Linear Algebra

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Contents

1 **Theorem 2.23. Change of coordinates: conjugation by change of coordinate matrix.** 5

2 **Corollary 2.23. Representing a matrix in a different basis / change of coordinate matrix.** 5

3 **Intuition. Change of coordinates** 6

4 **Definition 2.23. Similar matrices** 6

5 **Theorem 2.3. Rank nullity theorem / Dimension theorem** 6

6 **Theorem 6.1. Properties of inner products** 6

7 **Exercise 5.4.25.** ***Simultaneously diagonalizable if*** $UT = TU$ 6

8 **Theorem.** ***If two operators agree on a basis, they are equal.*** 7

9 **Schur's Theorem 6.14. Splitting characteristic polynomial and orthonormal basis s.t.** $[T]_\beta$ **upper triangular.** 7

10 **Def. Normal operators** 7

11 **E.g. of normal operators: unitary, selfadjoint, and real symmetric operators** 7

12 **Theorem 6.15.** ***Eigenvectors corresponding to distinct eigenvalues of a normal operator are orthogonal*** **7**

13 Definition. Adjoint operators 8

14 Theorem. ***Normal operators are diagonalizable*** **8**

15 Theorem *6.3. ***An operator T is diagonalizable iff there exists a basis of V consisting of eigenvectors of T*** **8**

16 Theorem 6.3. Representing a vector as a linear combination of orthogonal vectors using inner product projections 8

17 Theorem 6.10. Matrix of the adjoint and adjoint of the matrix under orthonormal basis 9

18 Corollary 6.10. Matrix version 9

19 Theorem 6.16. Complex case: normal operator and orthonormal basis consisting of eigenvectors 9

20 Theorem 6.17. Real case: self-adjoint operator and orthonormal basis consisting of eigenvectors 10

21 Corollary 6.17. Normal / selfadjoint implies diagonalizable 10

22 Summary of normality V.S. diagonalizability 10

23 TODO. Example of diagonalizable operator that isn't normal/selfadjoint 11

24 Example of a complex symmetric matrix that isn't normal 11

25 Proposition. Eigenvectors and eigenvalues of the adjoint of a normal operator 11

26 Conjecture. ***The inner product is unique up to an orthonormal basis.*** **11**

27 Exercise 6.4.17. Positive semi/definite operator **12**

28 Ex 6.4.18. Derived positive semidefinite matrices **13**

29 Ex 6.4.19. Properties of positive definite operators **13**

30 Unitary and orthogonal operators and their matrices **14**

31 Example 6.18. Rotation in R^2. **14**

32 Lemma 6.18. T_0 is the only self-adjoint operator that is orthogonal to all its inputs **15**

33 Theorem 6.18. Characterizing unitary / orthogonal / isometric operators on a fin dim inner product space **15**

34 Corollary 6.18.0. ***The adjoint of a unitary / orthogonal operator is its inverse*** **17**

35 Proposition 6.18.0. T adjoint is T inverse iff T maps orthonormal basis to orthonormal basis. **17**

36 Proposition. Equality of two operators in an inner product **18**

37 Exercise 2.4.9. ***AB invertible implies A and B are invertible for square matrices A and B*** **18**

38 Exercise 2.4.10. ***One-sided inverse is a two-sided inverse*** **19**

39 Corollary 6.18. Selfadjoint and orthogonal iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1 **19**

40 Corollary 6.18.1. Unitary iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1 **20**

41 Lemma *6.18. Orthonormal basis of eigenvectors with eigenvalues of absolute value 1 implies T maps orthonormal basis to orthonormal basis **20**

42 Definition 6.18. Reflection about a line in R^2 21

43 Example 6.5.5. Matrix representation of a reflection in R^2 21

44 Definition 6.18.1. Orthogonal and unitary matrices 22

45 Corollary 6.18.1.1 Square matrix is unitary / orthogonal iff its rows and columns form orthonormal bases for F^n. 22

46 Corollary 6.18.2. Operator is unitary / orthogonal iff its matrix under orthonormal basis is unitary / orthogonal 22

47 Note 6.18.3. Unitary / orthogonal equivalence by conjugation: $A = Q^{-1}DQ$. 23

48 Definition 6.18.3. Unitary / orthogonal equivalence by conjugation 23

49 Ex 6.5.18. Unitary / orthogonal equivalence is an equivalence relation on $M_{n\times n}(C)$ and $M_{n\times n}(R)$. 23

