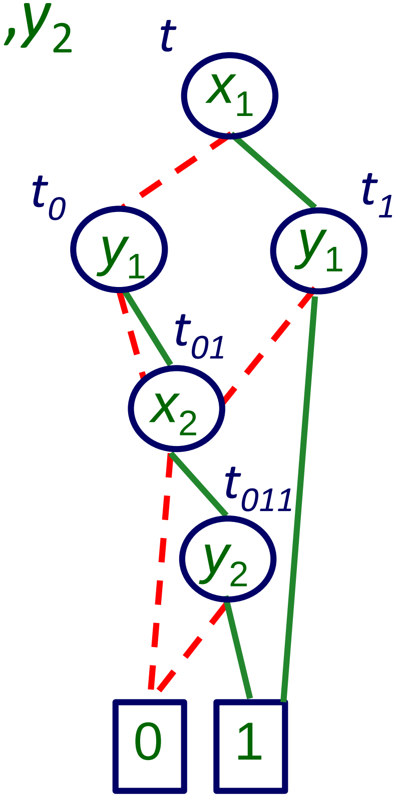


$$
\begin{aligned}
t = {} & x_1 \to (y_1 \to 1, (x_2 \to (y_2 \to 1,0), 0)), \\
& (y_1 \to (x_2 \to (y_2 \to 1,0), 0), (x_2 \to (y_2 \to 1,0), 0))
\end{aligned}
$$

Reduction I: substitute identical subtrees

- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1, y_1, x_2, y_2

$$t = x_1 \rightarrow t_1, t_0$$
$$t_0 = y_1 \rightarrow t_{01}, t_{01}$$
$$t_1 = y_1 \rightarrow 1, t_{01}$$
$$t_{01} = x_2 \rightarrow t_{011}, 0$$
$$t_{011} = y_2 \rightarrow 1,0$$

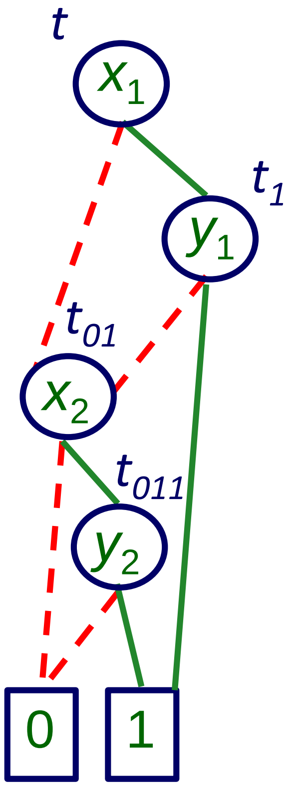


Result: an Ordered Binary Decision Diagram (OBDD)

Reduction II: remove redundant tests

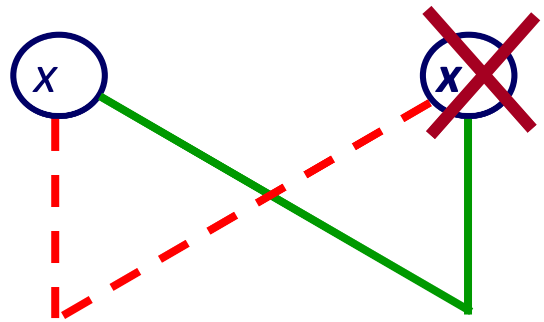
- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Shannon expansion of t in order x_1,y_1,x_2,y_2

$$
\begin{aligned}
t &= x_1 \to t_1,\ t_{01} \\
t_1 &= y_1 \to 1,\ t_{01} \\
t_{01} &= x_2 \to t_{011},\ 0 \\
t_{011} &= y_2 \to 1,0
\end{aligned}
$$

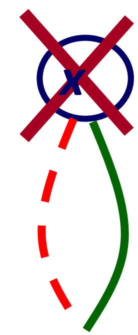


Result: a Reduced Ordered Binary Decision Diagram (ROBDD)
[often called a BDD]

