Linear Algebra

April 30, 2018

Contents

Part I
2017

1 Rank nullity theorem

Let V be a finite dimensional vector space, and W be a (not necessarily finite dimensional) vector space over some field and let $T : V \to W$ be a linear map. Then

$$\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \dim(V)$$

Part II
January 2018

2 Theorem 6.1. Properties of inner products

Let V $\langle\rangle$. If $\langle x, y\rangle = \langle x, z\rangle$ for all x, then $y = z$. Similarly $\langle y, x\rangle = \langle z, x\rangle$.

Part III
February

M19F18

3 Exercise 5.4.25. *Simultaneously diagonalizable if $UT = TU$*

Proposition. *If T and U are diagonalizable linear operators on a finite-dimensional vector space V s.t. $UT = TU$, then T and U are simultaneously diagonalizable.*

4 Theorem. *If two operators agree on a basis, they are equal.*

5 Theorem. *Eigenvectors corresponding to distinct eigenvalues of a normal operator are orthogonal*

Theorem. *Let T be normal on V, $\langle\cdot,\cdot\rangle$. Then eigenvectors corresponding to distinct eigenvalues of T are orthogonal.*

6 Theorem. *Normal operators are diagonalizable*

7 Theorem *6.3. *An operator T is diagonalizable iff there exists a basis of V consisting of eigenvectors of T*

Corollary. If T is a self-adjoint operator, then there is a basis of V consisting of eigenvectors of T.

Proof. Follows from Theorem 6.16 or 6.17.

8 Theorem 6.3

Theorem. Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \operatorname{span}(S)$ then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{||v_i||^2} v_i.$$

Corollary. *Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal subset of V and $y \in \operatorname{span}(S)$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle\, v_i.$$

Corollary. *Let V be an inner product space, $y \in V$, and $\beta = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal basis for V. Then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle\, v_i.$$

9 Theorem 6.16. Complex case: normal operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting of eigenvectors of T.

10 Theorem 6.17. Real case: self-adjoint operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint iff there exists an orthonormal basis for V consisting of eigenvectors of T.

Corollary 6.17. *Let T be a linear operator on a finite-dimensional complex [real] inner product space V. If T is normal [self-adjoint] then T is diagonalizable.*

Proof. In either case, V has an orthonormal basis consisting of eigenvectors of T. By Theorem *6.3, this happens iff T is diagonalizable. Oh, I already have this corollary as a corollary over there.

11 Proposition. Eigenvectors and eigenvalues of the adjoint of a normal operator

Proposition. *Let T be a normal linear operator on an inner product space V with eigenvalue λ and eigenvector x, then x is an eigenvector of T^* corresponding to eigenvalue $\overline{\lambda}$.*

Part IV
March

12 Conjecture. *The inner product is unique up to an orthonormal basis.*

Specifically, let V be a finite-dimensional inner product on C or R, and β and γ be any orthonormal bases for V, and x, y be vectors in V. Then

$$\langle x, y\rangle = \overline{y}^T x = [y]_\beta \cdot [x]_\beta = [y]_\gamma \cdot [x]_\gamma.$$

In other words, the dot product of two vectors is the same in any orthonormal basis.

Proof. TODO.

Definition. *A linear operator T on a finite dimensional inner product space V is called positive definite if T is self-adjoint and $\langle T(x), x\rangle > 0$ for all $x \neq 0$. It's called positive semidefinite if $\langle T(x), x\rangle \geq 0$ for all $x \neq 0$. Similarly for a square matrix A.*

13 Exercise 6.4.17. Positive semi/definite operator

Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove:

1. *T is positive definite (semidefinite) iff all of its eigenvalues are positive (nonnegative).*

2. *T is positive definite iff*

$$\sum_{i,j} A_{ij} a_j \overline{a}_i > 0$$

for all $(a_1, \ldots, a_n) \neq 0$. [What about semidefinite? Also true.]

3. *T is positive semidefinite iff $A = B^*B$ for some square matrix B.*

4. *If T and U are positive semidefinite operators s.t. $T^2 = U^2$, then $T = U$.*

5. *If T and U are positive definite (semidefinite?) operators s.t. $TU = UT$, then TU is positive definite (semidefinite?).*

6. *T is positive definite (semidefinite) iff A is.*

Proof 1. We'll show definite, semi is similar. Let T be positive definite, and let λ be an eigenvalue, and $x \neq 0$. Then

$$\begin{aligned} \langle T(x), x\rangle &> 0 \\ \langle \lambda x, x\rangle &> \\ \lambda \langle x, x\rangle &> \\ \lambda |x|^2 &> 0. \end{aligned}$$

Since $|x| > 0$, λ must also be > 0. Conversely, suppose that all eigenvalues of T are positive. Let v_i be the eigenvectors of T with corresponding eigenvalues λ_i. Then

$$T(v_i) = \lambda_i v_i.$$

Now let x be any vector, and $x = \sum a_i v_i$. Then

$$\langle T(x), x\rangle = \left\langle T\left(\sum a_i v_i\right), \sum a_i v_i\right\rangle = \left\langle \sum a_i \lambda_i v_i, \sum a_i v_i\right\rangle = \sum \lambda_i |v_i|^2$$

(because the v_i's are orthonormal).

