

Linear Algebra

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Be wary of gorgeous view.

Dark Souls

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Part I

Linear Transformations and Matrices

Just imagine \mathbf{R}^n , then let $n = 14$.

Well known math joke

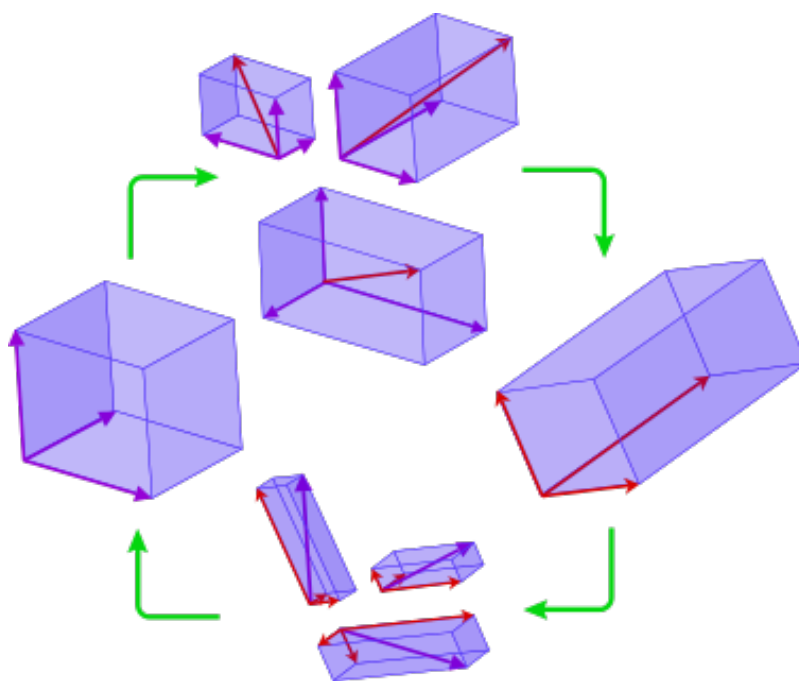
1 Special Matrix Functions

Proposition 1. *If A and B are square matrices, then $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr } A = \text{tr } A^t$.*

2 Change of Coordinate Matrix

Proposition 2. *Let B be an $n \times n$ invertible matrix. Then the map $\Phi_B : M_{n \times n}(F) \longrightarrow M_{n \times n}(F)$ defined by $\Phi_B(A) = B^{-1}AB$ is an isomorphism.*

Proposition 3. *For any invertible matrix B there exist bases β, γ s.t. $B = [I]_{\gamma}^{\beta}$, i.e. every invertible matrix is a change of coordinates matrix.*



Keywords

Change of coordinates matrix, basis representation, trace.

Part II

Eigenvectors and Eigenvalues

Eigenvectors: $E_m(K)$

The Big Bang Theory

Proposition 4. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof. Easy to show for two eigenvectors, then use induction. □

Theorem 5. *A linear operator T on a finite-dimensional vector space V is diagonalizable iff there exists an ordered basis β for V consisting of eigenvectors of T . Furthermore, if T is diagonalizable, $\beta = \{v_1, \dots, v_n\}$ is an ordered basis of eigenvectors of T , and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.*

Proof in the book.

Corollary 6. *A matrix A is diagonalizable iff the dimensions of its eigenspaces—i.e. the geometric multiplicities over all its eigenvalues—add up to the size of A . In this case the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.*

Proof. By Proposition 4, the eigenspaces are linearly independent. If their dimensions added up to less than n , we'd have too few eigenvectors to make a basis for V , and by Theorem 5, A would not be diagonalizable. If they added up to more than n , we'd have too many linearly independent vectors in V . Conversely if they do add up to n , then the union of bases for the eigenspaces forms an eigenbasis for V , and A is diagonalizable. \square

Proposition 7. *Let T be a linear operator on a finite-dimensional vector space V , and let β be an ordered basis for V . Then λ is an eigenvalue of T iff it is an eigenvalue of $[T]_\beta$.*

Corollary 8. *Similar matrices have the same eigenvalues, but not necessarily the same eigenvectors.*

Proposition 9. *If v is an eigenvector of A corresponding to eigenvalue λ , and B is similar to A under change of coordinates matrix Q , then Qv is an eigenvector of B corresponding to the same eigenvalue λ . Another way of saying this is that change of coordinates preserves eigenvalues and eigenvectors.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned} Av &= Q^{-1}BQv \\ QA v &= BQv \\ \lambda Qv &= BQv, \end{aligned}$$

so Qv is an eigenvector of B corresponding to λ . \square

Definition 10. *Let T be a linear operator on a finite-dimensional vector space V . Define the determinant of T to be $\det T = \det([T]_\beta)$ for any ordered basis β for V .*

Note 11. Since the determinant is multiplicative, for any two bases β and α we have

$$\begin{aligned}\det([T]_\beta) &= \det(Q^{-1}[T]_\alpha Q) \\ &= \det Q^{-1} \det([T]_\alpha) \det Q \\ &= \det([T]_\alpha),\end{aligned}$$

where $Q = [I]_\beta^\alpha$ is the change of basis matrix from β to α , and therefore $\det T$ is well defined, i.e. it's independent of the choice of basis.

Proposition 12. *Representation of a matrix with respect to a basis is a linear operation, i.e.*

$$[T + \lambda U]_\beta = [T]_\beta + \lambda[U]_\beta.$$

In fact it's an isomorphism. For a fixed basis β this transformation is usually written $\Phi_\beta : \mathcal{L}(V) \longrightarrow M_{n \times n}(F)$. See Proposition 2.

