Linear Algebra

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Part I
January

1 Theorem 6.1. Properties of inner products

Let V $\langle\rangle$. Then $\langle x, y\rangle = \langle x, z\rangle$ for all x, then $y = z$. Similarly $\langle y, x\rangle = \langle z, x\rangle$.

Part II
February

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2 Exercise 5.4.25. *Simultaneously diagonalizable if $UT = TU$*

Proposition. *If T and U are diagonalizable linear operators on a finite-dimensional vector space V s.t. $UT = TU$, then T and U are simultaneously diagonalizable.*

3 Theorem. *If two operators agree on a basis, they are equal.*

4 Theorem. *Eigenvectors correspondiong to distinct eigenvalues of a normal operator are orthogonal*

Theorem. *Let T be normal on V, $\langle\cdot,\cdot\rangle$. Then eigenvectors correspondiong to distinct eigenvalues of T are orthogonal.*

5 Theorem. *Normal operators are diagonalizable*

6 Theorem *6.3. *An operator T is diagonalizable iff there exists a basis of V consisting of eigenvectors of T*

Corollary. If T is a self-adjoint operator, then there is a basis of V consisting of eigenvectors of T.

Proof. Follows from Theorem 6.16 or 6.17.

7 Theorem 6.3

Theorem. Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \operatorname{span}(S)$ then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Corollary. *Let V be an inner product space and $S = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal subset of V and $y \in \operatorname{span}(S)$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle\, v_i.$$

Corollary. *Let V be an inner product space, $y \in V$, and $\beta = \{v_1, v_2 \ldots\ldots, v_k\}$ be an orthonormal basis for V. Then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle\, v_i.$$

8 Theorem 6.16. Complex case: normal operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting of eigenvectors of T.

9 Theorem 6.17. Real case: self-adjoint operator and orthonormal basis consisting of eigenvectors

Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint iff there exists an orthonormal basis for V consisting of eigenvectors of T.

Corollary 6.17. *Let T be a linear operator on a finite-dimensional complex [real] inner product space V. If T is normal [self-adjoint] then T is diagonalizable.*

Proof. In either case, V has an orthonormal basis consisting of eigenvectors of T. By Theorem *6.3, this happens iff T is diagonalizable. Oh, I already have this corollary as a corollary over there.

10 Eigenvectors of a normal operator

Proposition. *Let T be a normal linear operator on an inner product space V with eigenvalue λ and eigenvector x, then x is an eigenvector of T^* corresponding to eigenvalue $\overline{\lambda}$.*

Part III
March

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Theorem. *The inner product is unique up to an orthonormal basis. Specifically, let V be a finite-dimensional inner product on C or R, and β and γ be any orthonormal bases for V, and x, y be vectors in V. Then*

$$\langle x, y\rangle = \overline{y}^T x = [y]_\beta \cdot [x]_\beta = [y]_\gamma \cdot [x]_\gamma.$$

In other words, the dot product of two vectors is the same in any orthonormal basis.

Proof. TODO.

Definition. *A linear operator T on a finite dimensional inner product space V is called positive definite if T is self-adjoint and $\langle T(x), x\rangle > 0$ for all $x \neq 0$. It's called positive semidefinite if $\langle T(x), x\rangle \geq 0$ for all $x \neq 0$. Similarly for a square matrix A.*

12 Exercise 6.4.17. Positive semi/definite operator

Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove:

1. *T is positive definite (semidefinite) iff all of its eigenvalues are positive (nonnegative).*
2. *T is positive definite iff*
$$\sum_{i,j} A_{ij} a_j \overline{a}_i > 0$$
for all $(a_1, \ldots, a_n) \neq 0$. [What about semidefinite? Also true.]
3. *T is positive semidefinite iff $A = B^*B$ for some square matrix B.*
4. *If T and U are positive semidefinite operators s.t. $T^2 = U^2$, then $T = U$.*
5. *If T and U are positive definite (semidefinite?) operators s.t. $TU = UT$, then TU is positive definite (semidefinite?).*
6. *T is positive definite (semidefinite) iff A is.*

