Exercise 1

a)

The data type Strange is strange, because it lacks a base case.

b)

Provide an induction principle for Strange.

$$\frac{\begin{array}{c}\forall n\ s.\ P\ s \rightarrow P(C_1\ n\ s)\\ \forall b\ s.\ P\ s \rightarrow P(C_2\ b\ s)\end{array}}{P\ s}$$

c)

$$\frac{\begin{array}{c}\forall n\ s.\ False \rightarrow False\\ \forall b\ s.\ False \rightarrow False\end{array}}{False}$$

Assume S:Strange
Prove False
Proof by inductions on S
Case C_1
Assume IH C_1: False hence False Case C_2
Assume IH C_2: False hence False

Exercise 2

$$\frac{\begin{array}{l}(S, s) \Rightarrow \gamma\\ \forall s\ P\ \textbf{skip}\ s\ s\\ \forall b\ S\ s\ P(\textbf{while}\ b\ \textbf{do}\ S)\ s\ \langle \textbf{if}\ b\ \textbf{then}\ (S;\ \textbf{while}\ b\ \textbf{do}\ S\ \textbf{else}\ \textbf{skip},\ ,\rangle s) >\\ \forall a\ s.\ P(\mathtt{x} := a)\ s\ (s[x \rightarrow \mathcal{A}[a]_s])\\ \forall S_1\ S_2\ s\ s'.\ \langle S_1,\ s\rangle \Rightarrow s' \rightarrow P\ S_1\ s\ s' \rightarrow P\ (S_1; S_2)\ s\ \langle S_2,\ s'\rangle\\ \forall S_1\ S_2\ s\ s'.\ \langle S_1,\ s\rangle \Rightarrow \langle {S_1}'',\ s'\rangle \rightarrow P\ S_1\ s\ s'. \rightarrow P\ (S_1; S_2)\ s\ \langle S_2,\ s'\rangle\\ \forall b\ s\ s'\ S_1\ S_2.\ \mathcal{B}[b]_s = \mathbf{tt} \rightarrow P\ (\textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{else}\ S_2)\ s\ \langle S_1,\ s\rangle\\ \forall b\ s\ s'\ S_1\ S_2.\ \mathcal{B}[b]_s = \mathbf{ff} \rightarrow P\ (\textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{else}\ S_2)\ s\ \langle S_2,\ s\rangle\end{array}}{P\ S\ s}$$

$[\mathsf{if}^{\mathsf{ff}}_{\mathsf{sos}}]\forall s\ s'\ b\ S_1\ S_2.\ \mathcal{B}[b]_s = \mathbf{ff} \Rightarrow \langle S_2,\ s\rangle \rightarrow s' \Rightarrow P\ S_2\ s\ s' \Rightarrow P\langle \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{else}\ S_2,\ s\rangle \rightarrow s'$

$[\mathsf{while}_{\mathsf{sos}}]\forall s\ s'\ s''\ b\ S.\ \langle S,\ s\rangle \rightarrow s'' \Rightarrow \langle \textbf{while}\ b\ \textbf{do}\ S,\ s''\rangle \rightarrow s'$
$P\ S\ s\ s'' \Rightarrow P\langle \textbf{while}\ b\ \textbf{do}\ S,\ s''\rangle \rightarrow s' \Rightarrow P\ (\textbf{while}\ b\ \textbf{do}\ S)\ s''\ s' \Rightarrow \mathcal{B}[b]_s = \mathbf{tt} \Rightarrow P\ (\textbf{while}\ b\ \textbf{do}\ S)\ s\ s$

$[\mathsf{while}_{\mathsf{sos}}]\forall b\ S.\ \mathcal{B}[b]_s = \mathbf{ff} \Rightarrow P\ (\textbf{while}\ b\ \textbf{do}\ S)\ s\ s$

Exercise 3

Prove that if $(S_1, s) \Rightarrow^k s'$ then $(S_1; S_2, s) \Rightarrow^k (S_2, s')$
We can assume A:

$$(S_1, s) \Rightarrow^k s'$$

Only two derivation rules can be used to derive B:

$$(S_1; S_2, s) \Rightarrow^k (S_2, s')$$

The first is C:

$$[\mathsf{comp}^1_{\mathsf{sos}}]\ \frac{(S_1, s) \Rightarrow (S_1', s')}{(S_1; S_2, s) \Rightarrow (S_1'; S_2, s')}$$

and the second D:

$$[\mathsf{comp}^2_{\mathsf{sos}}]\ \frac{(S_1, s) \Rightarrow s'}{(S_1; S_2, s) \Rightarrow (S_2, s')}$$

Since we assume A, that S_1 terminates in s' in k steps, we have the assumptions needed to use D and therefore show that the execution of S_1 is not influence by the following statements.

Exercise 4