50 The ideal state of mathematics: mechanical manipulation of symbols 24

51 Question. What is the link between normal operators and normal subgroups? 24

52 Theorem 6.19. Normal iff unitarily equivalent to a diagonal matrix. 24

53 Theorem 6.20. Real symmetric iff orthogonally equivalent to a diagonal matrix. 24

54 TODO. Is R normal the same as R selfadjoint/symmetric? 25

55 Example 6.5.6. Diagonalizing a symmetric matrix by an orthogonal matrix 25

56 Question. Suppose P is uni/orthog, and A is normal/selfadjoint. Is P^*AP always diagonal? 26

57 Schur's Theorem 6.21 26

58 Rigid motions 26

1 Theorem 2.23. Change of coordinates: conjugation by change of coordinate matrix.

Let TnV, β, γ be ordered bases for V. Suppose that $Q = I_\gamma^\beta$ is the change of coordinate matrix that changes γ coordinates to β coordinates. Then

$$[T]_\gamma = Q^{-1} [T]_\beta Q.$$

2 Corollary 2.23. Representing a matrix in a different basis / change of coordinate matrix.

Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.

Trivial example: $[L_A]_\beta = I^{-1}AI = A$, where β is the standard ordered basis for F^n.

Given a γ, we can define a map $\Gamma : M_{n\times n}(F) \longrightarrow M_{n\times n}(F)$ given by

$$\Gamma : A \longmapsto [L_A]_\gamma = Q^{-1}AQ.$$

What can we say about this map? Does it preserve properties of A and $M_{n\times n}(F)$? First of all, is this a linear transformation? Yes:

$$\Gamma(aA + B) = Q^{-1}(aA + B)Q = aQ^{-1}AQ + Q^{-1}BQ = a\Gamma(A) + \Gamma(B).$$

Note that Γ maps operator to operator, not vectors in V.

3 Intuition. Change of coordinates

Change of coordinates basically maps each vector in the original basis to a vector in the new basis. Each matrix in the original space V is mapped to a new vector in the same space V, but we should think of it really as a new space.

4 Definition 2.23. Similar matrices

Let A and B be matrices in $M_{n\times n}(F^n)$. We say that B is similar to A if there exists an invertible matrix Q s.t. $B = Q^{-1}AQ$.

5 Theorem 2.3. Rank nullity theorem / Dimension theorem

Let V be a finite dimensional vector space, and W be a (not necessarily finite dimensional) vector space over some field and let $T : V \longrightarrow W$ be a linear map. Then

$$\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \dim(V)$$

6 Theorem 6.1. Properties of inner products

Let V. If $\langle x, y\rangle = \langle x, z\rangle$ for all x, then $y = z$. Similarly $\langle y, x\rangle = \langle z, x\rangle$.

7 Exercise 5.4.25. *Simultaneously diagonalizable if $UT = TU$*

Proposition. *If T and U are diagonalizable linear operators on a finite-dimensional vector space V s.t. $UT = TU$, then T and U are simultaneously diagonalizable.*

8 Theorem. *If two operators agree on a basis, they are equal.*

9 Schur's Theorem 6.14. Splitting characteristic polynomial and orthonormal basis s.t. $[T]_\beta$ upper triangular.

Let TnV. Suppose that the characteristic polynomial of T splits. Then there exists an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

10 Def. Normal operators

Let $A : V \longrightarrow V$. Then A is normal iff it commutes with its adjoint: $AA^* = A^*A$.

11 E.g. of normal operators: unitary, selfadjoint, and real symmetric operators

Unitary operators are normal: $A^* = A^{-1}$, which commutes with A. Selfadjoint [and therefore real symmetric] operators are normal: $A^* = A$.

12 Theorem 6.15. *Eigenvectors corresponding to distinct eigenvalues of a normal operator are orthogonal*

Theorem. *Let T be normal on V, $\langle \cdot, \cdot \rangle$. Then eigenvectors corresponding to distinct eigenvalues of T are orthogonal.*

13 Definition. Adjoint operators

are also called Hermitian adjoint, Hermitian conjugate or Hermitian transpose.

Let $A : V \longrightarrow W$ be linear. Then the adjoint of A is the unique linear operator $A^* : W \longrightarrow V$ s.t.

$$\langle Av, w \rangle_W = \langle v, A^* w \rangle_V .$$

Existence and uniqueness to be proved.

