Polynomial Approximation, Sequences, and Series

Nhan Trong

July 7, 2016

In mathematics you don't understand things. You just get used to them.

John von Neumann

Part I
Polynomial Approximation

Theorem 1 (Taylor's Theorem)**.** *If $f', \dots, f^{(n+1)}$ are defined on $[a, x]$, then*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$ for some t in (a, x).

Note 2. The Mean Value Theorem is a special case of Taylor's Theorem:

$$f(b) = f(a) + f'(c)(b-a)$$

for some c between a and b.

Keywords 1. Taylor's Theorem, Taylor polynomial, error / remainder term, Cauchy, Lagrange, integral form.

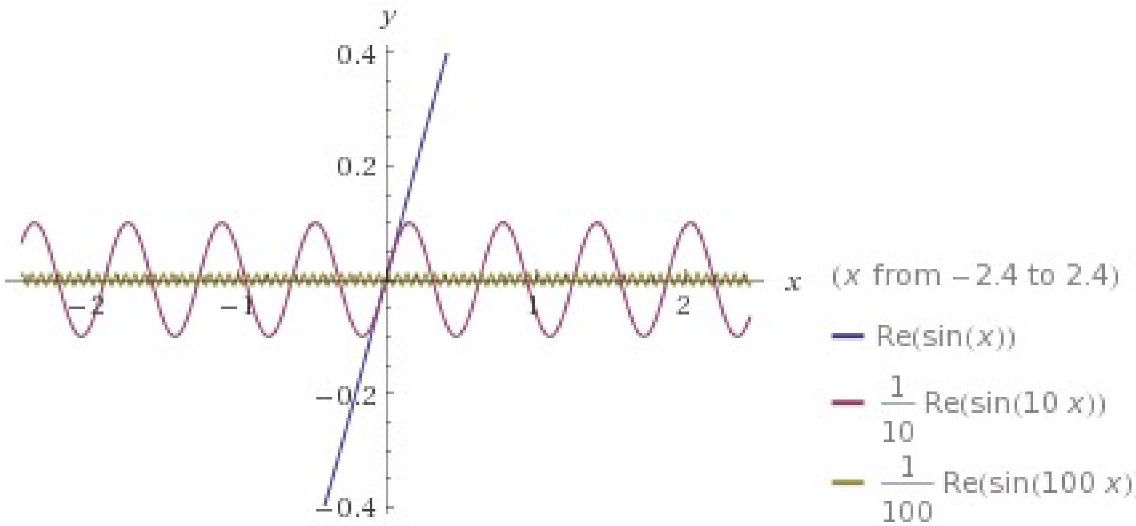
Part II
Sequences

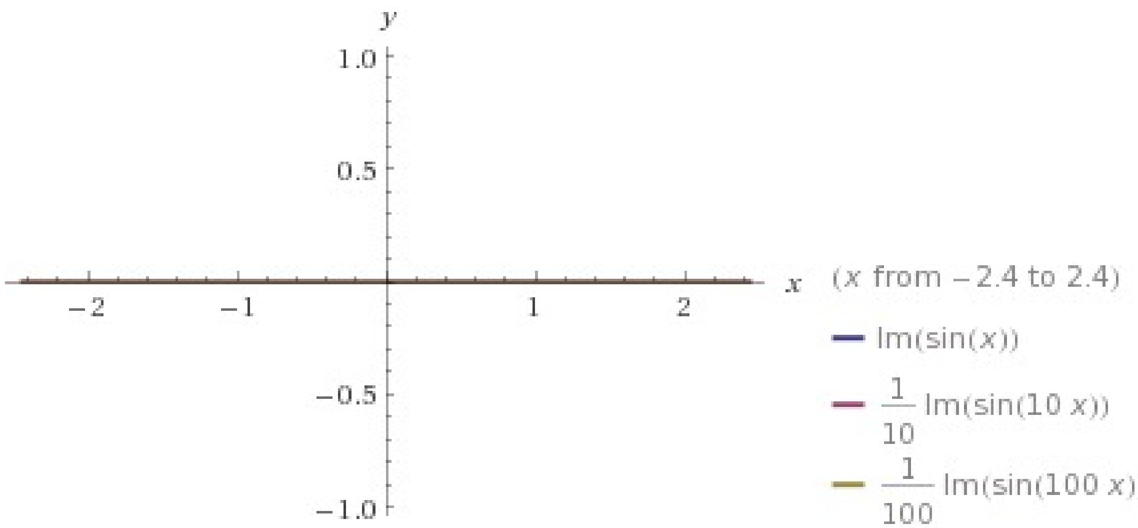
Theorem 3 (Uniform Limit Theorem)**.** *Uniform convergence of functions preserves continuity, i.e. if f_n are continuous and approach f uniformly, then f is continuous.*

