Linear Algebra

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Be wary of gorgeous view.

Dark Souls

Contents

Part I
Linear Transformations and Matrices

Just imagine $\mathbf{R}^n$, then let $n = 14$.

Well known math joke

1 Special Matrix Functions

Proposition 1. *If A and B are square matrices, then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *and* $\operatorname{tr} A = \operatorname{tr} A^t$.

2 Change of Coordinate Matrix

Proposition 2. *Let B be an $n \times n$ invertible matrix. Then the map $\Phi_B : M_{n\times n}(F) \longrightarrow M_{n\times n}(F)$ defined by $\Phi_B(A) = B^{-1}AB$ is an isomorphism.*

Proposition 3. *For any invertible matrix B there exist bases β, γ s.t. $B = [I]_\gamma^\beta$, i.e. every invertible matrix is a change of coordinates matrix.*

Keywords

Change of coordinates matrix, basis representation, trace.

Part II
Eigenvectors and Eigenvalues

Eigervalves: $E_m(K)$

The Big Bang Theory

Proposition 4. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof. Easy to show for two eigenvectors, then use induction. □

Theorem 5. *A linear operator T on a finite-dimensional vector space V is diagonalizable iff there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.*

Proof in the book.

Corollary 6. *A matrix A is diagonalizable iff the dimensions of its eigenspaces—i.e. the geometric multiplicities over all its eigenvalues—add up to the size of A. In this case the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.*

Proof. By Proposition 4, the eigenspaces are linearly independent. If their dimensions added up to less than n, we'd have too few eigenvectors to make a basis for V, and by Theorem 5, A would not be diagonalizable. If they added up to more than n, we'd have too many linearly independent vectors in V. Conversely if they do add up to n, then the union of bases for the eigenspaces forms an eigenbasis for V, and A is diagonalizable. □

Proposition 7. *Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Then λ is an eigenvalue of T iff it is an eigenvalue of $[T]_\beta$.*

Corollary 8. *Similar matrices have the same eigenvalues, but not necessarily the same eigenvectors.*

Proposition 9. *If v is an eigenvector of A corresponding to eigenvalue λ, and B is similar to A under change of coordinates matrix Q, then Qv is an eigenvector of B corresponding to the same eigenvalue λ. Another way of saying this is that change of coordinates preserves eigenvalues and eigenvectors.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned} Av &= Q^{-1}BQv \\ QAv &= BQv \\ \lambda Qv &= BQv, \end{aligned}$$

so Qv is an eigenvector of B corresponding to λ. □

Definition 10. *Let T be a linear operator on a finite-dimensional vector space V. Define the determinant of T to be* $\det T = \det([T]_\beta)$ *for any ordered basis β for V.*

Note 11. Since the determinant is multiplicative, for any two bases β and α we have

$$\begin{aligned}\det([T]_\beta) &= \det(Q^{-1}[T]_\alpha Q)\\ &= \det Q^{-1}\det([T]_\alpha)\det Q\\ &= \det([T]_\alpha),\end{aligned}$$

where $Q = [I]_\beta^\alpha$ is the change of basis matrix from β to α, and therefore $\det T$ is well defined, i.e. it's independent of the choice of basis.

Proposition 12. *Representation of a matrix with respect to a basis is a linear operation, i.e.*

$$[T + \lambda U]_\beta = [T]_\beta + \lambda[U]_\beta.$$

In fact it's an isomorphism. For a fixed basis β this transformation is usually written $\Phi_\beta : \mathcal{L}(V) \longrightarrow M_{n\times n}(F)$. See Proposition 2.

Proof. Let v be a vector, $\beta = \{v_1, \ldots, v_n\}$, and $T(v) = \sum a_i v_i, U(v) = \sum b_i v_i$. We want to show that

$$\begin{aligned}[T + \lambda U]_\beta[v]_\beta &= ([T]_\beta + \lambda[U]_\beta)[v]_\beta\\ [T(v) + \lambda U(v)]_\beta &= [T(v)]_\beta + \lambda[U(v)]_\beta.\end{aligned}$$

The RHS is $[a_i] + \lambda[b_i]$, which is the same as the LHS: $[a_i + \lambda b_i]$. □

Note 13. Analogously, the standard representation of a vector space V with respect to a basis β is $\phi_\beta : V \longrightarrow F^n$. And it's also an isomorphism.

Proposition 14. *For any scalar λ and any ordered basis β for V,* $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$.