14 Theorem. *Normal operators are diagonalizable*

15 Theorem *6.3. *An operator T is diagonalizable iff there exists a basis of V consisting of eigenvectors of T*

Corollary. If T is a selfadjoint operator, then there is a basis of V consisting of eigenvectors of T.

Proof. Follows from Theorem 6.16 or 6.17.

16 Theorem 6.3. Representing a vector as a linear combination of orthogonal vectors using inner product projections

Theorem. Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \mathrm{span}(S)$ then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{||v_i||^2} v_i .$$

Corollary. *Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal subset of V and $y \in \operatorname{span}(S)$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle \, v_i.$$

Corollary. *Let V be an inner product space, $y \in V$, and $\beta = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal basis for V. Then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle \, v_i.$$

17 Theorem 6.10. Matrix of the adjoint and adjoint of the matrix under orthonormal basis

Let $TVn\beta$ be orthonormal. Then $[T^*]_\beta = [T]_\beta^*$.

18 Corollary 6.10. Matrix version

Let A by an n by n matrix. Then $L_{A^*} = (L_A)^*$.

19 Theorem 6.16. Complex case: normal operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting of eigenvectors of T.

20 Theorem 6.17. Real case: self-adjoint operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint iff there exists an orthonormal basis for V consisting of eigenvectors of T.

Note that a real selfadjoint matrix is symmetric, $A^* = A^T = A$.

21 Corollary 6.17. Normal / selfadjoint implies diagonalizable

Let T be a linear operator on a finite-dimensional complex [real] inner product space V. If T is normal [self-adjoint] then T is diagonalizable.

Proof. In either case, V has an orthonormal basis consisting of eigenvectors of T. By Theorem *6.3, this happens iff T is diagonalizable. Oh, I already have this corollary as a corollary over there.

22 Summary of normality V.S. diagonalizability

We have

$$\begin{aligned}\text{Normal / selfadjoint} &\iff \text{Exists orthonormal eigenbasis}\\ &\implies \text{Exists eigenbasis}\\ &\iff \text{Diagonalizable.}\end{aligned}$$

and it seems that the two are not equivalent. QUESTION. Are there diagonalizable operators that aren't normal / selfadjoint? We just need to find one that has an eigenbasis that isn't orthonormal, How?

23 TODO. Example of diagonalizable operator that isn't normal/selfadjoint

24 Example of a complex symmetric matrix that isn't normal

Let

$$A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

Then A is symmetric complex, but isn't normal, because it is not diagonalizable [TODO. Show this]. If it were normal, then it would be diagonalizable by Corollary 6.17.

25 Proposition. Eigenvectors and eigenvalues of the adjoint of a normal operator

Proposition. *Let T be a normal linear operator on an inner product space V with eigenvalue λ and eigenvector x, then x is an eigenvector of T^* corresponding to eigenvalue $\overline{\lambda}$.*

26 Conjecture. *The inner product is unique up to an orthonormal basis.*

Specifically, let V be a finite-dimensional inner product on C or R, and β and γ be any orthonormal bases for V, and x, y be vectors in V. Then

$$\langle x, y \rangle = \overline{y}^T x = [y]_\beta \cdot [x]_\beta = [y]_\gamma \cdot [x]_\gamma.$$

In other words, the dot product of two vectors is the same in any orthonormal basis.

Proof. TODO.

Definition. *A linear operator T on a finite dimensional inner product space V is called positive definite if T is self-adjoint and $\langle T(x), x \rangle >$*

0 for all $x \neq 0$. It's called positive semidefinite if $\langle T(x), x\rangle \geq 0$ for all $x \neq 0$. Similarly for a square matrix A.

27 Exercise 6.4.17. Positive semi/definite operator

Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove:

1. *T is positive definite (semidefinite) iff all of its eigenvalues are positive (nonnegative).*

2. *T is positive definite iff*

$$\sum_{i,j} A_{ij} a_j \overline{a}_i > 0$$

   for all $(a_1, \ldots, a_n) \neq 0$. [What about semidefinite? Also true.]

