Analysis II

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December 1, 2018—December 2, 2018

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1 Differentiation

Definition 1 (Directional derivative)**.** Let $A \subset \mathbf{R}^m, f : A \longrightarrow \mathbf{R}^n$. Suppose A contains a neighbourhood of a. Given $u \in \mathbf{R}^m$ with $u \neq 0$, define the directional derivative of f at a in the direction of u to be

$$f'(a; u) = \lim_{t \to 0} \frac{f(a+u) - f(a)}{t}$$

if it exists.

Definition 2 (Derivative)**.** Let $A \subset \mathbf{R}^m, f : A \longrightarrow \mathbf{R}^n$. Suppose A contains a neighbourhood of a. We say that f is differentiable at a if there is an $n \times m$ matrix B s.t.

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \longrightarrow 0$$

as $h \longrightarrow 0$. The matrix B is unique and is denoted $Df(a)$. Sometimes people also call the derivative the gradient, and write $Df(a) = \nabla f(a)$.

Theorem 3 (Relating directional derivatives to the derivative of f)**.** *Let $A \subset \mathbf{R}^m, f : A \longrightarrow \mathbf{R}^n$. If f is differentiable at a, then all the directional derivatives of f at a exists, and*

$$f'(a; u) = Df(a) \cdot u.$$

Definition 4 (Partial derivative)**.** Let $A \subset \mathbf{R}^m, f : A \longrightarrow \mathbf{R}$. Define the j-th partial derivative of f at a to be directional derivative of f at a with respect to the vector e_j, provided it exists; and we denote it by $D_j f(a)$:

$$D_j f(a) = \lim_{t \to 0} \frac{f(a + te_j) - f(a)}{t}.$$

IOW, partial derivatives are directional derivatives along coordinate axes. Note that if we define $\phi(t) = f(a_1, \ldots, a_{j-1}, t, a_{j+1}, \ldots, a_m)$, then

$$D_j f(a) = \phi'(a_j).$$

Theorem 5 (Derivative of a real-valued function). *Let $A \subset \mathbf{R}^m$, $f : A \longrightarrow \mathbf{R}$. If f is differentiable at a, then the derivative of f is the row matrix*

$$Df(a) = [D_1 f(a) \quad \cdots \quad D_m f(a)].$$

Theorem 6. *Let $A \subset \mathbf{R}^m$, $f : A \longrightarrow \mathbf{R}^n$. Suppose A contains a neighbourhood of a. Let $f_i : A \longrightarrow R$ be the i-th component function of f, so that*

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

- *Then f is differentiable at a iff each component f_i is differentiable at a.*

- *If f is differentiable at a, then its derivative is the $n \times m$ matrix whose i-th row is the derivative of f_i, i.e.*

$$Df(a) = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_n(a) \end{bmatrix} = \begin{bmatrix} D_1 f_1(a) & \cdots & D_m f_1(a) \\ \vdots & & \vdots \\ D_1 f_n(a) & \cdots & D_m f_n(a) \end{bmatrix}.$$

This matrix is called the Jacobian matrix of f.

Roughly: Differentiability of $f : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ is equivalent to differentiability of each component, because the components are independent of each other as far as taking limits is concerned. Note that this doesn't imply that the partial derivatives of the components must be continuous, only that they exist.

2 Continuously differentiable functions

Theorem 7 (Mean value theorem)**.** *If $\phi : [a,b] \longrightarrow \mathbf{R}$ is continuous at each point of the closed interval $[a,b]$, and differentiable at each point of the interval (a,b), then there exists a point c of (a,b) s.t.*

$$\phi(b) - \phi(a) = \phi'(c)(b-a).$$

Theorem 8 (Continuously differentiable functions)**.** *Let A be open in $\mathbf{R}^m$. Suppose that the partial derivatives $D_j f_i(x)$ of the component functions of f exist at each point $x \in A$ and are continuous on A. Then f is differentiable at each point of A.*

This theorem guarantees differentiability of f if its partial derivatives exist and are continuous. Such a function is called continuously differentiable, or C^1 on A.

Theorem 9. *Let A be open in $\mathbf{R}^m, f : A \longrightarrow \mathbf{R}$ be a function of class C^2. Then for each $a \in A$, the mixed second* *order partial derivatives are equal:*

$$D_k D_j f(a) = D_j D_k f(a).$$