2.3:

Derivation tree for :
$\mathtt{z} := 0;\ \textbf{while}\ \mathtt{y} \leq \mathtt{x}\ \textbf{do}\ (\mathtt{z} := \mathtt{z}+1;\ \mathtt{x} := \mathtt{x} - \mathtt{y})$

$$
\begin{array}{l}
s_0 = [x \mapsto 17 y \mapsto 5] \\
s_1 = [x \mapsto 17,\ y \mapsto 5; y \mapsto 0] \\
s_2 = [x \mapsto 17,\ y \mapsto 5; y \mapsto 1] \\
s_3 = [x \mapsto 12,\ y \mapsto 5; y \mapsto 1] \\
s_4 = [x \mapsto 12,\ y \mapsto 5; y \mapsto 2] \\
s_5 = [x \mapsto 7,\ y \mapsto 5; y \mapsto 2] \\
s_6 = [x \mapsto 7,\ y \mapsto 5; y \mapsto 3] \\
s_7 = [x \mapsto 2,\ y \mapsto 5; y \mapsto 3]
\end{array}
\qquad
\begin{array}{rcl}
b & = & \mathtt{y} \leq \mathtt{x} \\
A & = & \mathtt{z} := 0;\ B \\
B & = & \textbf{while}\ b\ \textbf{do}\ C \\
C & = & \mathtt{z} := \mathtt{z}+1;\ D \\
D & = & \mathtt{x} := \mathtt{x} - \mathtt{y}
\end{array}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{z} := 0,\ s\rangle \to s_1} \quad [\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_1\rangle \to s_2} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x}-y,\ s_2\rangle \to s_3}}{\langle C,\ s_1\rangle \to s_3} \quad [\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_3\rangle \to s_4} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x}-y,\ s_4\rangle \to s_5}}{\langle C,\ s_3\rangle \to s_5} \quad [\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_5\rangle \to s_6} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x}-y,\ s_6\rangle \to s_7}}{\langle C,\ s_5\rangle \to s_7} \quad [\mathsf{while^{ff}_{ns}}]\ \frac{}{\langle B,\ s_7\rangle \to s_7}\ \mathcal{B}[b]_{s_7} = \mathbf{ff}}{\langle B,\ s_5\rangle \to s_7}\ \mathcal{B}[b]_{s_5} = \mathbf{tt}}{\langle B,\ s_3\rangle \to s_7}\ \mathcal{B}[b]_{s_3} = \mathbf{tt}}{\langle B,\ s_1\rangle \to s_7}\ \mathcal{B}[b]_{s_1} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_7}
$$

2.4:

Program one will not terminate, because $\neg(\mathtt{x} = 1)$ will never be false when the initial state of x is 0. So the derivation tree will expand forever.
$\textbf{while}\ \neg(\mathtt{x} = 1)\ \textbf{do}\ (\mathtt{y} := y * x;\ \mathtt{x} := x - 1)$

$$
\begin{array}{l}
s_0 = [x \mapsto 0, y \mapsto 0] \\
s_1 = [x \mapsto 0, y \mapsto -1] \\
s_2 = [x \mapsto 0, y \mapsto -1] \\
s_3 = [x \mapsto 0, y \mapsto -2] \\
s_4 = [x \mapsto 0, y \mapsto -2]
\end{array}
\qquad
\begin{array}{rcl}
b & = & \neg(\mathtt{x} = 1) \\
A & = & \textbf{while}\ b\ \textbf{do}\ B \\
B & = & \mathtt{y} := y * x;\ C \\
C & = & \mathtt{x} := x - 1
\end{array}
$$

$$
[\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_0\rangle \to s_1} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_1\rangle \to s_2}}{\langle B,\ s_0\rangle \to s_2} \quad [\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_2\rangle \to s_3} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_3\rangle \to s_4}}{\langle B,\ s_2\rangle \to s_4} \quad \langle \textbf{while}\ b\ \textbf{do}\ B,\ s_4\rangle \to s_x}{\langle A,\ s_2\rangle \to s_x}\ \mathcal{B}[b]_{s_2} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}
$$