Proof 2. First note that

$$\sum_{i,j} A_{ij} a_j \overline{a}_i = \overline{a}^T A a = \overline{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}} [A_{ij}] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This is equal to

$$\langle T(x), x\rangle$$

since β is an orthonormal basis (recall that *the dot product of two vectors is the same in any orthonormal basis*).

14 Ex 6.4.18. Derived positive semidefinite matrices

Let $T : V \longrightarrow W$ be a linear transformation, where V and W are finite-dim. Then

1. T^*T and TT^* are positive semidefinite.
2. $\text{rank}(T^*T) = \text{rank}(TT^*) = \text{rank}(T)$.

15 Ex 6.4.19. Properties of positive definite operators

Let T and U be positive definite operators on an inner product space V. Then

1. $T + U$ is positive definite.
2. If $c > 0$, then cT is p.d.
3. T^{-1} is p.d.

16 Unitary and orthogonal operators and their matrices

Definition. *Let T, n, $\langle\rangle, V, F$. If $||T(x)|| = ||x||$ for all x, we call T a unitary operator if $F = C$, and an orthogonal operator if $F = R$. T is also called an isometry, or length-preserving operator.*

17 Example 6.18. Rotation in R^2.

E.g. any rotation or reflection in R^2 preserves length and hence is an orthogonal operator. Rotation by θ given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Rotation by $-\theta$ is its inverse:

$$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T.$$

Since rotation by θ followed by rotation by $-\theta$ is the identity, we have

$$R_\theta^T R_\theta = R_\theta^{-1} R_\theta = I.$$

By Theorem 6.18 below, R_θ is orthogonal. By Theorem 6.18.b, it preserves the inner product and hence preserves the angle between two vectors. Since $R_\theta \neq R_\theta^T$, it is not selfadjoint.

E.g. Recall the space H of continuous complex-valued functions defined on $[0, 2\pi]$ with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt.$$

Let $h \in H$ satisfy $|h(x)| = 1$ for all x. Define T on H by $T(f) = hf$. Then

$$||T(f)||^2 = ||hf||^2 = \frac{1}{2\pi} \int_0^{2\pi} h(t)f(t)\overline{h(t)f(t)}dt = ||f||^2$$

since $|h(t)|^2 = 1$. So T is a unitary operator.

18 Lemma 6.18. T_0 is the only self-adjoint operator that is orthogonal to all its inputs

Let U be self adjoint on $n, \langle\rangle, V$. If $\langle x, U(x) \rangle = 0$ for all x, then $U = T_0$, the zero operator.

Proof. By Theorem 6.16 or 6.17, there exists an orthonormal basis β for V consisting of eigenvectors of U. Let $x \in \beta$. Then $U(x) = \lambda x$ for some λ. Thus

$$0 = \langle x, U(x) \rangle = \langle x, \lambda x \rangle = \overline{\lambda} \langle x, x \rangle = \overline{\lambda} \, ||x||^2,$$

and $\overline{\lambda} = 0$. Hence $U(x) = 0$ for all $x \in \beta$ and $U = T_0$.

Nonexample of a nonselfadjoint operator that has $\langle x, U(x) \rangle = 0$ but is not the zero op: the rotation U by 90 degrees in the plane.

19 Theorem 6.18. Characterizing unitary / orthogonal / isometric operators on a fin dim inner product space

Let T, n, $\langle\rangle$, V, F. Then the following statements are equivalent:

1. $TT^* = T^*T = I$. *In particular, T is normal and there exists an orthonormal basis for V consisting of eigenvectors of T.*

2. $\langle T(x), T(y)\rangle = \langle x, y\rangle$ *for all* x, y.

3. *If β is an orthonormal basis, then $T(\beta)$ is an orthonormal basis.*

4. *There exists an orthonormal basis β s.t. $T(\beta)$ is an orthonormal basis.*

5. $||T(x)|| = ||x||$ *for all x, i.e. T is unitary / orthogonal.*

In other words, an operator is unitary / orthogonal iff it is normal and its "norm" TT^* is 1.