3. *T is positive semidefinite iff $A = B^* B$ for some square matrix B.*

4. *If T and U are positive semidefinite operators s.t. $T^2 = U^2$, then $T = U$.*

5. *If T and U are positive definite (semidefinite?) operators s.t. $TU = UT$, then TU is positive definite (semidefinite?).*

6. *T is positive definite (semidefinite) iff A is.*

Proof 1. We'll show definite, semi is similar. Let T be positive definite, and let λ be an eigenvalue, and $x \neq 0$. Then

$$\begin{aligned} \langle T(x), x\rangle &> 0 \\ \langle \lambda x, x\rangle &> \\ \lambda \langle x, x\rangle &> \\ \lambda |x|^2 &> 0. \end{aligned}$$

Since $|x| > 0$, λ must also be > 0. Conversely, suppose that all eigenvalues of T are positive. Let v_i be the eigenvectors of T with corresponding eigenvalues λ_i. Then

$$T(v_i) = \lambda_i v_i.$$

Now let x be any vector, and $x = \sum a_i v_i$. Then

$$\langle T(x), x\rangle = \left\langle T\left(\sum a_i v_i\right), \sum a_i v_i\right\rangle = \left\langle \sum a_i \lambda_i v_i, \sum a_i v_i\right\rangle = \sum \lambda_i \left|v_i\right|^2$$

(because the v_i's are orthonormal).

Proof 2. First note that

$$\sum_{i,j} A_{ij} a_j \overline{a}_i = \overline{a}^T A a = \overline{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}} \left[A_{ij}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This is equal to

$$\langle T\left(x\right), x\rangle$$

since β is an orthonormal basis (recall that *the dot product of two vectors is the same in any orthonormal basis*).

28 Ex 6.4.18. Derived positive semidefinite matrices

Let $T : V \longrightarrow W$ be a linear transformation, where V and W are finite-dim. Then

1. T^*T and TT^* are positive semidefinite.

2. $\operatorname{rank}(T^*T) = \operatorname{rank}(TT^*) = \operatorname{rank}(T)$.

29 Ex 6.4.19. Properties of positive definite operators

Let T and U be positive definite operators on an inner product space V. Then

1. $T+U$ is positive definite.
2. If $c > 0$, then cT is p.d.
3. T^{-1} is p.d.

30 Unitary and orthogonal operators and their matrices

Definition. *Let T, n, $\langle\rangle, V, F$. If $||T(x)|| = ||x||$ for all x, we call T a unitary operator if $F = C$, and an orthogonal operator if $F = R$. T is also called an isometry, or length-preserving operator.*

31 Example 6.18. Rotation in R^2.

E.g. any rotation or reflection in R^2 preserves length and hence is an orthogonal operator. Rotation by θ given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Rotation by $-\theta$ is its inverse:

$$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T.$$

Since rotation by θ followed by rotation by $-\theta$ is the identity, we have

$$R_\theta^T R_\theta = R_\theta^{-1} R_\theta = I.$$

By Theorem 6.18 below, R_θ is orthogonal. By Theorem 6.18.b, it preserves the inner product and hence preserves the angle between two vectors. By Corollary 6.18.1, its rows and columns form orthonormal bases for R^2. Since $R_\theta \neq R_\theta^T$, it is not selfadjoint.

E.g. Recall the space H of continuous complex-valued functions defined on $[0, 2\pi]$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt.$$

Let $h \in H$ satisfy $|h(x)| = 1$ for all x. Define T on H by $T(f) = hf$. Then

$$||T(f)||^2 = ||hf||^2 = \frac{1}{2\pi}\int_0^{2\pi} h(t)f(t)\overline{h(t)f(t)}dt = ||f||^2$$

since $|h(t)|^2 = 1$. So T is a unitary operator.

32 Lemma 6.18. T_0 is the only self-adjoint operator that is orthogonal to all its inputs

Let U be self adjoint on $n, \langle\rangle, V$. If $\langle x, U(x)\rangle = 0$ for all x, then $U = T_0$, the zero operator.

Proof. By Theorem 6.16 or 6.17, there exists an orthonormal basis β for V consisting of eigenvectors of U. Let $x \in \beta$. Then $U(x) = \lambda x$ for some λ. Thus

$$0 = \langle x, U(x)\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle = \overline{\lambda}\,||x||^2,$$

and $\overline{\lambda} = 0$. Hence $U(x) = 0$ for all $x \in \beta$ and $U = T_0$.

Nonexample of a nonselfadjoint operator that has $\langle x, U(x)\rangle = 0$ but is not the zero op: the rotation U by 90 degrees in the plane.