Proposition 4. *The uniform limit of uniformly continuous functions is uniformly continuous.*

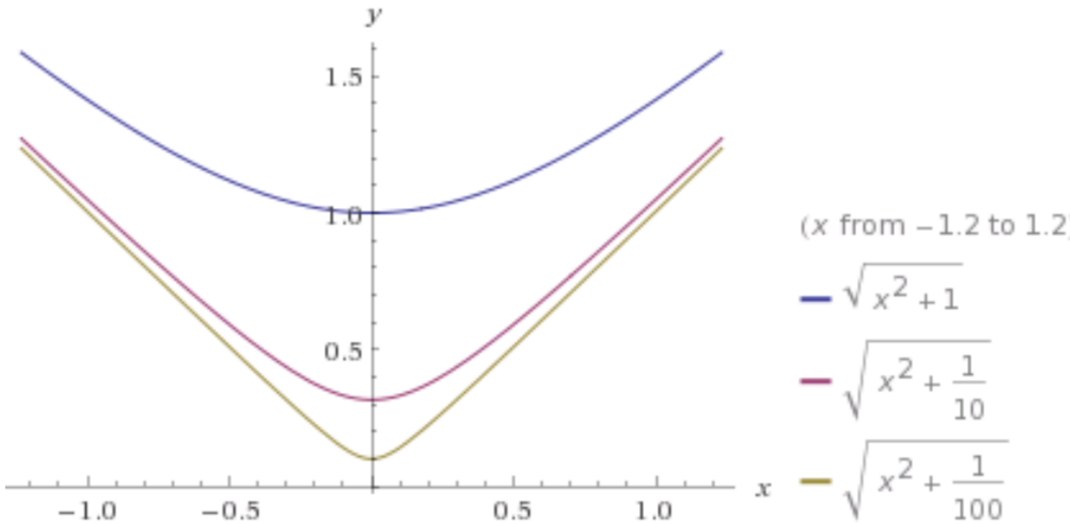
Question 5. *What about differentiability, i.e. if f_n are differentiable and approach f uniformly, is f always differentiable, and is $\lim f_n' = f'$?*

Example 6. No to the second question: the functions $f_n(x) = \frac{1}{n}\sin(nx)$ converge uniformly to the zero function, which *is* differentiable. But, the limit of the derivatives don't exist. What about just differentiability?





Example 7. Still No, e.g. the functions $f_n(x) = \sqrt{x^2 + 1/n}$ converge uniformly to $f = |x|$, which is not differentiable at zero.



