

Trong

December 19, 2018

Contents

1	Orthogonal Matrix	2
2	Normal Matrix	4

1 Orthogonal Matrix

Definition 1. *A matrix Q is orthogonal iff its columns are orthonormal, IOW,*

$$Q^T Q = I.$$

Corollary 2. *If Q is an orthogonal matrix, then its inverse is its transpose.*

Corollary 3. *If Q is an orthogonal matrix, then*

$$\det Q = \pm 1.$$

Proof.

$$\begin{aligned} 1 &= \det I \\ &= \det(Q^T Q) \\ &= \det Q^T \det Q \\ &= \det Q \det Q \\ &= (\det Q)^2 \end{aligned}$$

Therefore $\det Q = \pm 1$. □

Note 4. This is why people like orthogonal matrices, because they're easy to invert.

Theorem 5. *If Q is orthogonal, then*

$$Q^T Q = Q Q^T = I.$$

Proof. Recall that $Q^T = Q^{-1}$, and inverses commute with each other, i.e.

$$Q^T Q = Q Q^T.$$
 □

Theorem 6. *If the columns of a square matrix are orthonormal, then its rows are also orthonormal, and vice versa.*

Problem 7 (6.11.). *Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.*

Solution. Find two other independent vectors by trial and error, then use Gram-Schmidt to orthogonalize the set, then normalize to get orthonormal set. \square

2 Normal Matrix

Theorem 8. *A matrix is normal iff it is unitarily diagonalizable.*

Exercise 9 (6.12.). *Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that*

$$\det A = \prod_{i=1}^n \lambda_i,$$

where the λ_i s are the [not necessarily distinct] eigenvalues of A .

Proof. Recall that a symmetric / normal matrix is diagonalizable, i.e. it is similar to a diagonal matrix.

IOW there are matrices P and P^* s.t.

$$PAP^* = D.$$

Then

$$\det A = \det(PAP^*) = \det D = \prod \lambda_i. \quad \square$$

Exercise 10 (6.13.). *Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B iff A and B are unitarily equivalent.*

Proof. The converse is true: if A and B are unitarily equivalent, then immediately they are similar. The forward direction is false: e.g. the two matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are similar, but they are not unitarily equivalent, because one is symmetric and the other is not. The conclusion follows from the following Proposition: \square

Proposition 11. *If A and B are orthogonally equivalent on a vector space over R , then either they are both symmetric or neither is. IOW, orthogonal equivalence preserves symmetry.*

Proof. Let $A = W^t B W = W^{-1} B W$ with $W^t = W^{-1}$. Suppose A is symmetric: $A = A^t$. Then

$$\begin{aligned} B &= W A W^{-1} \\ B^t &= (W A W^{-1})^t \\ &= (W^{-1})^t A^t W^t \\ &= W A W^{-1} \\ &= B. \end{aligned}$$

\square

Proposition 12. *If A and B are unitarily equivalent, then either they are both self-adjoint or neither*

is. IOW, unitary equivalence preserves self-adjointness.

Exercise 13 (Ex. 6.14. Unitary equivalence preserves positive semi/definiteness). *Prove that if A and B are Hermitian matrices and unitarily equivalent, then A is positive semi/definite iff B is.*

Proof. Recall that a Hermitian matrix is positive semi/definite iff all its eigenvalues are positive/non-negative; also recall that similar matrices have the same eigenvalues. Since A and B are unitarily equivalent, they are similar: there exists a matrix P with $P^{-1} = P^*$ s.t.

$$A = P^{-1}BP.$$

Therefore A and B have the same eigenvalues, so they are either both positive semi/definite or neither is. \square

Proposition 14 (Ex. 6.15.). *Let U be a unitary operator on an inner product space V , and let W*

be a finite-dimensional U -invariant subspace of V .
Then:

1. $U(W) = W$.
2. W^\perp is also U -invariant.

Proof. (1) follows immediately from the facts that U is invertible and W has finite dimension, and the Rank-Nullity Theorem. Note that this means W is also U^{-1} -invariant:

$$W = U^{-1}(W). \quad (*)$$

To prove (2), let $v \in W^\perp$; we want to show that $U(v) \in W^\perp$, i.e. that $U(v)$ is perpendicular to vectors in W . Let w be any vector in W . Then

$$\langle U(v), w \rangle = \langle v, U^*(w) \rangle = \langle v, U^{-1}(w) \rangle = \langle v, x \rangle$$

for some $x \in W$, since by $(*)$ we know that W is U^{-1} -invariant. Finally

$$\langle U(v), w \rangle = \langle v, x \rangle = 0$$

since $v \in W^\perp$ and $x \in W$, therefore $U(v)$ and w are perpendicular. \square