Program two terminates for all inputs: $\textbf{while}\ 1 \leq \mathtt{x}\ \textbf{do}\ (\mathtt{y} := y * x;\ \mathtt{x} := x - 1)$

$$
\begin{array}{l}
s_0 = [x \mapsto 2, y \mapsto 3] \\
s_1 = [x \mapsto 2, y \mapsto 6] \\
s_2 = [x \mapsto 1, y \mapsto 6] \\
s_3 = [x \mapsto 1, y \mapsto 6] \\
s_4 = [x \mapsto 0, y \mapsto 6]
\end{array}
\qquad
\begin{array}{rcl}
b & = & 1 \leq x \\
A & = & \textbf{while}\ b\ \textbf{do}\ B \\
B & = & \mathtt{y} := y * x;\ C \\
C & = & \mathtt{x} := x - 1
\end{array}
$$

$$
[\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_0\rangle \to s_1} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_1\rangle \to s_2}}{\langle B,\ s_0\rangle \to s_2} \quad [\mathsf{while^{tt}_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_2\rangle \to s_3} \quad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_3\rangle \to s_4}}{\langle B,\ s_2\rangle \to s_4} \quad [\mathsf{while^{ff}_{ns}}]\ \frac{\langle \textbf{skip},\ s_4\rangle \to s_4}{\langle A,\ s_2\rangle \to s_4}\ \mathcal{B}[b]_{s_4} = \mathbf{ff}}{\langle A,\ s_2\rangle \to s_4}\ \mathcal{B}[b]_{s_2} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_4}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}
$$

Program three will never terminate because **true** will never be false, so the derivation tree will expand forever.
$\textbf{while}\ \mathit{true}\ \textbf{do skip}$

$$
s_0 = [] \qquad
\begin{array}{rcl}
b & = & \mathbf{tt} \\
A & = & \textbf{while}\ b\ \textbf{do skip}
\end{array}
$$

$$
[\mathsf{while^{tt}_{ns}}]\ \frac{\langle \textbf{skip},\ s_0\rangle \to s_0, \qquad \langle \textbf{while}\ b\ \textbf{do skip},\ s_0\rangle \to s_x}{\langle A,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}
$$

2.6:

Show that $S1; (S2; S3)$ and $(S1; S2); S3$ are semantically equivalent:

$$
\begin{array}{rcl}
C_1 & = & S_1;\ C_2 \\
C_2 & = & S_2;\ S_3 \\
C_3 & = & C_4;\ S_3 \\
C_4 & = & S_1;\ S_2
\end{array}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{\langle S_1,\ s_0\rangle \to s_1, \qquad [\mathsf{comp_{ns}}]\ \frac{\langle S_2,\ s_1\rangle \to s_2, \qquad \langle S_3,\ s_2\rangle \to s_3}{\langle C_2,\ s_1\rangle \to s_3}}{\langle S_1;\ C_2,\ s_0\rangle \to s_3}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{[\mathsf{comp_{ns}}]\ \frac{\langle S_1,\ s_0\rangle \to s_1, \qquad \langle S_2,\ s_1\rangle \to s_2}{\langle C_4,\ s_0\rangle \to s_2,} \qquad \langle S_3,\ s_2\rangle \to s_3}{\langle C_3;\ S_1,\ s_0\rangle \to s_3}
$$

Show that $S1; S2$ and $S2; S1$ are not always semantically equivalent:

$$
\begin{array}{l}
s_0 = [] \\
s_1 = [x \mapsto 3] \\
s_2 = [x \mapsto 5] \\
s_3 = [x \mapsto 5] \\
s_4 = [x \mapsto 3]
\end{array}
\qquad
\begin{array}{rcl}
C_1 & = & S_1;\ S_2 \\
C_2 & = & S_2;\ S_1 \\
S_1 & = & \mathtt{x} := 3 \\
S_2 & = & \mathtt{x} := 5
\end{array}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := 3,\ s_0\rangle \to s_1}, \qquad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := 5,\ s_1\rangle \to s_2}}{\langle C_1,\ s_0\rangle \to s_2}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{[\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := 5,\ s_0\rangle \to s_3}, \qquad [\mathsf{ass_{ns}}]\ \frac{}{\langle \mathtt{x} := 3,\ s_3\rangle \to s_4}}{\langle C_2,\ s_0\rangle \to s_4}
$$