Reductions

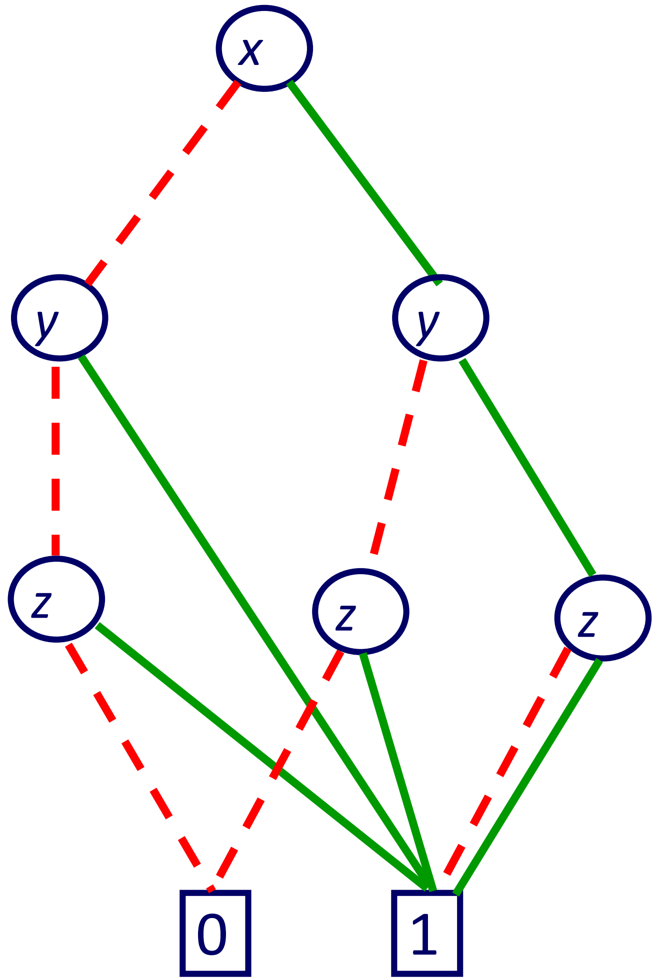


Uniqueness requirement



Non-redundant tests requirement

Another reduction example