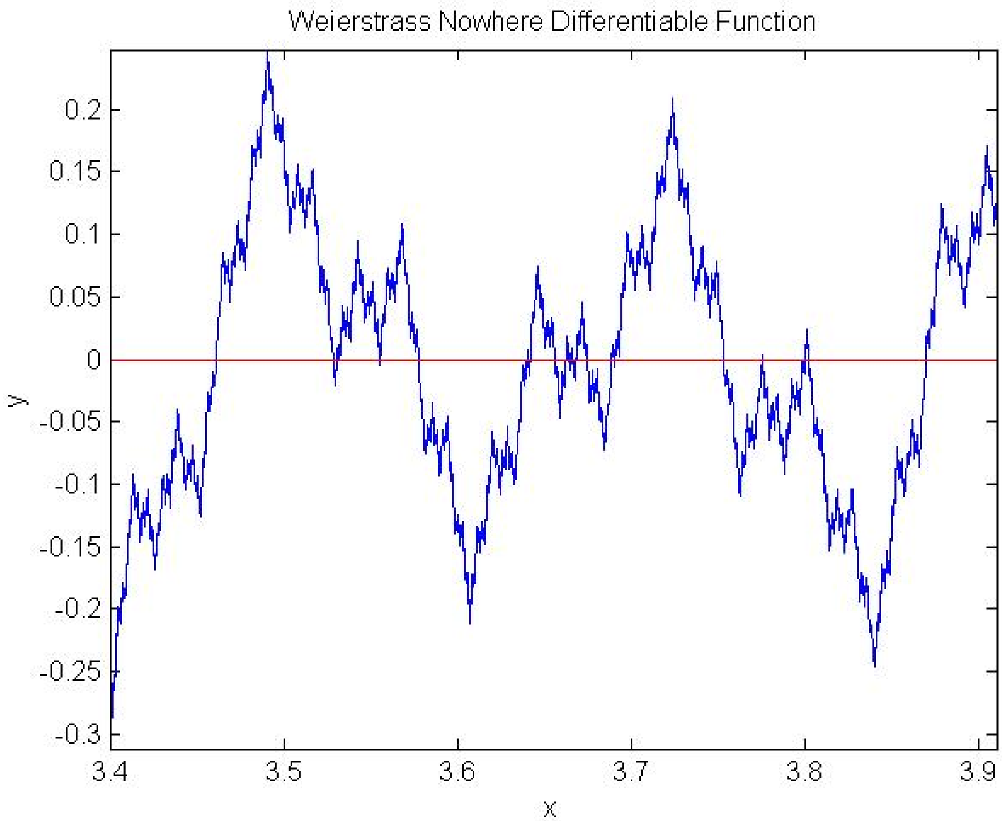
Example 8 (Weierstrass function)**.** The Weierstrass function

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

for appropriate values a and b, is the uniform limit of

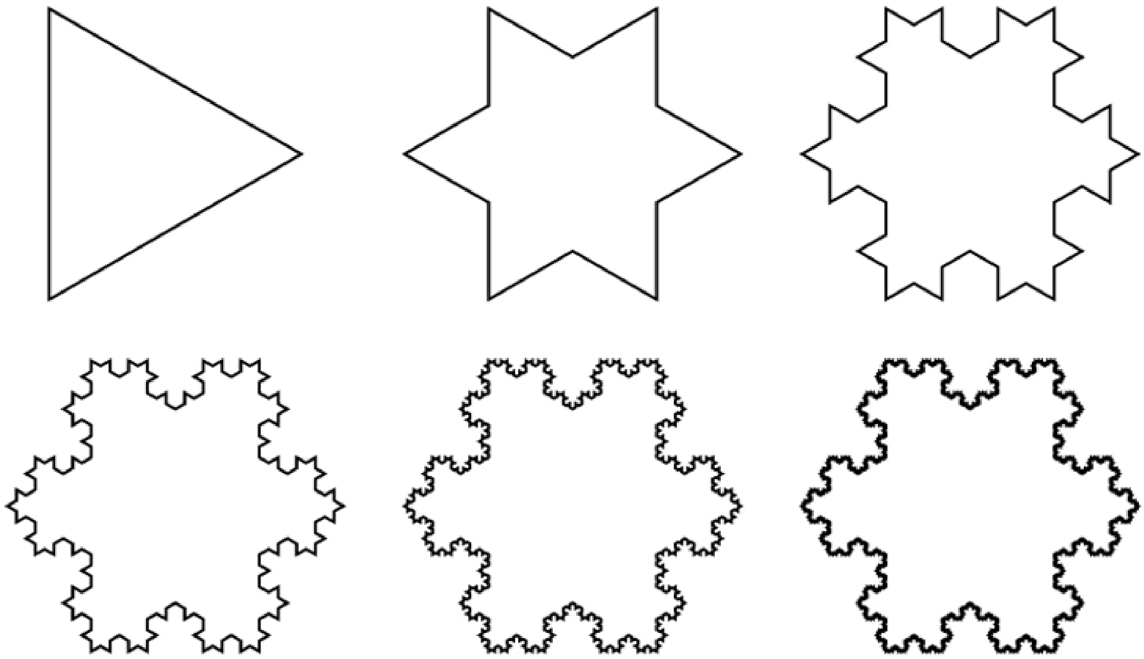
$$f_n = \sum_{k=0}^{n} a^k \cos(b^k \pi x),$$