Proof (1) implies (2). For any x, y,

$$\langle x, y\rangle = \langle T^*T(x), y\rangle = \langle T(x), T(y)\rangle .$$

Proof (2) implies (3). Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Then $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ and

$$\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij},$$

so $T(\beta)$ is an orthonormal basis for V.

Proof (3) implies (4). [This one is a little odd?] Any orthonormal basis β satisfies this property, and there must be one because V is fin dim.

Proof (4) implies (5). Let $x \in V, \beta = \{v_1, \ldots, v_n\}$. Then

$$x = \sum_{i=1}^{n} a_i v_i$$

for some a_i, and

$$||x||^2 = \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle v_i, v_j \rangle = \sum_{i=1}^{n} |a_i|^2 .$$

Similarly,

$$||T(x)||^2 = \left\langle \sum_{i=1}^{n} a_i T(v_i), \sum_{i=1}^{n} a_i T(v_i) \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle T(v_i), T(v_j) \rangle = \sum_{i=1}^{n} |a_i|^2 ,$$

since $T(\beta)$ is also orthonormal.

Proof (5) implies (1). For any x,

$$\langle x, x \rangle = \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle$$
$$\langle x, (I - T^*T)(x) \rangle = 0.$$

Let $U = I - T^*T$. Then U is self-adjoint and $\langle x, U(x) \rangle = 0$ for all x. By the previous lemma, $I - T^*T = U = T_0$ and $I = T^*T$. [Why does this imply that $TT^* = I$? The referenced Exercise 2.4.10 is about invertible matrices, not adjoint operators.... Ah, See next.]

20 Proposition *6.18. *The adjoint of a unitary / orthogonal operator is its inverse*

Let $T n \langle\rangle V F$, and let β be an orthonormal basis for V, and suppose T^{-1} exists. Then $T^* = T^{-1}$ iff $T(\beta)$ is also an orthonormal basis for V.

If that were true we can apply Exercise 2.4.10 and say that $TT^* = TT^{-1} = I$ and therefore $T^{-1}T = T^*T = I$.

Proof. Suppose $T^* = T^{-1}$. Then $TT^* = T^*T = I$ and (3) implies that $T(\beta)$ is an orthonormal basis. Conversely suppose that $T(\beta)$ is an orthonormal basis,

$$\beta = \{v_1, \ldots, v_n\}, T(\beta) = \{T(v_1), \ldots, T(v_n)\} .$$

To show that $T^* = T^{-1}$, we want to show that

$$\left\langle T^{-1}(x), y \right\rangle = \langle T^*(x), y \rangle = \langle x, T(y) \rangle$$

for all $x, y \in V$. It suffices to show that this holds for all $x \in T(\beta), y \in \beta$. [Why? This feels right, but it's not quite the result I'm thinking about.] There are two cases: either (1) $x = T(y)$ or (2) $x \neq T(y)$. In case (1),

$$\left\langle T^{-1}(x), y \right\rangle = \langle y, y \rangle = 1 = \langle x, T(y) \rangle .$$

In case (2),

$$\langle x, T(y) \rangle = 0 = \left\langle T^{-1}(x), y \right\rangle .$$

Therefore $T^* = T^{-1}$.

21 Proposition. Equality of two operators in an inner product

Let $T, n \langle\rangle VF$, and let β and $T(\beta)$ be orthonormal bases for V and suppose that T^{-1} exists. If

$$\left\langle T^{-1}(x), y \right\rangle = \langle x, T(y) \rangle$$

for all $y \in \beta, x \in T(\beta)$, then $\langle T^{-1}(x), y \rangle = \langle x, T(y) \rangle$ for all $x, y \in V$. In particular $\langle T^{-1}(x), y \rangle = \langle T^*(x), y \rangle$ for all x, y, and therefore $T^{-1} = T^*$.

Proof. Let $\beta = \{v_1, \ldots, v_n\}, T(\beta) = \{T(v_1), \ldots, T(v_n)\}$, and let

$$\begin{aligned} x &= \sum a_i T(v_i) \\ y &= \sum b_i v_i. \end{aligned}$$

Expanding $\langle x, T(y) \rangle$ and $\langle T^{-1}(x), y \rangle$ we get

$$\begin{aligned} \langle x, T(y) \rangle &= \left\langle \sum a_i T(v_i), T\left(\sum b_i v_i\right) \right\rangle \\ &= \left\langle \sum a_i T(v_i), \sum b_i T(v_i) \right\rangle \\ &= \sum a_i \overline{b_i} \\ \left\langle T^{-1}(x), y \right\rangle &= \left\langle T^{-1}\left(\sum a_i T(v_i)\right), \sum b_i v_i \right\rangle \\ &= \sum a_i \overline{b_i}. \end{aligned}$$

We should apply abstract results on concrete examples.

