2.3:

Derivation tree for :
$\mathtt{z} := 0;$ **while** $\mathtt{y} \le \mathtt{x}$ **do** $(\mathtt{z} := \mathtt{z} + 1;\ \mathtt{x} := \mathtt{x} - \mathtt{y})$

$$\begin{array}{l} s_0 = [x \mapsto 17 y \mapsto 5] \\ s_1 = [x \mapsto 17,\ y \mapsto 5; y \mapsto 0] \\ s_2 = [x \mapsto 17,\ y \mapsto 5; y \mapsto 1] \\ s_3 = [x \mapsto 12,\ y \mapsto 5; y \mapsto 1] \\ s_4 = [x \mapsto 12,\ y \mapsto 5; y \mapsto 2] \\ s_5 = [x \mapsto 7,\ y \mapsto 5; y \mapsto 2] \\ s_6 = [x \mapsto 7,\ y \mapsto 5; y \mapsto 3] \\ s_7 = [x \mapsto 2,\ y \mapsto 5; y \mapsto 3] \end{array} \qquad \begin{array}{rcl} b & = & \mathtt{y} \le \mathtt{x} \\ A & = & \mathtt{z} := 0;\ B \\ B & = & \textbf{while } b \textbf{ do } C \\ C & = & \mathtt{z} := \mathtt{z} + 1;\ D \\ D & = & \mathtt{x} := \mathtt{x} - \mathtt{y} \end{array}$$

$$\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{z} := 0,\ s\rangle \to s_1} \quad \mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_1\rangle \to s_2} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - y,\ s_2\rangle \to s_3}}{\langle C,\ s_1\rangle \to s_3} \quad \mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_3\rangle \to s_4} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - y,\ s_4\rangle \to s_5}}{\langle C,\ s_3\rangle \to s_5} \quad \mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{z} := \mathtt{z}+1,\ s_5\rangle \to s_6} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - y,\ s_6\rangle \to s_7}}{\langle C,\ s_5\rangle \to s_7} \quad \mathsf{[while^{ff}_{ns}]}\ \dfrac{}{\langle B,\ s_7\rangle \to s_7}\ \mathcal{B}[b]_{s_7} = \mathbf{ff}}{\langle B,\ s_5\rangle \to s_7}\ \mathcal{B}[b]_{s_5} = \mathbf{tt}}{\langle B,\ s_3\rangle \to s_7}\ \mathcal{B}[b]_{s_3} = \mathbf{tt}}{\langle B,\ s_1\rangle \to s_7}\ \mathcal{B}[b]_{s_1} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_7}$$

2.4:

Program one will not terminate, because $\neg(\mathtt{x} = 1)$ will never be false when the initial state of x is 0. So the derivation tree will expand forever.
while $\neg(\mathtt{x} = 1)$ **do** $(\mathtt{y} := y * x;\ \mathtt{x} := x - 1)$

$$\begin{array}{l} s_0 = [x \mapsto 0, y \mapsto 0] \\ s_1 = [x \mapsto 0, y \mapsto -1] \\ s_2 = [x \mapsto 0, y \mapsto -1] \\ s_3 = [x \mapsto 0, y \mapsto -2] \\ s_4 = [x \mapsto 0, y \mapsto -2] \end{array} \qquad \begin{array}{rcl} b & = & \neg(\mathtt{x} = 1) \\ A & = & \textbf{while } b \textbf{ do } B \\ B & = & \mathtt{y} := y * x;\ C \\ C & = & \mathtt{x} := x - 1 \end{array}$$

$$\mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_0\rangle \to s_1} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_1\rangle \to s_2}}{\langle B,\ s_0\rangle \to s_2} \quad \mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_2\rangle \to s_3} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_3\rangle \to s_4}}{\langle B,\ s_2\rangle \to s_4} \quad \langle \textbf{while } b \textbf{ do } B,\ s_4\rangle \to s_x}{\langle A,\ s_2\rangle \to s_x}\ \mathcal{B}[b]_{s_2} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}$$

Program two terminates for all inputs: **while** $1 \le \mathtt{x}$ **do** $(\mathtt{y} := y * x;\ \mathtt{x} := x - 1)$

$$\begin{array}{l} s_0 = [x \mapsto 2, y \mapsto 3] \\ s_1 = [x \mapsto 2, y \mapsto 6] \\ s_2 = [x \mapsto 1, y \mapsto 6] \\ s_3 = [x \mapsto 1, y \mapsto 6] \\ s_4 = [x \mapsto 0, y \mapsto 6] \end{array} \qquad \begin{array}{rcl} b & = & 1 \le x \\ A & = & \textbf{while } b \textbf{ do } B \\ B & = & \mathtt{y} := y * x;\ C \\ C & = & \mathtt{x} := x - 1 \end{array}$$

$$\mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_0\rangle \to s_1} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_1\rangle \to s_2}}{\langle B,\ s_0\rangle \to s_2} \quad \mathsf{[while^{tt}_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{y} := \mathtt{y} * \mathtt{x},\ s_2\rangle \to s_3} \quad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := \mathtt{x} - 1,\ s_3\rangle \to s_4}}{\langle B,\ s_2\rangle \to s_4} \quad \mathsf{[while^{ff}_{ns}]}\ \dfrac{\langle \textbf{skip},\ s_4\rangle \to s_4}{\langle A,\ s_2\rangle \to s_4}\ \mathcal{B}[b]_{s_4} = \mathbf{ff}}{\langle A,\ s_2\rangle \to s_4}\ \mathcal{B}[b]_{s_2} = \mathbf{tt}}{\langle A,\ s_0\rangle \to s_4}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}$$