but is nowhere differentiable.



Question 9. *Is the Koch snowflake nowhere differentiable?*

Yes. Proof?



Definition 10. Let $\{a_n\}$ be a sequence, and $0 \le a < b \le 1$. Let $N(n; a, b)$ be the number of integers $j \le n$ s.t. $a_j \in [a, b]$. A sequence $\{a_n\}$ of numbers in $[0, 1]$ is called uniformly distributed in $[0, 1]$ if

$$\lim_{n\to\infty} \frac{N(n; a, b)}{n} = b - a$$

for all a, b, s.t. $0 \le a < b \le 1$.

Proposition 11. *If s is a step function on $[0, 1]$, and $\{a_n\}$ is uniformly distributed in $[0, 1]$, then*

$$\int_0^1 s = \lim_{n\to\infty} \frac{s(a_1) + \cdots + s(a_n)}{n}.$$

Proof. Let $\Delta_1, \ldots, \Delta_m$ be a partition of $[0, 1]$ corresponding to the steps in s. Then (with a slight abuse of notation) we have

$$\begin{aligned}\int_0^1 s &= \sum_{i=1}^m s(\Delta_i)\Delta_i \\ &= \sum_{i=1}^m s(\Delta_i)\lim_{n\to\infty}\frac{1}{n}N(n;\Delta_i) \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^m s(\Delta_i)N(n;\Delta_i) \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n s(a_i). \qquad \square\end{aligned}$$

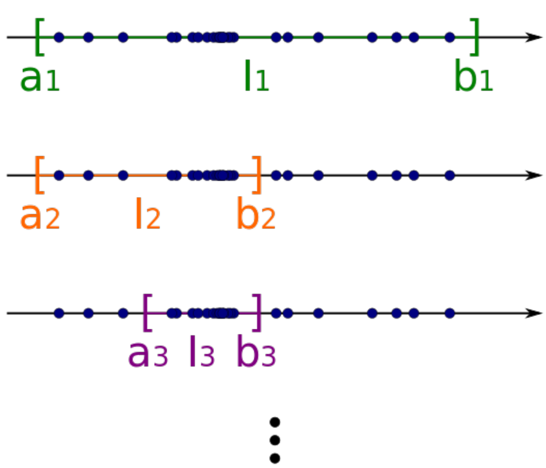
Proposition 12. *If f is integrable on $[0,1]$, and $\{a_n\}$ is uniformly distributed in $[0,1]$, then*

$$\int_0^1 f = \lim_{n\to\infty}\frac{f(a_1)+\cdots+f(a_n)}{n}.$$

Sketch. Since f is integrable, there is a step function s such that $\int_0^1 f$ is close to $\int_0^1 s$, which is close to $\frac{s(a_1)+\cdots+s(a_n)}{n}$, which is close to $\frac{f(a_1)+\cdots+f(a_n)}{n}$. $\square$