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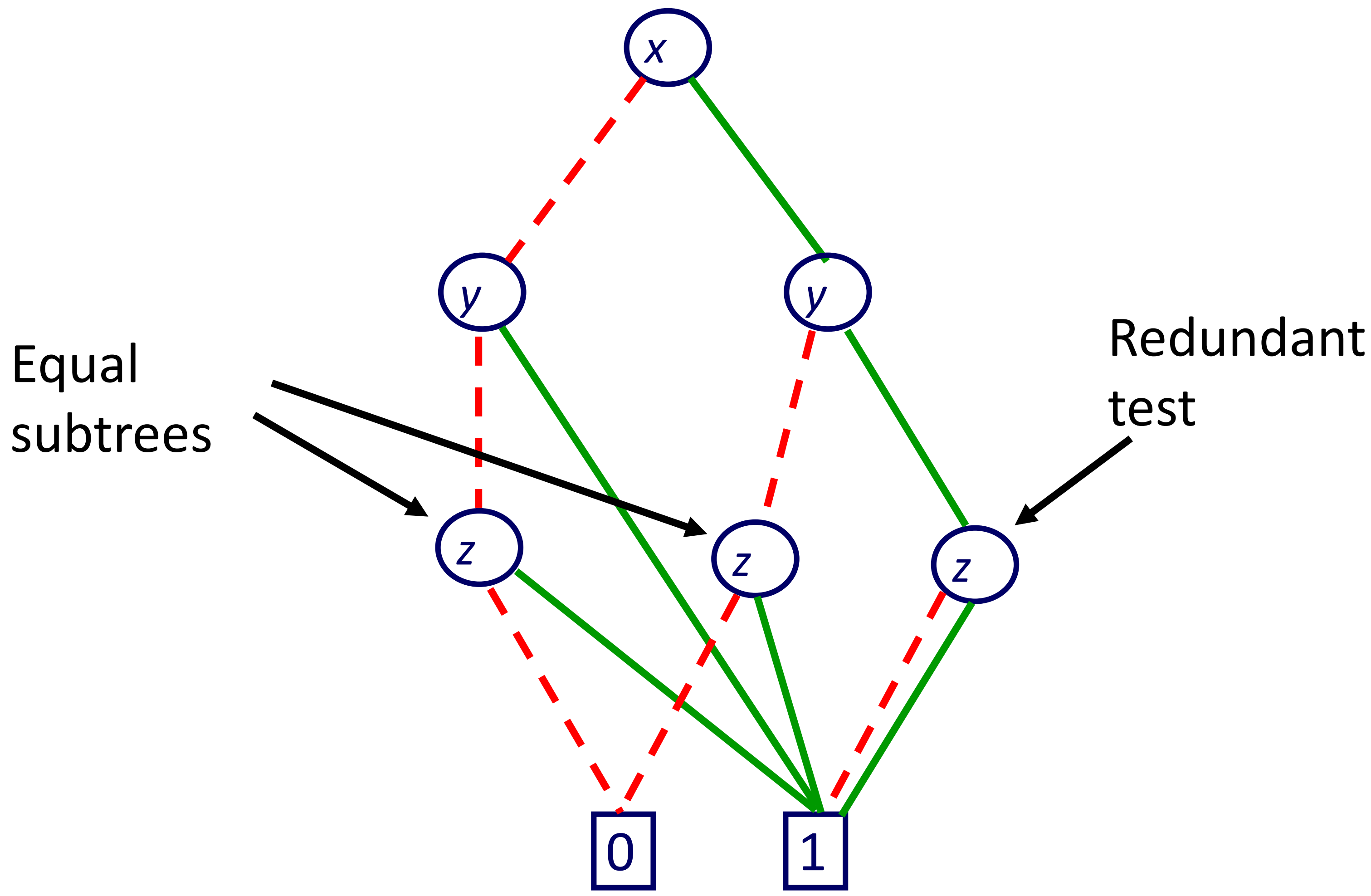
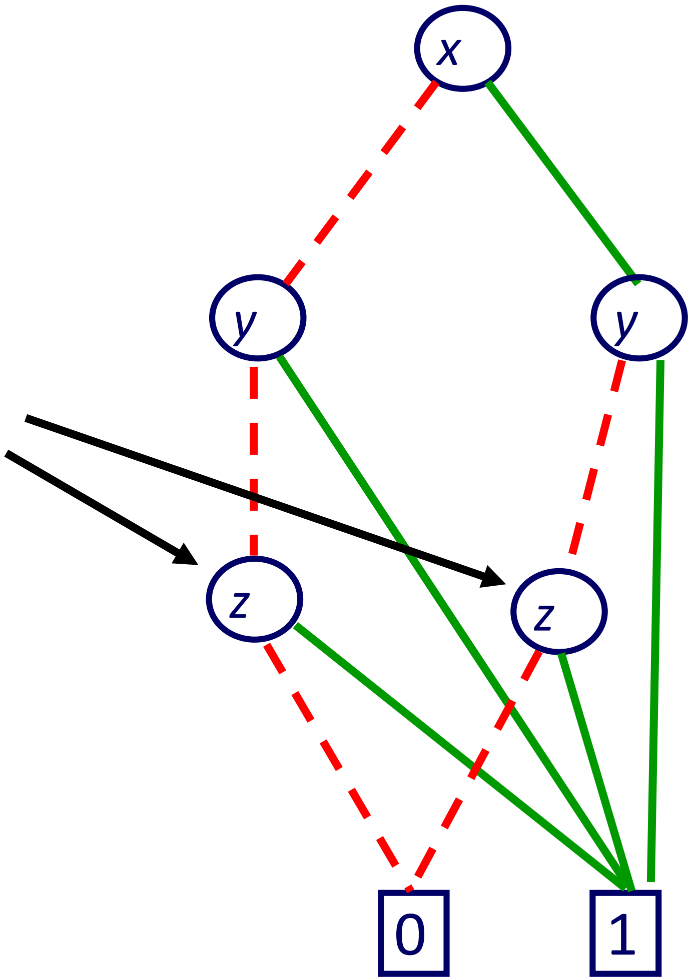
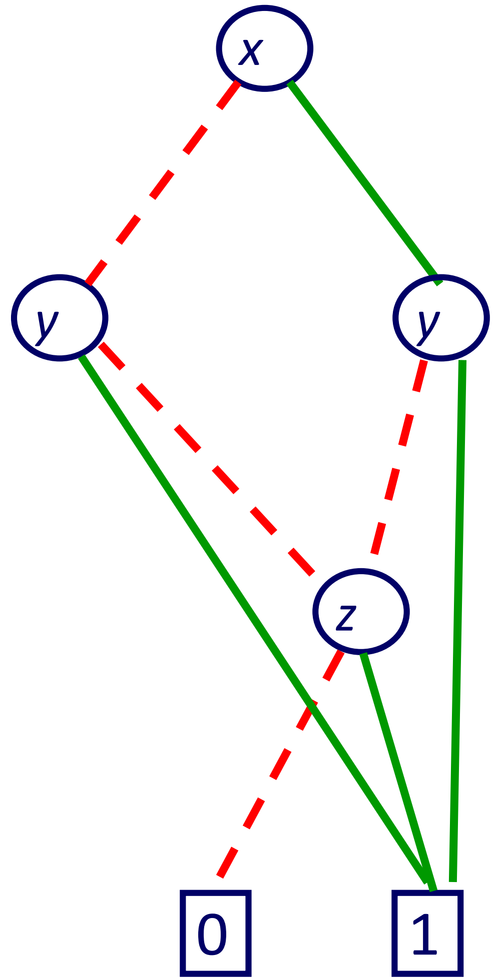


