

Putting it all Together:

Recursive equations:

$$\begin{aligned}a &= \perp \\ b &= f_{x=1} \\ c &= b \\ d &= f_{x=0} \\ e &= d \sqcup f \\ f &= f_{swap}\end{aligned}$$

Transfer functions:

$$\begin{aligned}f_{x=1}(l) &= + \\ f_{x=0}(l) &= 0 \\ f &= f_{swap}(l) =\end{aligned}$$

Monotone Functions and Fixed-Points:

i)

$$\begin{aligned}X &= \{a, b\} \\ Y &= X \cup Y\end{aligned}$$

In equation form:

$$\begin{aligned}X &= f(X, Y) = \{a, b\} \\ Y &= g(X, Y) = X \cup Y\end{aligned}$$

Can be written as:

$$\begin{aligned}X &= f() = \{a, b\} \\ Y &= g(X, Y) = X \cup Y\end{aligned}$$

Monotonicity

The function $f()$ is monotone, as it is constant. Therefore $f() \sqsubseteq f()$ will always hold. The function $g(X, Y)$ is also monotone, as no matter which sets use give the function it will combine them and the resulting set will always be greater or equal to the input sets.

ii)

$$\begin{aligned}X &= \{a, b\} \cup Y \\ Y &= X \setminus \{b\}\end{aligned}$$

In equation form:

$$\begin{aligned}X &= f(X, Y) = \{a, b\} \cup Y \\ Y &= g(X, Y) = X \setminus \{b\}\end{aligned}$$

Can be written as:

$$\begin{aligned}X &= f(Y) = \{a, b\} \cup Y \\ Y &= g(X) = X \setminus \{b\}\end{aligned}$$

The function $f(Y)$ is monotone, as it return $\{a, b\} \cup Y$ which is always a superset of Y . The function $g(X)$ is also monotone. It is hard to give an example of why, however intuitively removing a constant element preserves monotonicity, as the same element is removed on both sides of \sqsubseteq .

iii)

$$\begin{aligned} X &= \{a, b\} \cup Z \\ Y &= \{a, c\} \setminus X \\ Z &= X^C \end{aligned}$$

In equation form:

$$\begin{aligned} X &= f(X, Y, Z) = \{a, b\} \cup Z \\ Y &= g(X, Y, Z) = \{a, c\} \setminus X \\ Z &= h(X, Y, Z) = X^C \end{aligned}$$

Can be written as:

$$\begin{aligned} X &= f(Z) = \{a, b\} \cup Z \\ Y &= g(X) = \{a, c\} \setminus X \\ Z &= h(X) = X^C \end{aligned}$$

The function $f(Z)$ is monotone, as it return $\{a, b\} \cup Z$ which is always a superset of Z . If $g(X)$ is monotone then $\forall Y, Y' \ P(\{a, b, c\}) : Y \subseteq Y'$. However this is not the case:

$$\begin{aligned} \emptyset &\subseteq \{a, b, c\} \\ g(\emptyset) &\not\subseteq g(\{a, b, c\}) \\ \{a, c\} &\not\subseteq \emptyset \end{aligned}$$

$g(X)$ is therefore not monotone. $h(X)$ is not monotone either, as:

$$\begin{aligned} \emptyset &\subseteq \{a, b, c\} \\ g(\emptyset) &\not\subseteq g(\{a, b, c\}) \\ \{a, b, c\} &\not\subseteq \emptyset \end{aligned}$$