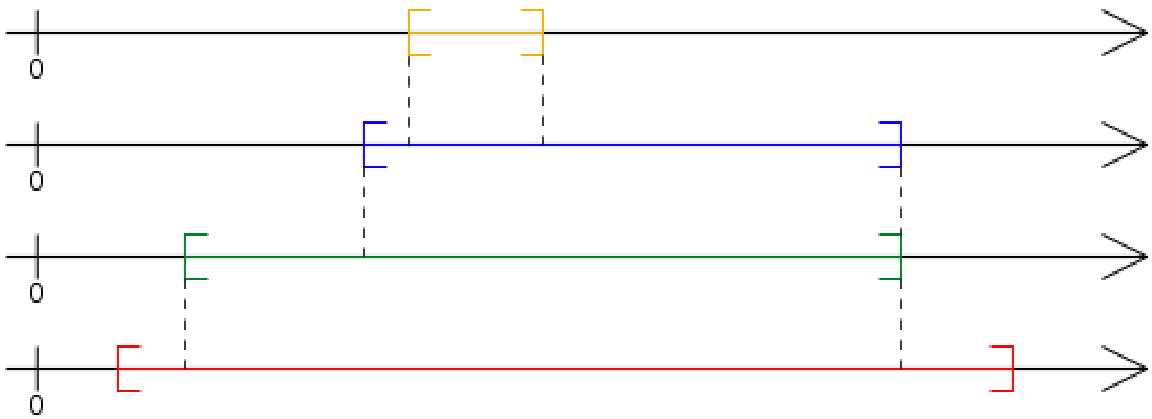
1 Bolzano-Weierstrass Theorem

Theorem 13 (Bolzano-Weierstrass Theorem)**.** *An infinite sequence contained in a closed interval I has a limit point in I.*



Proof uses the Nested Interval Theorem:

Theorem 14 (Nested Interval Theorem)**.** *The intersection of closed nested intervals is nonempty. If the interval lengths tend to zero, then the intersection is a point. Otherwise it's a closed interval.*



Definition 15. A function f defined on an interval I is called limitful if $\lim_{y\to a} f(y)$ exists for all $a \in I$.

Proposition 16. *Let f be a limitful function on $[0,1]$. Then for any $\epsilon > 0$ there are only finitely many points $a \in [0,1]$ with*

$$|\lim_{y\to a} f(y) - f(a)| > \epsilon.$$

Sketch 1. Suppose that there are infinitely many such points a. Then by the Bolzano-Weierstrass Theorem, these points have a limit $x \in [0,1]$. Let

$$L \stackrel{\text{def}}{=} \lim_{y\to x} f(y) = \lim_{a\to x} f(a).$$

The condition

$$|\lim_{y\to a} f(y) - f(a)| > \epsilon$$

means that for y close to a, $f(y)$ is far from $f(a)$. Similarly $\lim_{a\to x} f(a) = L$ means that for a close to x, $f(a)$ is close to L. Together this means that for y close to x and y close to a for some a, we have that $f(y)$ is far from L, but this contradicts the fact that $L = \lim_{y\to x} f(y)$, i.e. for all y close to x, $f(y)$ is close to L. $\square$

Sketch 2. Another way to see this is to let a_n be the convergent subsequence given by Bolzano-Weierstrass, and choose y_n close to a_n so that $|f(y_n) - f(a_n)|$ is big. Since $|f(a_n) - L|$ is small, the triangle inequality

$$|f(y_n) - L| \geq |f(y_n) - f(a_n)| - |f(a_n) - L|$$

says $|f(y_n) - L|$ is big, thus contradiction. □

Theorem 17. *A limitful function on* $[0, 1]$ *has at most countably many discontinuities.*

Proof. By the previous Proposition, for each $\epsilon_q > 0$ there are at most finitely many points a s.t.

$$|\lim_{y \to a} f(y) - f(a)| > \epsilon_q.$$

Taking a sequence $\epsilon_q \in \mathbf{Q}$ converging to zero, we have countably many such points a. □

Corollary 18. *If* f *has only removable discontinuities, then* f *is continuous except at countably many points. In particular,* f *cannot be discontinuous everywhere.*

2 Keywords

Uniform Limit Theorem, point-wise, uniform convergence, metric space, Cauchy criterion, Koch snowflake, Weierstrass function, uniformly distributed / equidistributed sequence, Bolzano-Weierstrass / Sequential Compactness Theorem, limitful function, removable discontinuity.

Part III
Infinite Series

Infinite growth of material consumption in a finite world is an impossibility.

E. F. Schumacher

References

[1] Spivak's Calculus.