Another reduction example

Equal subtrees

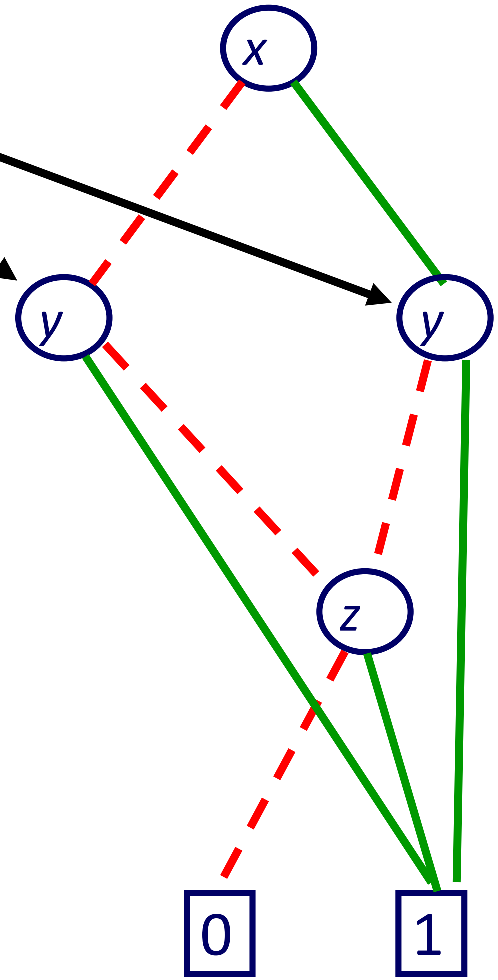


Another reduction example



Another reduction example

Equal subtrees



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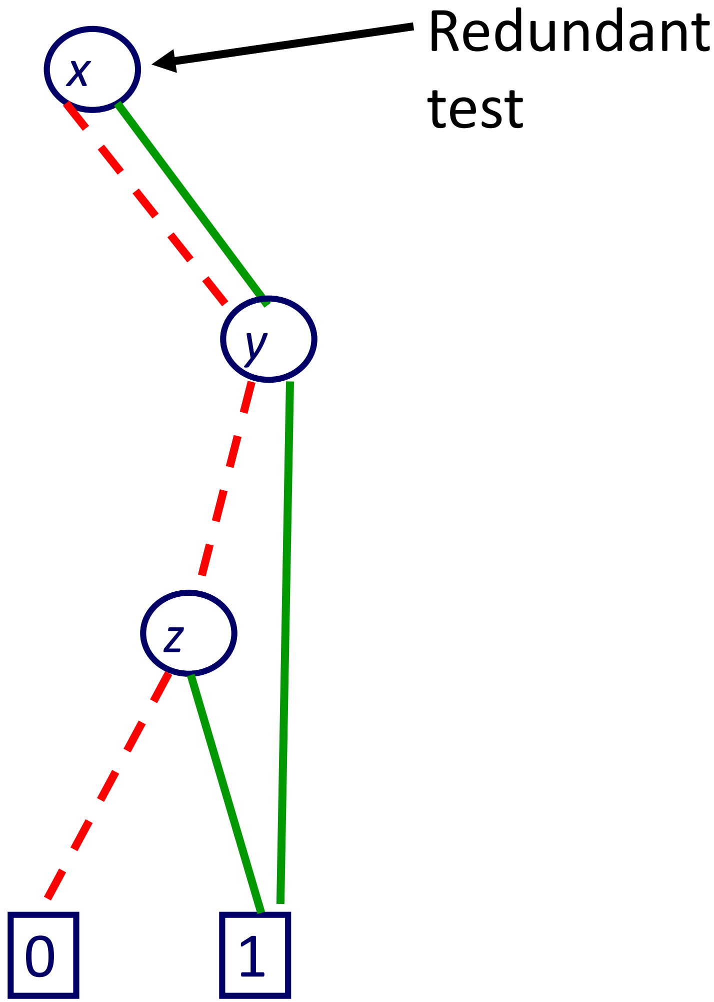
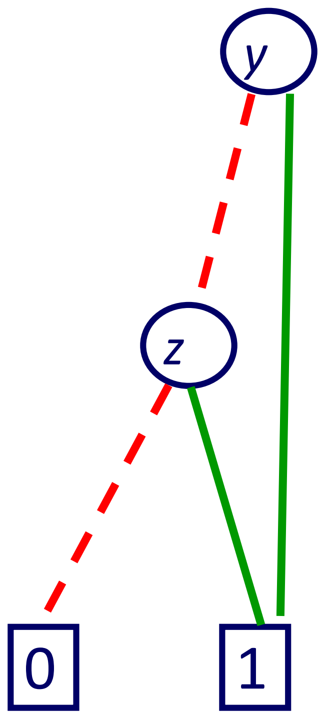