33 Theorem 6.18. Characterizing unitary / orthogonal / isometric operators on a fin dim inner product space

Let T, n, $\langle\rangle, V, F$. Then the following statements are equivalent:

1. *$TT^* = T^*T = I$. In particular, T is normal and there exists an orthonormal basis for V consisting of eigenvectors of T.*
2. *$\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all x, y.*
3. *If β is an orthonormal basis, then $T(\beta)$ is an orthonormal basis.*
4. *There exists an orthonormal basis β s.t. $T(\beta)$ is an orthonormal basis.*

5. $||T(x)|| = ||x||$ *for all* x*, i.e.* T *is unitary / orthogonal.*

In other words, an operator is unitary / orthogonal iff it is normal and its "norm" TT^* is 1.

Proof (1) implies (2). For any x, y,

$$\langle x, y\rangle = \langle T^*T(x), y\rangle = \langle T(x), T(y)\rangle .$$

Proof (2) implies (3). Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Then $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ and

$$\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij},$$

so $T(\beta)$ is an orthonormal basis for V.

Proof (3) implies (4). [This one is a little odd?] Any orthonormal basis β satisfies this property, and there must be one because V is fin dim.

Proof (4) implies (5). Let $x \in V, \beta = \{v_1, \ldots, v_n\}$. Then

$$x = \sum_{i=1}^{n} a_i v_i$$

for some a_i, and

$$||x||^2 = \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle v_i, v_j\rangle = \sum_{i=1}^{n} |a_i|^2 .$$

Similarly,

$$||T(x)||^2 = \left\langle \sum_{i=1}^{n} a_i T(v_i), \sum_{i=1}^{n} a_i T(v_i) \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle T(v_i), T(v_j)\rangle = \sum_{i=1}^{n} |a_i|^2 ,$$

since $T(\beta)$ is also orthonormal.

Proof (5) implies (1). For any x,

$$\langle x, x\rangle = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle$$
$$\langle x, (I - T^*T)(x)\rangle = 0.$$

Let $U = I - T^*T$. Then U is self-adjoint and $\langle x, U(x)\rangle = 0$ for all x. By the previous lemma, $I - T^*T = U = T_0$ and $I = T^*T$. [Why does this imply that $TT^* = I$? The referenced Exercise 2.4.10 is about invertible matrices, not adjoint operators.... Ah, See next.]

34 Corollary 6.18.0. *The adjoint of a unitary / orthogonal operator is its inverse*

Proof. Suppose T is uni./orthog. Then $TT^* = I$, hence $T^* = T^{-1}$, by Exercise 2.4.10.

35 Proposition 6.18.0. T adjoint is T inverse iff T maps orthonormal basis to orthonormal basis.

Let $TnVF$, and let β be an orthonormal basis for V, and suppose T^{-1} exists. Then $T^* = T^{-1}$ iff $T(\beta)$ is also an orthonormal basis for V.

If that were true we can apply Exercise 2.4.10 and say that $TT^* = TT^{-1} = I$ and therefore $T^{-1}T = T^*T = I$.

Proof. Suppose $T^* = T^{-1}$. Then $TT^* = T^*T = I$ and (3) implies that $T(\beta)$ is an orthonormal basis. Conversely suppose that $T(\beta)$ is an orthonormal basis,

$$\beta = \{v_1, \dots, v_n\}, T(\beta) = \{T(v_1), \dots, T(v_n)\}.$$

To show that $T^* = T^{-1}$, we want to show that

$$\left\langle T^{-1}(x), y \right\rangle = \langle T^*(x), y\rangle = \langle x, T(y)\rangle$$

for all $x, y \in V$. It suffices to show that this holds for all $x \in T(\beta), y \in \beta$. [Why? This feels right, but it's not quite the result I'm thinking about.] There are two cases: either (1) $x = T(y)$ or (2) $x \neq T(y)$. In case (1),

$$\left\langle T^{-1}(x), y \right\rangle = \langle y, y\rangle = 1 = \langle x, T(y)\rangle .$$

In case (2),

$$\langle x, T(y)\rangle = 0 = \left\langle T^{-1}(x), y \right\rangle .$$

Therefore $T^* = T^{-1}$.

36 Proposition. Equality of two operators in an inner product

Let $T, n \langle\rangle V F$, and let β and $T(\beta)$ be orthonormal bases for V and suppose that T^{-1} exists. If

$$\left\langle T^{-1}(x), y\right\rangle = \langle x, T(y)\rangle$$

for all $y \in \beta, x \in T(\beta)$, then $\langle T^{-1}(x), y\rangle = \langle x, T(y)\rangle$ for all $x, y \in V$. In particular $\langle T^{-1}(x), y\rangle = \langle T^{*}(x), y\rangle$ for all x, y, and therefore $T^{-1} = T^{*}$.