2.7:

The While language can be extended with the **repeat** S **until** b statement:

$$
[\mathsf{repeat^{ff}_{ns}}]\ \frac{\langle S,\ s_0\rangle \to s_1}{\langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s_0\rangle \to s_1}\ \mathcal{B}[b]_{s_1} = \mathbf{ff}
$$

$$
[\mathsf{repeat^{tt}_{ns}}]\ \frac{\langle S,\ s_0\rangle \to s_1, \qquad \langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s_1\rangle \to s_x}{\langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_1} = \mathbf{tt}
$$

Showing that **repeat** S **until** b (A) and S; **if** b **then skip else** (**repeat** S **until** b) (B) are semantically equivalent:
Assuming that b is false we have that:

$$
[\mathsf{repeat^{ff}_{ns}}]\ \frac{\langle S,\ s\rangle \to s',}{\langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s\rangle \to s'}\ \mathcal{B}[b]_{s'} = \mathbf{ff}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{\langle S,\ s\rangle \to s', \qquad [\mathsf{if^{ff}_{ns}}]\ \frac{\textbf{skip}}{\textbf{if}\ b\ \textbf{then skip else}\ (\textbf{repeat}\ S\ \textbf{until}\ b)}\ \mathcal{B}[b]_{s'} = \mathbf{ff}}{\langle S;\ \textbf{if}\ b\ \textbf{then skip else}\ (\textbf{repeat}\ S\ \textbf{until}\ b),\ s\rangle \to s'}
$$

Assuming that b is true we have that:

$$
[\mathsf{repeat^{tt}_{ns}}]\ \frac{\langle S,\ s\rangle \to s', \qquad \langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s'\rangle \to s_x}{\langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s\rangle \to s_x}\ \mathcal{B}[b]_{s'} = \mathbf{tt}
$$

$$
[\mathsf{comp_{ns}}]\ \frac{\langle S,\ s\rangle \to s', \qquad [\mathsf{if^{tt}_{ns}}]\ \frac{\langle \textbf{repeat}\ S\ \textbf{until}\ b,\ s'\rangle \to s_x}{\textbf{if}\ b\ \textbf{then skip else}\ (\textbf{repeat}\ S\ \textbf{until}\ b)}\ \mathcal{B}[b]_{s'} = \mathbf{tt}}{\langle S;\ \textbf{if}\ b\ \textbf{then skip else}\ (\textbf{repeat}\ S\ \textbf{until}\ b),\ s\rangle \to s_x}
$$

2.8:

The While language can be extended with the: $\textbf{for}\ \mathtt{A} := a_1\ \textbf{to}\ a_2\ \textbf{do}\ S$, statement:

$$
[\mathsf{for^{ff}_{ns}}]\ \frac{\langle \mathtt{A} := a_1,\ s_0\rangle \to s_0[A \to a_1], \qquad \langle S,\ s_0[A \to a_1]\rangle \to s_1, \qquad \langle \mathtt{A} := A+1,\ s_1\rangle \to s_1[A \to \mathtt{A}+1], \qquad \langle \textbf{for}\ \mathtt{A} := a_1\ \textbf{to}\ a_2\ \textbf{do}\ S,\ s_1[A \to \mathtt{A}+1]\rangle \to s_2}{\langle \textbf{for}\ \mathtt{A} := a_1\ \textbf{to}\ a_2\ \textbf{do}\ S,\ s_0\rangle \to s_2}\ \mathcal{B}[a_1 \leq a_2]_{s_0} = \mathbf{tt}
$$

$$
[\mathsf{for^{tt}_{ns}}]\ \frac{\langle \mathtt{A} := a_1,\ s_0\rangle \to s_0[A \to a_1]}{\langle \textbf{for}\ \mathtt{A} := a_1\ \textbf{to}\ a_2\ \textbf{do}\ S,\ s_0\rangle \to s_1}\ \mathcal{B}[a_1 \leq a_2]_{s_0} = \mathbf{ff}
$$