22 Exercise 2.4.9. *AB invertible implies A and B are invertible for square matrices A and B*

Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Proof. The columns of AB are of the form $[Ab_1 \cdots Ab_n]$ where b_i are the columns of B. Since AB is invertible, its columns are linearly independent. By the rank-nullity theorem ($\dim N_A + \dim R_A = \dim V = n$), we have $\dim R_A = n$, so $\dim N_A = 0$, and T is invertible. This also means the b_i are linearly independent, so B is invertible.

23 Exercise 2.4.10. *One-sided inverse is a two-sided inverse*

Let A and B be $n \times n$ matrices s.t. $AB = I_n$. (a) Use previous to conclude that A and B are invertible. (b) Prove $A = B^{-1}$ and $B = A^{-1}$, i.e. for square matrices, a one-sided inverse is a two-sided inverse.

Proof. (a) By previous, A and B are invertible. (b) Multiply on the left by A^{-1}

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1} \\ B &= A^{-1}. \end{aligned}$$

Similarly for the other one.

24 Corollary 6.18. Selfadjoint and orthogonal iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1

Corollary. *Let $TnV\mathbf{R}\langle\rangle$. Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 iff T is both selfadjoint and orthogonal.*

Proof. ($\Longrightarrow$) Suppose $\beta = \{v_i\}$ is an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1. By Theorem 6.17 T is selfadjoint. Let $x = \sum a_i v_i$. We want to show T is orthogonal, i.e. $||T(x)|| = ||x||$:

$$||T(x)||^2 = \langle T(x), T(x)\rangle = \left\langle \sum a_i T(v_i), \sum a_i T(v_i)\right\rangle = \sum a_i^2 = ||x||$$

because the $T(v_i)$'s are orthonormal, thanks to a lemma we'll prove below.

($\Longleftarrow$) Suppose T is selfadjoint and orthogonal. By Theorem 6.17 V has an orthonormal basis $\beta = \{v_i\}$ of eigenvectors of T. WTS $|\lambda_i| = 1$. We have $T(v_i) = \lambda_i v_i$, so

$$\begin{aligned} ||T(v_i)|| = \langle T(v_i), T(v_i)\rangle = \langle \lambda_i v_i, \lambda_i v_i\rangle = \lambda_i^2 \langle v_i, v_i\rangle &= \lambda_i^2 ||v_i|| \\ 1 &= \lambda_i^2, \end{aligned}$$

because T is orthogonal. Therefore $|\lambda_i| = 1$. [NOTE. We could've written the previous equation using norms instead of inner products: $||T(v_i)|| = ||\lambda_i v_i|| = |\lambda_i| \, ||v||$.]

25 Corollary 6.18.1. Unitary iff orthonormal basis of eigenvectors with eigenvalues of absolute value 1

Corollary. *Let $TnV\mathbf{C}\langle\rangle$. Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 iff T is unitary.*

Proof similar to the real case.

26 Lemma *6.18. Orthonormal basis of eigenvectors with eigenvalues of absolute value 1 implies T maps orthonormal basis to orthonormal basis

Lemma. *Let $TnV\mathbf{R}\langle\rangle$. If V has an orthonormal basis β of eigenvectors with eigenvalues of absolute value 1, then $T(\beta)$ is also an orthonormal basis.*

Proof. Let $\beta = \{v_i\}$. Then

$$\langle T(v_i), T(v_j)\rangle = \langle \lambda_i v_i, \lambda_j v_j\rangle = \lambda_i \lambda_j \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Therefore the $T(v_i)$'s are orthonormal and form an orthonormal basis.

27 Definition 6.18. Reflection about a line in R^2

Let L be a one dimensional subspace of R^2. We may view L as a line in the plane through the origin. A linear operator T on R^2 is called a reflection of R^2 about L if $T(x) = x$ for all $x \in L$ and $T(x) = -x$ for all $x \in L^\perp$.

T is an orthogonal operator: let $v_1 \in L, v_2 \in L^\perp$ with length 1. Then $T(v_1) = v_1$ and $T(v_2) = -v_2$, thus v_i are eigenvectors with eigenvalues 1 and -1. By Corollary 6.18 T is orthogonal. We can also see that $\beta = \{v_i\}$ is an orthonormal basis for V, as is $T(\beta) = \{T(v_i)\}$.

28 Definition 6.18.1. Orthogonal and unitary matrices

Definition. *A square matrix A is called an orthogonal matrix if $A^tA = AA^t = I$ and unitary if $A^*A = AA^* = I$.*