Linear Algebra II

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1 Orthogonal Matrix

Definition 1. *A matrix Q is orthogonal iff its columns are orthonormal, IOW,*

$$Q^T Q = I.$$

Corollary 2. *If Q is an orthogonal matrix, then its inverse is its transpose.*

Corollary 3. *If Q is an orthogonal matrix, then*

$$\det Q = \pm 1.$$

Proof.

$$\begin{aligned} 1 &= \det I \\ &= \det(Q^T Q) \\ &= \det Q^T \det Q \\ &= \det Q \det Q \\ &= (\det Q)^2 \end{aligned}$$

Therefore $\det Q = \pm 1$. $\square$

Note 4. This is why people like orthogonal matrices, because they're easy to invert.

Theorem 5. *If Q is orthogonal, then*

$$Q^T Q = Q Q^T = I.$$

Proof. Recall that $Q^T = Q^{-1}$, and inverses commute with each other, i.e.

$$Q^T Q = QQ^T. \qquad \square$$

Theorem 6. *If the columns of a square matrix are orthonormal, then its rows are also orthonormal, and vice versa.*

Problem 7 (Ex. 6.5.11.)**.** *Find an orthogonal matrix whose first row is* $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

Solution. Find two other independent vectors by trial and error, then use Gram-Schmidt to orthogonalize the set, then normalize to get orthonormal set. $\square$

Normal Matrix

Theorem 8. *A matrix is normal iff it is unitarily diagonalizable.*

Exercise 9 (Ex. 6.5.12.)**.** *Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that*

$$\det A = \prod_{i=1}^{n} \lambda_i,$$

where the λ_is are the [not necessarily distinct] eigenvalues of A.

Proof. Recall that a symmetric / normal matrix is diagonalizable, i.e. it is similar to a diagonal matrix. IOW there are matrices P and P^* s.t.

$$PAP^* = D.$$

Then

$$\det A = \det(PAP^*) = \det D = \prod \lambda_i. \qquad \square$$

Exercise 10 (Ex. 6.5.13.)**.** *Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B iff A and B are unitarily equivalent.*

Proof. The converse is true: if A and B are unitarily equivalent, then immediately they are similar. The forward direction is false: e.g. the two matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are similar, but they are not unitarily equivalent, because one is symmetric and the other is not. The conclusion follows from the following Proposition: □

Proposition 11. *If A and B are orthogonally equivalent on a vector space over R, then either they are both symmetric or neither is. IOW, orthogonal equivalence preserves symmetry.*

Proof. Let $A = W^t BW = W^{-1}BW$ with $W^t = W^{-1}$. Suppose A is symmetric: $A = A^t$. Then

$$\begin{aligned} B &= WAW^{-1} \\ B^t &= (WAW^{-1})^t \\ &= (W^{-1})^t A^t W^t \\ &= WAW^{-1} \\ &= B. \end{aligned}$$

□

Proposition 12. *If A and B are unitarily equivalent, then either they are both self-adjoint or neither is. IOW, unitary equivalence preserves self-adjoint-ness.*

Exercise 13 (Ex. 6.5.14. Unitary equivalence preserves positive semi/definiteness)**.** *Prove that if A and B are Hermitian matrices and unitarily equivalent, then A is positive semi/definite iff B is.*

Proof. Recall that a Hermitian matrix is positive semi/definite iff all its eigenvalues are positive/non-negative; also recall that similar matrices have the same eigenvalues. Since A and B are unitarily equivalent, they are similar: there exists a matrix P with $P^{-1} = P^*$ s.t.

$$A = P^{-1}BP.$$

Therefore A and B have the same eigenvalues, so they are either both positive semi/definite or neither is. □

Proposition 14 (Ex. 6.5.15.)**.** *Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Then:*

1. *$U(W) = W$.*
2. *$W^\perp$ is also U-invariant.*

Proof. (1) follows immediately from the facts that U is invertible and W has finite dimension, and the Rank-Nullity Theorem. Note that this means W is also U^{-1}-invariant:

$$W = U^{-1}(W). \tag{$*$}$$

To prove (2), let $v \in W^\perp$; we want to show that $U(v) \in W^\perp$, i.e. that $U(v)$ is perpendicular to vectors in W. Let w be any vector in W. Then

$$\langle U(v), w\rangle = \langle v, U^*(w)\rangle = \langle v, U^{-1}(w)\rangle = \langle v, x\rangle$$

for some $x \in W$, since by $(*)$ we know that W is U^{-1}-invariant. Finally

$$\langle U(v), w\rangle = \langle v, x\rangle = 0$$

since $v \in W^\perp$ and $x \in W$, therefore $U(v)$ and w are perpendicular. □

Proposition 15 (Ex. 6.5.21. Negative unitary equivalence test)**.** *Let A and B be unitarily equivalent $n \times n$ matrices. Then*

$$\mathrm{tr}(A^*A) = \mathrm{tr}(B^*B).$$

Note that this may alternatively be written as

$$\sum_{ij} |A_{ij}|^2 = \sum_{ij} |B_{ij}|^2.$$

Proof. Since A and B are unitarily equivalent, there is a unitary matrix P s.t. $A = P^*BP$ and $P^* = P^{-1}$. Then

$$\begin{aligned}
\mathrm{tr}(A^*A) &= \mathrm{tr}((P^*BP)^*(P^*BP)) \\
&= \mathrm{tr}(P^*B^*PP^*BP) \\
&= \mathrm{tr}(P^*B^*BP) \\
&= \mathrm{tr}(B^*BPP^*) && (*) \\
&= \mathrm{tr}(B^*B).
\end{aligned}$$

$(*)$ follows by the cyclic permutation property of the trace operator. □

This provides a negative test of unitary equivalence given two matrices. E.g. the following two matrices are not unitarily equivalent since their entries don't add up:

$$\begin{bmatrix} 1 & 2 \\ 2 & i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i & 4 \\ 1 & 1 \end{bmatrix}.$$