Proof. Let $\beta = \{v_1, \ldots, v_n\}, T(\beta) = \{T(v_1), \ldots, T(v_n)\}$, and let

$$\begin{aligned} x &= \sum a_i T(v_i) \\ y &= \sum b_i v_i. \end{aligned}$$

Expanding $\langle x, T(y)\rangle$ and $\langle T^{-1}(x), y\rangle$ we get

$$\begin{aligned} \langle x, T(y)\rangle &= \left\langle \sum a_i T(v_i), T\left(\sum b_i v_i\right)\right\rangle \\ &= \left\langle \sum a_i T(v_i), \sum b_i T(v_i)\right\rangle \\ &= \sum a_i \overline{b_i} \\ \left\langle T^{-1}(x), y\right\rangle &= \left\langle T^{-1}\left(\sum a_i T(v_i)\right), \sum b_i v_i\right\rangle \\ &= \sum a_i \overline{b_i}. \end{aligned}$$

We should apply abstract results on concrete examples.

37 Exercise 2.4.9. *AB invertible implies A and B are invertible for square matrices A and B*

Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Proof. The columns of AB are of the form $[Ab_1 \cdots Ab_n]$ where b_i are the columns of B. Since AB is invertible, its columns are linearly independent.

By the rank-nullity theorem ($\dim N_A + \dim R_A = \dim V = n$), we have $\dim R_A = n$, so $\dim N_A = 0$, and T is invertible. This also means the b_i are linearly independent, so B is invertible.

38 Exercise 2.4.10. *One-sided inverse is a two-sided inverse*

Let A and B be $n \times n$ matrices s.t. $AB = I_n$. (a) Use previous to conclude that A and B are invertible. (b) Prove $A = B^{-1}$ and $B = A^{-1}$, i.e. for square matrices, a one-sided inverse is a two-sided inverse.

Proof. (a) By previous, A and B are invertible. (b) Multiply on the left by A^{-1}

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1} \\ B &= A^{-1}. \end{aligned}$$

Similarly for the other one.

39 Corollary 6.18. Selfadjoint and orthogonal iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1

Corollary. *Let $TnV\mathbf{R}\langle\rangle$. Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 iff T is both selfadjoint and orthogonal.*

Proof. ($\Longrightarrow$) Suppose $\beta = \{v_i\}$ is an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1. By Theorem 6.17 T is selfadjoint. Let $x = \sum a_i v_i$. We want to show T is orthogonal, i.e. $||T(x)|| = ||x||$:

$$||T(x)||^2 = \langle T(x), T(x)\rangle = \left\langle \sum a_i T(v_i), \sum a_i T(v_i)\right\rangle = \sum a_i^2 = ||x||$$

because the $T(v_i)$'s are orthonormal, thanks to a lemma we'll prove below.

($\Longleftarrow$) Suppose T is selfadjoint and orthogonal. By Theorem 6.17 V has an orthonormal basis $\beta = \{v_i\}$ of eigenvectors of T. WTS $|\lambda_i| = 1$. We have $T(v_i) = \lambda_i v_i$, so

$$||T(v_i)|| = \langle T(v_i), T(v_i) \rangle = \langle \lambda_i v_i, \lambda_i v_i \rangle = \lambda_i^2 \langle v_i, v_i \rangle = \lambda_i^2 ||v_i||$$
$$1 = \lambda_i^2,$$

because T is orthogonal. Therefore $|\lambda_i| = 1$. [NOTE. We could've written the previous equation using norms instead of inner products: $||T(v_i)|| = ||\lambda_i v_i|| = |\lambda_i| \, ||v|| $.]

40 Corollary 6.18.1. Unitary iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1

Corollary. *Let $TnV\mathbf{C}\langle\rangle$. Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 iff T is unitary.*

Proof similar to the real case.

41 Lemma *6.18. Orthonormal basis of eigenvectors with eigenvalues of absolute value 1 implies T maps orthonormal basis to orthonormal basis

Lemma. *Let $TnV\mathbf{R}\langle\rangle$. If V has an orthonormal basis β of eigenvectors with eigenvalues of absolute value 1, then $T(\beta)$ is also an orthonormal basis.*

Proof. Let $\beta = \{v_i\}$. Then

$$\langle T(v_i), T(v_j) \rangle = \langle \lambda_i v_i, \lambda_j v_j \rangle = \lambda_i \lambda_j \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Therefore the $T(v_i)$'s are orthonormal and form an orthonormal basis.