Proof 1. We'll show definite, semi is similar. Let T be positive definite, and let λ be an eigenvalue, and $x \neq 0$. Then

$$\begin{aligned} \langle T(x), x\rangle &> 0 \\ \langle \lambda x, x\rangle &> \\ \lambda \langle x, x\rangle &> \\ \lambda |x|^2 &> 0. \end{aligned}$$

Since $|x| > 0$, λ must also be > 0. Conversely, suppose that all eigenvalues of T are positive. Let v_i be the eigenvectors of T with corresponding eigenvalues λ_i. Then

$$T(v_i) = \lambda_i v_i.$$

Now let x be any vector, and $x = \sum a_i v_i$. Then

$$\langle T(x), x\rangle = \left\langle T\left(\sum a_i v_i\right), \sum a_i v_i\right\rangle = \left\langle \sum a_i \lambda_i v_i, \sum a_i v_i\right\rangle = \sum \lambda_i \left|v_i\right|^2$$

(because the v_i's are orthonormal).

Proof 2. First note that

$$\sum_{i,j} A_{ij} a_j \overline{a}_i = \overline{a}^T A a = \overline{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}} \left[A_{ij}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This is equal to

$$\langle T(x), x \rangle$$

since β is an orthonormal basis (recall that *the dot product of two vectors is the same in any orthonormal basis*).

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14 Ex 6.4.18

Let $T : V \longrightarrow W$ be a linear transformation, where V and W are finite-dim. Then

1. T^*T and TT^* are positive semidefinite.
2. $\text{rank}(T^*T) = \text{rank}(TT^*) = \text{rank}(T)$.

15 Ex 6.4.19

Let T and U be positive definite operators on an inner product space V. Then

1. $T + U$ is positive definite.
2. If $c > 0$, then cT is p.d.
3. T^{-1} is p.d.

16 18149 2592 600. Unitary and orthogonal operators and their matrices

Definition. *Let T, n, $\langle\rangle, V, F$. If $||T(x)|| = ||x||$ for all x, we call T a unitary operator if $F = C$, and an orthogonal operator if $F = R$. T is also called an isometry, or length-preserving operator.*

E.g. any rotation or reflection in R^2 preserves length and hence is an orthogonal operator.

E.g. Recall the space H of continuous complex-valued functions defined on $[0, 2\pi]$ with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} dt.$$

Let $h \in H$ satisfy $|h(x)| = 1$ for all x. Define T on H by $T(f) = hf$. Then

$$||T(f)||^2 = ||hf||^2 = \frac{1}{2\pi} \int_0^{2\pi} h(t)f(t)\overline{h(t)f(t)} dt = ||f||^2$$

since $|h(t)|^2 = 1$. So T is a unitary operator.

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18 Lemma 6.18

Let U be self adjoint on $n, \langle\rangle, V$. If $\langle x, U(x)\rangle = 0$ for all x, then $U = T_0$, the zero operator.

Proof. By Theorem 6.16 or 6.17, there exists an orthonormal basis β for V consisting of eigenvectors of U. Let $x \in \beta$. Then $U(x) = \lambda x$ for some λ. Thus

$$0 = \langle x, U(x)\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle = \overline{\lambda}\,||x||^2,$$

and $\overline{\lambda} = 0$. Hence $U(x) = 0$ for all $x \in \beta$ and $U = T_0$.

Nonexample of a non-self adjoint operator that has $\langle x, U(x)\rangle = 0$ but is not the zero op: U is the rotation by 90 degrees in the plane.

19 Theorem 6.18. Characterizing unitary / orthogonal / isometric operators on a fin dim inner product space

Let T, n, $\langle\rangle, V, F$. Then the following statements are equivalent:

1. *$TT^* = T^*T = I$. In particular, T is normal and there exists an orthonormal basis for V consisting of eigenvectors of T.*
2. *$\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all x, y.*
3. *If β is an orthonormal basis, then $T(\beta)$ is an orthonormal basis.*
4. *There exists an orthonormal basis β s.t. $T(\beta)$ is an orthonormal basis.*
5. *$||T(x)|| = ||x||$ for all x.*

In other words, an operator is unitary / orthogonal iff it is normal and its "norm" TT^* is 1.

