Eigenvectors and Eigenvalues

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Proposition 1. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof. Easy to show for two eigenvectors, then use induction. □

Theorem 2. *A linear operator T on a finite-dimensional vector space V is diagonalizable iff there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.*

Proof in the book.

Corollary 3. *A matrix A is diagonalizable iff the dimensions of its eigenspaces—i.e. the geometric multiplicities over all its eigenvalues—add up to the size of A. In this case the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.*

Proof. By Proposition 1, the eigenspaces are linearly independent. If their dimensions added up to less than n, we'd have too few eigenvectors to make a basis for V, and by Theorem 2 A would not be diagonalizable. If they added up to more than n, we'd have too many linearly independent vectors in V. Conversely if they do add up to n, then the union of bases for the eigenspaces forms an eigenbasis for V, and A is diagonalizable. □

Proposition 4. *Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Then λ is an eigenvalue of T iff it is an eigenvalue of $[T]_\beta$.*

Corollary 5. *Similar matrices have the same eigenvalues, but not necessarily the same eigenvectors.*

Proposition 6. *If v is an eigenvector of A corresponding to eigenvalue λ, and B is similar to A under change of coordinates matrix Q, then Qv is an eigenvector of B corresponding to the same eigenvalue λ. Another way of saying this is that change of coordinates preserves eigenvalues and eigenvectors.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned} Av &= Q^{-1}BQv \\ QAv &= BQv \\ \lambda Qv &= BQv, \end{aligned}$$

so Qv is an eigenvector of B corresponding to λ. $\square$

Definition 7. Let T be a linear operator on a finite-dimensional vector space V. Define the determinant of T to be $\det T = \det([T]_\beta)$ for any ordered basis β for V.

Note 8. Since the determinant is multiplicative, for any two bases β and α we have

$$\begin{aligned} \det([T]_\beta) &= \det(Q^{-1}[T]_\alpha Q) \\ &= \det Q^{-1} \det([T]_\alpha) \det Q \\ &= \det([T]_\alpha), \end{aligned}$$

where $Q = [I]_\beta^\alpha$ is the change of basis matrix from β to α, and therefore $\det T$ is well defined, i.e. it's independent of the choice of basis.

Proposition 9. *Change of basis is a linear operation, i.e.*

$$[T + \lambda U]_\beta = [T]_\beta + \lambda [U]_\beta.$$

Proof. Let v be a vector, $\beta = \{v_1, \dots, v_n\}$, and $T(v) = \sum a_i v_i, U(v) = \sum b_i v_i$. We want to show that

$$\begin{aligned} [T + \lambda U]_\beta [v]_\beta &= ([T]_\beta + \lambda [U]_\beta)[v]_\beta \\ [T(v) + \lambda U(v)]_\beta &= [T(v)]_\beta + \lambda [U(v)]_\beta. \end{aligned}$$

The RHS is $[a_i] + \lambda [b_i]$, which is the same as the LHS: $[a_i + \lambda b_i]$. $\square$

Proposition 10. *For any scalar λ and any ordered basis β for V,* $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$.

Proof. Follows from Definition 7 and Proposition 9. $\square$

Proposition 11. *A linear operator T on a finite-dimensional vector space is invertible iff zero is not an eigenvalue of T.*

Proof. If zero is an eigenvalue of T, then $\det(T - 0 \cdot I) = 0$, and T is not invertible. Conversely, if T is not invertible, then there exists a nonzero vector v s.t. $T(v) = 0 = 0 \cdot v$, and so 0 is an eigenvalue and v is an eigenvector of T. $\square$

Proposition 12. *Let T be an invertible linear operator. Then a scalar λ is an eigenvalue of T iff λ^{-1} is an eigenvalue of T^{-1}. Note that by the previous proposition λ is nonzero, so λ^{-1} exists.*

Proof. Apply T^{-1} to both sides of $T(v) = \lambda v$. □

Proposition 13. *The eigenvalues of an upper triangular matrix M are the diagonal entries of M.*

Proof. Follows from the fact that the determinant of an upper triangular matrix is the product of its diagonal entries. □

Definition 14. A scalar matrix is a square matrix of the form λI for some scalar λ.

Proposition 15. *If square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.*

Proof. If A is similar to λI, that means there is an invertible matrix Q s.t. $A = Q^{-1}\lambda I Q = \lambda I$. □

Proposition 16. *A diagonalizable matrix A having only one eigenvalue is a scalar matrix.*

Proof. If A is diagonalizable, this means A is similar to a diagonal matrix, whose diagonal entries are its eigenvalues. Since A only has one eigenvalue λ, the diagonal entries are all equal to λ. □

Example 17. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Proposition 18. *Similar matrices have the same characteristic polynomial.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned} \det(A - tI) &= \det(Q^{-1}BQ - Q^{-1}tIQ) \\ &= \det(Q^{-1}(B - tI)Q) \\ &= \det(Q^{-1})\det(B - tI)\det(Q) \\ &= \det(B - tI). \end{aligned}$$

□

Corollary 19. *The definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.*

Proof. Follows immediately from the fact that similar matrices are the same linear operator expressed under different bases. □

Keywords. Eigenvectors, eigenvalues, eigenspace, eigenbasis, differential operator, eigenfunctions, algebraic and geometric multiplicities, change of coordinates matrix, scalar matrix, characteristic polynomial.