42 Definition 6.18. Reflection about a line in R^2

Let L be a one dimensional subspace of R^2. We may view L as a line in the plane through the origin. A linear operator T on R^2 is called a reflection of R^2 about L if $T(x) = x$ for all $x \in L$ and $T(x) = -x$ for all $x \in L^\perp$.

T is an orthogonal operator: let $v_1 \in L, v_2 \in L^\perp$ with length 1. Then $T(v_1) = v_1$ and $T(v_2) = -v_2$, thus v_i are eigenvectors with eigenvalues 1 and -1. By Corollary 6.18 T is orthogonal. We can also see that $\beta = \{v_i\}$ is an orthonormal basis for V, as is $T(\beta) = \{T(v_i)\}$.

43 Example 6.5.5. Matrix representation of a reflection in R^2

Let T be a reflection about a line through the origin in R^2, let β be the standard basis for R^2, and let $A = [T]_\beta$. Then $T = L_A$. Since [Corollary 6.18.2.] *T is an orthogonal operator and β is an orthonormal basis, A is an orthogonal matrix. We want to know what A looks like.*

Let α be the angle from the positive x-axis to L. Let $v_1 = (\cos\alpha, \sin\alpha)$ and $v_2 = (-\sin\alpha, \cos\alpha)$. Then $||v_1|| = ||v_2|| = 1, v_1 \in L, v_2 \in L^\perp$. Hence $\gamma = \{v_1, v_2\}$ is an orthonormal basis for R^2. Since $T(v_1) = v_1, T(v_2) = -v_2$, we have

$$[T]_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

be the change of coordinates matrix from the standard basis to γ. By Corollary 2.23,

$$\begin{aligned} A &= Q\,[T]_\gamma\,Q^{-1} \\ &= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}. \end{aligned}$$

44 Definition 6.18.1. Orthogonal and unitary matrices

Definition. *A square matrix A is called an orthogonal matrix if $A^tA = AA^t = I$ and unitary if $A^*A = AA^* = I$.*

45 Corollary 6.18.1.1 Square matrix is unitary / orthogonal iff its rows and columns form orthonormal bases for F^n.

$AA^* = I$ is equivalent to the statement that the rows of A form an orthonormal basis for F^n, because

$$AA^* = I = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} \overline{A_1^t} & \cdots & \overline{A_n^t} \end{bmatrix},$$

and so

$$\langle A_i, A_j \rangle = A_i \overline{A_j^t} = \delta_{ij}.$$

Similarly the condition $A^*A = I$ is equivalent to the statement that the columns of A form an orthonormal basis for F^n. Therefore a square matrix is orthogonal iff its rows and columns form orthonormal bases for F^n.

46 Corollary 6.18.2. Operator is unitary / orthogonal iff its matrix under orthonormal basis is unitary / orthogonal

Let TnV. By Theorem 6.10, T is unitary / orthogonal iff $[T]_\beta$ is unitary / orthogonal for some orthonormal basis β for V.

47 Note 6.18.3. Unitary / orthogonal equivalence by conjugation: $A = Q^{-1}DQ$.

For a complex normal [R selfadjoint/symmetric] matrix A, there exists an orrthonormal basis β consisting of eigenvectors of A [Theorem 6.17 and 6.18], so A is diagonalizable and is similar to a diagonal matrix D: $A = Q^{-1}DQ$, where Q is the matrix whose columns are the vectors in β [Theorem 2.23]. Since the columns of Q form an orthonormal basis, by Corollary 6.18.1 Q is unitary [orthogonal]. In this case, we say that A is unitarily / orthogonally equivalent to D.

48 Definition 6.18.3. Unitary / orthogonal equivalence by conjugation

A and B are unitarily / orthogonally equivalent iff there exists a unitary / orthogonal matrix P s.t. $A = P^*BP$. Since P is unitary/orthogonal, we know by Corollary 6.18.0 *that $P^* = P^{-1}$, then by Proposition 6.18.1 we also have $A = P^*BP = P^{-1}BP$.*

49 Ex 6.5.18. Unitary / orthogonal equivalence is an equivalence relation on $M_{n\times n}(C)$ and $M_{n\times n}(R)$.

Proof. We need to show reflexivity, symmetry, and transitivity. Reflexivity: A unitarily equivalent to B means $A = Q^{-1}BQ$, so $QAQ^{-1} = B$ and B u.eq. A. Symmetry: A u.eq. with itself since $A = I^{-1}AI$. Transitivity: A u.eq. B and B u.eq. C means $A = Q^{-1}BQ$ and $B = P^{-1}CP$, therefore

$$A = Q^{-1}P^{-1}CPQ = (PQ)^{-1}CPQ,$$

so A u.eq. C.

