Linear Algebra

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Part I
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1 Eigenvectors of a normal operator

Proposition. *Let T be a normal linear operator on an inner product space V with eigenvalue λ and eigenvector x, then x is an eigenvector of T^* corresponding to eigenvalue $\overline{\lambda}$.*

Part II
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Theorem. *The inner product is unique up to an orthonormal basis. Specifically, let V be a finite-dimensional inner product on C or R, and β and γ be any orthonormal bases for V, and x, y be vectors in V. Then*

$$\langle x, y \rangle = \overline{y}^T x = [y]_\beta [x]_\beta = [y]_\gamma [x]_\gamma.$$

In other words, the dot product of two vectors is the same in any orthonormal basis.

Proof. TODO. Might need the fact that

$$w_i = \sum_{j=1}^{n} \langle w_i, v_j \rangle \, v_j$$

from Theorem 6.3 and cor in the book.

Definition. *A linear operator T on a finite dimensional inner product space V is called positive definite if T is self-adjoint and $\langle T(x), x\rangle > 0$ for all $x \neq 0$. It's called positive semidefinite if $\langle T(x), x\rangle \geq 0$ for all $x \neq 0$. Similarly for a square matrix A.*

Problem. *Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove:*

1. *T is positive definite (semidefinite) iff all of its eigenvalues are positive (nonnegative).*
2. *T is positive definite iff*
$$\sum_{i,j} A_{ij} a_j \overline{a}_i > 0$$
for all $(a_1, \ldots, a_n) \neq 0$. [What about semidefinite? Also true.]
3. *T is positive definite semidefinite iff $A = B^* B$ for some square matrix B.*
4. *If T and U are positive semidefinite operators s.t. $T^2 = U^2$, then $T = U$.*
5. *If T and U are positive definite operators s.t. $TU = UT$, then TU is positive definite.*
6. *T is positive definite (semidefinite) iff A is.*

Proof 1. We'll show definite, semi is similar. Let T be positive definite, and let λ be an eigenvalue, and $x \neq 0$. Then

$$\begin{aligned} \langle T(x), x\rangle &> 0 \\ \langle \lambda x, x\rangle &> \\ \lambda \langle x, x\rangle &> \\ \lambda |x|^2 &> 0. \end{aligned}$$

Since $|x| > 0$, λ must also be > 0. Conversely, suppose that all eigenvalues of T are positive. Let v_i be the eigenvectors of T with corresponding eigenvalues λ_i. Then

$$T(v_i) = \lambda_i v_i.$$

Now let x be any vector, and $x = \sum a_i v_i$. Then

$$\langle T(x), x\rangle = \left\langle T\left(\sum a_i v_i\right), \sum a_i v_i\right\rangle = \left\langle \sum a_i \lambda_i v_i, \sum a_i v_i\right\rangle = \sum \lambda_i \left|v_i\right|^2$$

(because the v_i's are orthonormal).

Proof 2. The result follows immediately once we realize that

$$\sum_{i,j} A_{ij} a_j \overline{a}_i = \overline{a}^T A a = \overline{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}} \left[A_{ij}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This is equal to

$$\langle T(x), x \rangle$$

since β is an orthonormal basis (recall that *the dot product of two vectors is the same in any orthonormal basis*).