Another reduction example



Canonicity of ROBDDs

- **Canonicity Lemma**: for any function $f : \mathbf{B}^n \rightarrow \mathbf{B}$ there is exactly one ROBDD u with a variable ordering $x_1 < x_2 < \ldots < x_n$ such that $f_u = f(x_1,\ldots,x_n)$

Proof (by induction on n)

Read on your own!

Practice

- What are the ROBDDs of
  - x
  - 1
  - 0
  - $x \wedge y$ order x, y
  - $(x \Rightarrow y) \wedge z$ order x, y, z

Size of ROBDDs

- ROBDDs of many practically important Boolean functions are small
- Do all functions have polynomial ROBDD size? NO
  - ROBDDs do not escape the curse of Boolean function representation

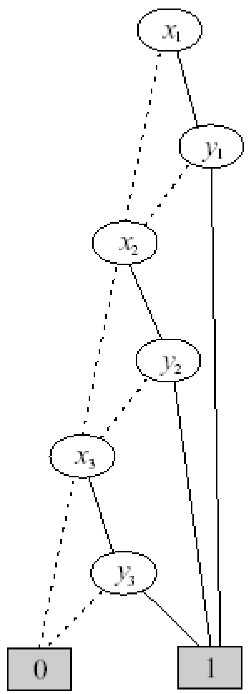
Size of ROBDDs

- The size of an ROBDD depends heavily on the variable ordering
- Example: $t = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$
- Build ROBDD of t in order x_1, x_2, y_1, y_2

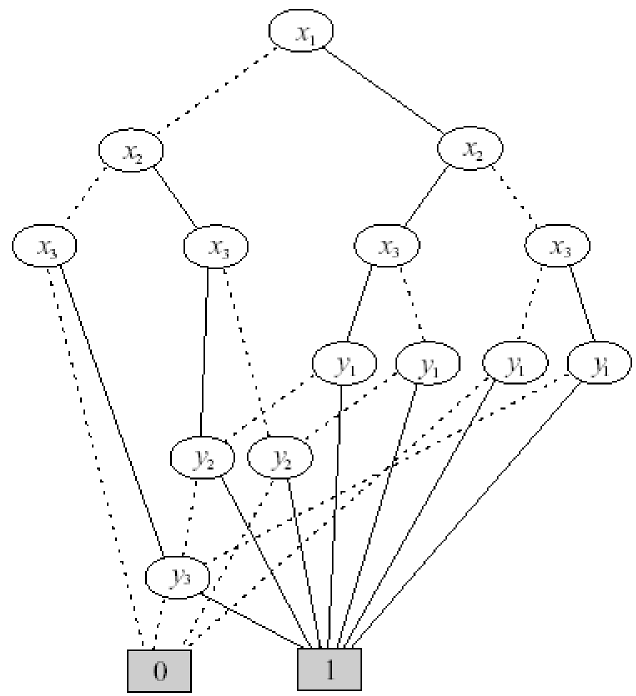
Size of ROBDDs

- The size of an ROBDD depends heavily on the variable ordering

$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$



$x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n$



$x_1 < x_2 < \ldots < x_n < y_1 < x_2 < \ldots < y_n$