Proof (1) implies (2). For any x, y,

$$\langle x, y\rangle = \langle T^*T(x), y\rangle = \langle T(x), T(y)\rangle.$$

Proof (2) implies (3). Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Then $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ and

$$\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij},$$

so $T(\beta)$ is an orthonormal basis for V.

Proof (3) implies (4). [This one is a little odd?] Any orthonormal basis β satisfies this property, and there must be one because V is fin dim.

Proof (4) implies (5). Let $x \in V, \beta = \{v_1, \ldots, v_n\}$. Then

$$x = \sum_{i=1}^{n} a_i v_i$$

for some a_i, and

$$||x||^2 = \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle v_i, v_j \rangle = \sum_{i=1}^{n} |a_i|^2 .$$

Similarly,

$$||T(x)||^2 = \left\langle \sum_{i=1}^{n} a_i T(v_i), \sum_{i=1}^{n} a_i T(v_i) \right\rangle = \sum_i \sum_j a_i \overline{a_j} \langle T(v_i), T(v_j) \rangle = \sum_{i=1}^{n} |a_i|^2 ,$$

since $T(\beta)$ is also orthonormal.

Proof (5) implies (1). For any x,

$$\langle x, x \rangle = \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle$$
$$\langle x, (I - T^*T)(x) \rangle = 0.$$

Let $U = I - T^*T$. Then U is self-adjoint and $\langle x, U(x) \rangle = 0$ for all x. By the previous lemma, $I - T^*T = U = T_0$ and $I = T^*T$. [Why does this imply that $TT^* = I$? The referenced Exercise 2.4.10 is about invertible matrices, not adjoint operators.... Ah, See next.]

20 Proposition. *The adjoint of a unitary / orthogonal operator is its inverse*

Let $T n \langle\rangle V F$, and let β be an orthonormal basis for V. Then $T^* = T^{-1}$ iff $T(\beta)$ is also an orthonormal basis for V.

If that were true we can apply Exercise 2.4.10 and say that $TT^* = TT^{-1} = I$ and therefore $T^{-1}T = T^*T = I$.

Proof. Suppose $T^* = T^{-1}$. Then $TT^* = T^*T = I$ and (3) implies that $T(\beta)$ is an orthonormal basis. Conversely suppose that $T(\beta)$ is an orthonormal basis,

$$\beta = \{v_1, \ldots, v_n\}, T(\beta) = \{T(v_1), \ldots, T(v_n)\}.$$

To show that $T^* = T^{-1}$, we want to show that

$$\left\langle T^{-1}(x), y \right\rangle = \langle T^*(x), y \rangle = \langle x, T(y) \rangle$$

for all $x, y \in V$. It suffices to show that this holds for all $x \in T(\beta), y \in \beta$. [Why? This feels right, but it's not quite the result I'm thinking about.] There are two cases: either (1) $x = T(y)$ or (2) $x \neq T(y)$. In case (1),

$$\left\langle T^{-1}(x), y \right\rangle = \langle y, y \rangle = 1 = \langle x, T(y) \rangle .$$

In case (2),

$$\langle x, T(y) \rangle = 0 = \left\langle T^{-1}(x), y \right\rangle .$$

Therefore $T^* = T^{-1}$.

21 Conjecture. Equality of two operators in an inner product

Let $T, n \langle\rangle V F$, and let β and $T(\beta)$ be orthonormal bases for V. If

$$\left\langle T^{-1}(x), y \right\rangle = \langle x, T(y) \rangle$$

for all $y \in \beta, x \in T(\beta)$, then $\left\langle T^{-1}(x), y \right\rangle = \langle x, T(y) \rangle$ for all $x, y \in V$. In particular $\left\langle T^{-1}(x), y \right\rangle = \langle T^{*}(x), y \rangle$ for all x, y, and therefore $T^{-1} = T^{*}$.

Proof. TODO.

We should apply abstract results on concrete examples.

22 Exercise 2.4.10. TODO