Proof. Let v be a vector, $\beta = \{v_1, \dots, v_n\}$, and $T(v) = \sum a_i v_i$, $U(v) = \sum b_i v_i$. We want to show that

$$\begin{aligned}[T + \lambda U]_\beta[v]_\beta &= ([T]_\beta + \lambda[U]_\beta)[v]_\beta \\ [T(v) + \lambda U(v)]_\beta &= [T(v)]_\beta + \lambda[U(v)]_\beta.\end{aligned}$$

The RHS is $[a_i] + \lambda[b_i]$, which is the same as the LHS: $[a_i + \lambda b_i]$. □

Note 13. Analogously, the standard representation of a vector space V with respect to a basis β is $\phi_\beta : V \longrightarrow F^n$. And it's also an isomorphism.

Proposition 14. *For any scalar λ and any ordered basis β for V , $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$.*

Proof. Follows from Definition 10 and Proposition 12. □

Proposition 15 (Eigenvalues and invertibility). *A linear operator T on a finite-dimensional vector space is invertible iff its eigenvalues are nonzero.*

Proof. If zero is an eigenvalue of T , then $\det(T - 0 \cdot I) = 0$, and T is not invertible. Conversely, if T is not invertible, then there exists a nonzero vector v s.t. $T(v) = 0 = 0 \cdot v$, and so 0 is an eigenvalue and v is an eigenvector of T . □

Proposition 16. *Let T be an invertible linear operator. Then a scalar λ is an eigenvalue of T iff λ^{-1} is an eigenvalue of T^{-1} . Note that by the previous proposition λ is nonzero, so λ^{-1} exists.*

Proof. Apply T^{-1} to both sides of $T(v) = \lambda v$. □

Proposition 17. *The eigenvalues of an upper triangular matrix M are the diagonal entries of M .*

Proof. Follows from the fact that the determinant of an upper triangular matrix is the product of its diagonal entries. □

Proposition 18. *Similar matrices have the same characteristic polynomial.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned}\det(A - tI) &= \det(Q^{-1}BQ - Q^{-1}tIQ) \\ &= \det(Q^{-1}(B - tI)Q) \\ &= \det(Q^{-1}) \det(B - tI) \det(Q) \\ &= \det(B - tI).\end{aligned}$$
□

Corollary 19. *The definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V .*

Proof. Follows immediately from the fact that similar matrices are the same linear operator expressed under different bases. □

Proposition 20. *Let T be a linear operator on a finite dimensional vector space V over a field F . Let β be an ordered basis for V , and let $A = [T]_{\beta}$. Then a vector $v \in V$ is an eigenvector of T corresponding to λ iff $\phi_{\beta}(v)$ is an eigenvector of A corresponding to λ .*

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \phi_{\beta} \downarrow & & \downarrow \phi_{\beta} \\ F^n & \xrightarrow{L_A} & F^n \end{array}$$

Proof. TODO. □

3 Eigenvectors and Some Special Functions and Matrices

Lemma 21. *A square matrix has the same determinant as its transpose.*

Proposition 22. *A square matrix has the same characteristic polynomial as its transpose.*

Proposition 23. *If x is an eigenvector of T corresponding to λ , then for any positive integer m , x is an eigenvector of T^m corresponding to λ^m .*

Proof. Linearity of T and induction. □

Note 24. The same holds for matrices. It's always the same!

Proposition 25. *Similar matrices have the same trace.*

Proof. Follows from Proposition 1. □

Corollary 26. *Define the trace of a linear operator T on a finite dimensional vector space as $\text{tr}[T]_\beta$ for any basis β . This is well defined by the previous Proposition.*

Proposition 27. *Let T be the linear operator on $M_{n \times n}(\mathbf{R})$ defined by $T(A) = A^t$. Then ± 1 are the only eigenvalues of T . The eigenvectors of T corresponding to ± 1 are symmetric and antisymmetric matrices, respectively.*

Example 28. In two dimensions, an ordered basis for $M_{2 \times 2}(\mathbf{R})$ consisting of eigenvectors of T so that $[T]_\beta$ is a diagonal matrix is

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

The first three matrices are symmetric and correspond to the eigenvalue $+1$, while the last is skew-symmetric and corresponds to -1 .

Example 29. More generally, in n dimensions an eigenbasis consists of symmetric matrices of the form A that has zeroes everywhere except a single 1 along the diagonal, and symmetric matrices of the form B with zeroes everywhere except a 1 in two opposite entries B_{ij} and B_{ji} , and finally anti-symmetric matrices of the form C with zeroes everywhere except a -1 and a $+1$ in two opposite entries C_{ij} and C_{ji} , where the -1 is in the lower left half and the $+1$ in the upper right half of C . There are n matrices of type A , $(n^2 - n)/2$ each of type B and C , for a total of n^2 , as expected.

3.1 Scalar Matrices

Definition 30. A scalar matrix is a square matrix of the form λI for some scalar λ .

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \equiv \lambda \mathbf{I}_3$$

Proposition 31. If a square matrix A is similar to a scalar matrix λI , then $A = \lambda I$.

Proof. If A is similar to λI , that means there is an invertible matrix Q s.t. $A = Q^{-1}\lambda I Q = \lambda I$. \square

Proposition 32. A diagonalizable matrix A having only one eigenvalue is a scalar matrix.

Proof. If A is diagonalizable, this means A is similar to a diagonal matrix, whose diagonal entries are its eigenvalues. Since A only has one eigenvalue λ , the diagonal entries are all equal to λ . \square

Example 33. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Keywords

Eigenvectors, eigenvalues, eigenspace, eigenbasis, differential operator, eigenfunctions, algebraic and geometric multiplicities, scalar matrix, characteristic polynomial, skew / anti-symmetric matrix.

References

- [1] Linear Algebra by Friedberg, Insel, and Spence.