Program three will never terminate because **true** will never be false, so the derivation tree will expand forever.
while *true* **do skip**

$$s_0 = [] \qquad \begin{array}{rcl} b & = & \mathbf{tt} \\ A & = & \textbf{while } b \textbf{ do skip} \end{array}$$

$$\mathsf{[while^{tt}_{ns}]}\ \dfrac{\langle \textbf{skip},\ s_0\rangle \to s_0, \qquad \langle \textbf{while } b \textbf{ do skip},\ s_0\rangle \to s_x}{\langle A,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_0} = \mathbf{tt}$$

2.6:

Show that $S1; (S2; S3)$ and $(S1; S2); S3$ are semantically equivalent:

$$\begin{array}{rcl} C_1 & = & S_1;\ C_2 \\ C_2 & = & S_2;\ S_3 \\ C_3 & = & C_4;\ S_3 \\ C_4 & = & S_1;\ S_2 \end{array}$$

$$\mathsf{[comp_{ns}]}\ \dfrac{\langle S_1,\ s_0\rangle \to s_1, \qquad \mathsf{[comp_{ns}]}\ \dfrac{\langle S_2,\ s_1\rangle \to s_2, \qquad \langle S_3,\ s_2\rangle \to s_3}{\langle C_2,\ s_1\rangle \to s_3}}{\langle S_1;\ C_2,\ s_0\rangle \to s_3}$$

$$\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[comp_{ns}]}\ \dfrac{\langle S_1,\ s_0\rangle \to s_1, \qquad \langle S_2,\ s_1\rangle \to s_2}{\langle C_4,\ s_0\rangle \to s_2,} \qquad \langle S_3,\ s_2\rangle \to s_3}{\langle C_3;\ S_1,\ s_0\rangle \to s_3}$$

Show that $S1; S2$ and $S2; S1$ are not always semantically equivalent:

$$\begin{array}{l} s_0 = [] \\ s_1 = [x \mapsto 3] \\ s_2 = [x \mapsto 5] \\ s_3 = [x \mapsto 5] \\ s_4 = [x \mapsto 3] \end{array} \qquad \begin{array}{rcl} C_1 & = & S_1;\ S_2 \\ C_2 & = & S_2;\ S_1 \\ S_1 & = & \mathtt{x} := 3 \\ S_2 & = & \mathtt{x} := 5 \end{array}$$

$$\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := 3,\ s_0\rangle \to s_1}, \qquad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := 5,\ s_1\rangle \to s_2}}{\langle C_1,\ s_0\rangle \to s_2}$$

$$\mathsf{[comp_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := 5,\ s_0\rangle \to s_3}, \qquad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{x} := 3,\ s_3\rangle \to s_4}}{\langle C_2,\ s_0\rangle \to s_4}$$

2.7:

The While language can be extended with the **repeat** S **until** b statement:

$$\mathsf{[repeat^{ff}_{ns}]}\ \dfrac{\textbf{skip}}{\langle \textbf{repeat } S \textbf{ until } b,\ s_0\rangle \to s_0}\ \mathcal{B}[b]_{s_0} = \mathbf{ff}$$

$$\mathsf{[repeat^{tt}_{ns}]}\ \dfrac{\langle S,\ s_0\rangle \to s_1, \qquad \langle \textbf{repeat } S \textbf{ until } b,\ s_1\rangle \to s_x}{\langle \textbf{repeat } S \textbf{ until } b,\ s_0\rangle \to s_x}\ \mathcal{B}[b]_{s_1} = \mathbf{tt}$$

2.8:

The While language can be extended with the: **for** $\mathtt{A} := a_1$ **to** a_2 **do** S, statement:

$$\mathsf{[for^{ff}_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{A} := a_0,\ s_0\rangle \to s_1}, \qquad \langle S,\ s_1\rangle \to s_2, \qquad \mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{A} := \mathsf{a}_0 + 1,\ s_2\rangle \to s_3}, \qquad \langle \textbf{for } \mathtt{A} := a_1 \textbf{ to } a_2 \textbf{ do } S,\ s_3\rangle \to s_4}{\langle \textbf{for } \mathtt{A} := a_1 \textbf{ to } a_2 \textbf{ do } S,\ s_0\rangle \to s_1}\ \mathcal{B}[a_1 = a_2]_{s_1} = \mathbf{ff}$$

$$\mathsf{[for^{tt}_{ns}]}\ \dfrac{\mathsf{[ass_{ns}]}\ \dfrac{}{\langle \mathtt{A} := a_0,\ s_0\rangle \to s_1}, \qquad \langle \textbf{skip},\ s_1\rangle \to s_1}{\langle \textbf{for } \mathtt{A} := a_1 \textbf{ to } a_2 \textbf{ do } S,\ s_0\rangle \to s_4}\ \mathcal{B}[a_1 = a_2]_{s_1} = \mathbf{tt}$$