50 The ideal state of mathematics: mechanical manipulation of symbols

You want to develop mathematics to a stage where all you need to do is apply some mechanical rule and execute a rote calculation. Remove the need to think, and reduce mathematics to programming. That might never happen in full, but that's the end goal of any small corner of mathematics.

51 Question. What is the link between normal operators and normal subgroups?

52 Theorem 6.19. Normal iff unitarily equivalent to a diagonal matrix.

Let A be a complex $n \times n$ matrix. Then A is normal iff A is u.eq. to a diagonal matrix.

Proof. The forward direction is already proved in Note 6.18.3: if A is normal, then it is u.eq. to a diagonal matrix D. Conversely, suppose that A is u.eq. to a diagonal matrix D. Then there exists a unitary matrix P s.t. $A = P^*DP$.

$$AA^* = P^*DP(P^*DP)^* = P^*DPP^*D^*P = P^*DD^*P.$$

Similarly

$$A^*A = P^*D^*PP^*DP = P^*D^*DP = P^*DD^*P.$$

The last equality holds because D is diagonal, hence $D^*D = DD^*$. Therefore A is normal.

53 Theorem 6.20. Real symmetric iff orthogonally equivalent to a diagonal matrix.

Let A be a real $n \times n$ matrix. Then A is selfadjoint i.e. symmetric iff A is orthogonally equivalent to a diagonal matrix D.

Proof. The forward direction is already proved in Note 6.18.3. Conversely, suppose that A is ortho. eq. to a diagonal matrix D. Then there exists an orthogonal matrix P s.t. $A = P^T DP$. We want to show that A is symmetric:

$$A^T = (P^T DP)^T = P^T D^T P = P^T DP = A,$$

since D is diagonal.

54 TODO. Is R normal the same as R selfadjoint/symmetric?

In that case we can restate the last two theorems as simply that *A normal iff A un. eq. diagonal matrix.*

55 Example 6.5.6. Diagonalizing a symmetric matrix by an orthogonal matrix

Let

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Since A is symmetric, Theorem 6.20 says that A is orthog. eq. to a diagonal matrix. WTF orthogonal P and diagonal D s.t. $P^T AP = D$.

By Corollary 6.18.1.1, P is orthogonal iff its columns and rows form orthonormal bases for R^3. To find P, we find an orthonormal basis for V. It's easy to show that the eigenvalues of A are 2 and 8 (TODO. Find λ s.t. $\det(A - \lambda I) = 0$ by expanding the eq into a polynomial eq of degree 3 and solve.) Once we know the eigenvalues, we can find the eigenvectors by solving $(A - \lambda I)x = 0$ using Gaussian elimination. Two eigenvectors corresponding to 2 are $\{(-1, 1, 0), (-1, 0, 1)\}$. This set is not orthogonal, so we apply Gram-Schmidt to obtain the orthogonal set $\{(-1, 1, 0), (1, 1, -2)\}$. An eigenvector for $\lambda = 8$ is $(1, 1, 1)$. Note that it is orthogonal to the two eigenvectors corresponding to 2, by Theorem 6.15. Normalizing all 3, we get the orthonormal basis for R^3 consisting of eigenvectors of A

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2), \frac{1}{\sqrt{3}}(1.1,1) \right\}.$$

Thus one choice for P is

$$P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

56 Question. Suppose P is uni/orthog, and A is normal/selfadjoint. Is P^*AP always diagonal?

57 Schur's Theorem 6.21

Let $A \in M_{n\times n}(F)$ be a matrix whose characteristic polynomial splits over F. If $F = C$, then A is unitarily eq. to a complex upper triangular matrix. IF $F = R$, then A is orthogonally eq. to a real upper triangular matrix.

58 Rigid motions

Let VR. A function $f : V \longrightarrow V$ is called a rigid motion if

$$||f(x) - f(y)|| = ||x - y||$$

for all x, y in V.

E.g. Any orthogonal operator on a finite dimensional reall inner product space is a rigid motion, e.g. rotations, reflection by a line through the origin.