Polynomial Approximation and Sequences of Functions

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Theorem 1 (Taylor's Theorem). *If $f', \dots, f^{(n+1)}$ are defined on $[a, x]$, then*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$ for some t in (a, x).

Note 2. The Mean Value Theorem is a special case of Taylor's Theorem:

$$f(b) = f(a) + f'(c)(b-a)$$

for some c between a and b.

Theorem 3. *Uniform convergence of functions preserves continuity, i.e. if f_n are continuous and approach f uniformly, then f is continuous.*

Question 4. *What about differentiability, i.e. if f_n are differentiable and approach f uniformly, is f always differentiable, and is $\lim f_n' = f'$?*

Example 5. No to the second question: the functions $f_n(x) = \frac{1}{n}\sin(nx)$ converge uniformly to the zero function, which *is* differentiable. But, the limit of the derivatives don't exist. What about just differentiability?

Keywords. Taylor's Theorem, Taylor polynomial, error / remainder term, Cauchy, Lagrange, integral form, point-wise, uniform convergence, metric space, Cauchy criterion, Koch snowflake.