Proof. Follows from Definition 10 and Proposition 12. □

Proposition 15 (Eigenvalues and invertibility)**.** *A linear operator T on a finite-dimensional vector space is invertible iff its eigenvalues are nonzero.*

Proof. If zero is an eigenvalue of T, then $\det(T - 0 \cdot I) = 0$, and T is not invertible. Conversely, if T is not invertible, then there exists a nonzero vector v s.t. $T(v) = 0 = 0 \cdot v$, and so 0 is an eigenvalue and v is an eigenvector of T. □

Proposition 16. *Let T be an invertible linear operator. Then a scalar λ is an eigenvalue of T iff λ^{-1} is an eigenvalue of T^{-1}. Note that by the previous proposition λ is nonzero, so λ^{-1} exists.*

Proof. Apply T^{-1} to both sides of $T(v) = \lambda v$. □

Proposition 17. *The eigenvalues of an upper triangular matrix M are the diagonal entries of M.*

Proof. Follows from the fact that the determinant of an upper triangular matrix is the product of its diagonal entries. □

Proposition 18. *Similar matrices have the same characteristic polynomial.*

Proof. Let $A = Q^{-1}BQ$. Then

$$\begin{aligned}\det(A - tI) &= \det(Q^{-1}BQ - Q^{-1}tIQ) \\ &= \det(Q^{-1}(B - tI)Q) \\ &= \det(Q^{-1})\det(B - tI)\det(Q) \\ &= \det(B - tI).\end{aligned}$$

□

Corollary 19. *The definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.*

Proof. Follows immediately from the fact that similar matrices are the same linear operator expressed under different bases. □

Proposition 20. *Let T be a linear operator on a finite dimensional vector space V over a field F. Let β be an ordered basis for V, and let $A = [T]_\beta$. Then a vector $v \in V$ is an eigenvector of T corresponding to λ iff $\phi_\beta(v)$ is an eigenvector of A corresponding to λ.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & V \arrow[d, "\phi_\beta"] \\ F^n \arrow[r, "L_A"'] & F^n \end{tikzcd}$$

Proof. TODO. □

3 Eigenvectors and Some Special Functions and Matrices

Lemma 21. *A square matrix has the same determinant as its transpose.*

Proposition 22. *A square matrix has the same characteristic polynomial as its transpose.*

Proposition 23. *If x is an eigenvector of T corresponding to λ, then for any positive integer m, x is an eigenvector of T^m corresponding to λ^m.*

Proof. Linearity of T and induction. □

Note 24. The same holds for matrices. It's always the same!

Proposition 25. *Similar matrices have the same trace.*

Proof. Follows from Proposition 1. □

Corollary 26. *Define the trace of a linear operator T on a finite dimensional vector space as $\operatorname{tr}[T]_\beta$ for any basis β. This is well defined by the previous Proposition.*

Proposition 27. *Let T be the linear operator on $M_{n\times n}(\mathbf{R})$ defined by $T(A) = A^t$. Then ± 1 are the only eigenvalues of T. The eigenvectors of T corresponding to ± 1 are symmetric and antisymmetric matrices, respectively.*

Example 28. In two dimensions, an ordered basis for $M_{2\times 2}(\mathbf{R})$ consisting of eigenvectors of T so that $[T]_\beta$ is a diagonal matrix is

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

The first three matrices are symmetric and correspond to the eigenvalue $+1$, while the last is skew-symmetric and corresponds to -1.

Example 29. More generally, in n dimensions an eigenbasis consists of symmetric matrices of the form A that has zeroes everywhere except a single 1 along the diagonal, and symmetric matrices of the form B with zeroes everywhere except a 1 in two opposite entries B_{ij} and B_{ji}, and finally anti-symmetric matrices of the form C with zeroes everywhere except a -1 and a $+1$ in two opposite entries C_{ij} and C_{ji}, where the -1 is in the lower left half and the $+1$ in the upper right half of C. There are n matrices of type A, $(n^2 - n)/2$ each of type B and C, for a total of n^2, as expected.

3.1 Scalar Matrices

Definition 30. *A scalar matrix is a square matrix of the form λI for some scalar λ.*

Proposition 31. *If a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.*

Proof. If A is similar to λI, that means there is an invertible matrix Q s.t. $A = Q^{-1}\lambda I Q = \lambda I$. $\square$

Proposition 32. *A diagonalizable matrix A having only one eigenvalue is a scalar matrix.*

Proof. If A is diagonalizable, this means A is similar to a diagonal matrix, whose diagonal entries are its eigenvalues. Since A only has one eigenvalue λ, the diagonal entries are all equal to λ. $\square$

Example 33. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.