Chapter 3:
Spectrum Analysis

References

[1] Simon Haykin, *Adaptive Filter Theory*, Prentice Hall, 1996 (3rd Ed.), 2001 (4th Ed.).

[2] Steven M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice Hall, 1993.

[3] Alan V. Oppenheim, Ronald W. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, 1989.

[4] Athanasios Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, 1991 (3rd Ed.), 2001 (4th Ed.).

3. Power Spectrum (1)

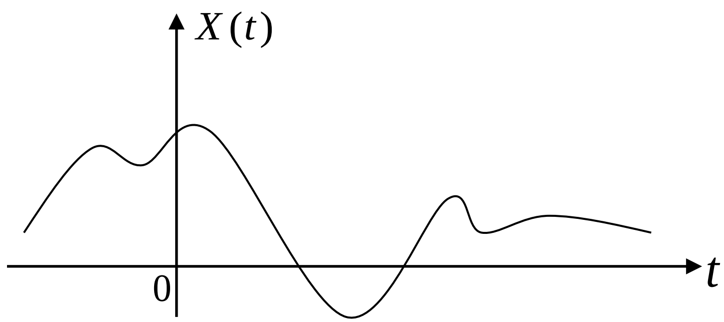
❑ For a deterministic signal $x(t)$, the spectrum is well defined: If $X(\omega)$ represents its Fourier transform, i.e., if

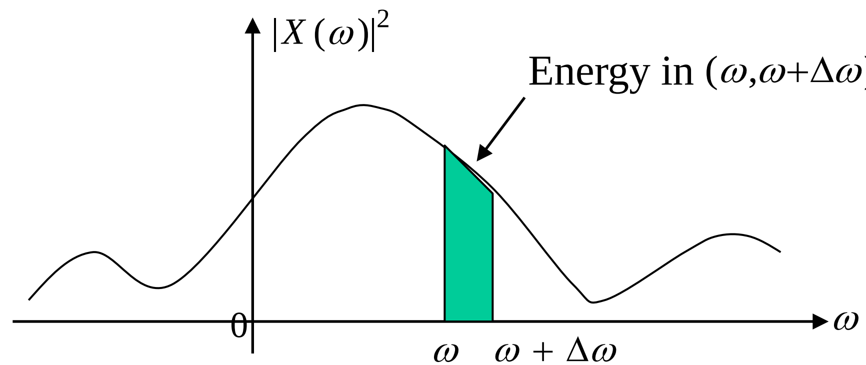
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \qquad (3\text{-}1)$$

then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega = E. \qquad (3\text{-}2)$$

Thus, $|X(\omega)|^2 \Delta\omega$ represents the signal energy in the band $(\omega, \ \omega + \Delta\omega)$.





3. Power Spectrum (2)

❑ However for stochastic processes, a direct application of (3-1) generates a sequence of random variables for every ω. Moreover, for a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant t.

To obtain the spectral distribution of power versus frequency for stochastic processes, it is best to avoid infinite intervals to begin with, and start with a finite interval $(-T, T)$ in (3-1). Formally, partial Fourier transform of a process $X(t)$ based on $(-T, T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt \qquad (3\text{-}3)$$

so that

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2 \qquad (3\text{-}4)$$

represents the power distribution associated with that realization based on $(-T, T)$. Notice that (3-4) represents a random variable for every ω and its ensemble average gives, the average power distribution based on $(-T, T)$. Thus

3. Power Spectrum (3)

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2 \qquad (3\text{-}5)$$

represents the power distribution of $X(t)$ based on $(-T, T)$. If $X(t)$ is assumed to be w.s.s, then $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$, and (3-5) simplifies to

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

Let $\tau = t_1 - t_2$, we get

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T-|\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(1-\tfrac{|\tau|}{2T})d\tau \ \geq \ 0 \qquad (3\text{-}6)$$

to be the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$. Finally letting $T \to \infty$ in (3-6), we obtain

3. Power Spectrum (4)

$$S_{XX}(\omega) = \lim_{T \to \infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \geq 0 \qquad (3\text{-}7)$$

to be the *power spectral density* of the w.s.s process $X(t)$. Notice that

$$R_{XX}(\tau) \xleftrightarrow{\text{F·T}} S_{XX}(\omega) \geq 0. \qquad (3\text{-}8)$$

i.e., the autocorrelation function and the power spectrum of a w.s.s process form a Fourier transform pair, a relation known as the **Wiener-Khinchin Theorem**. From (3-8), the inverse formula gives

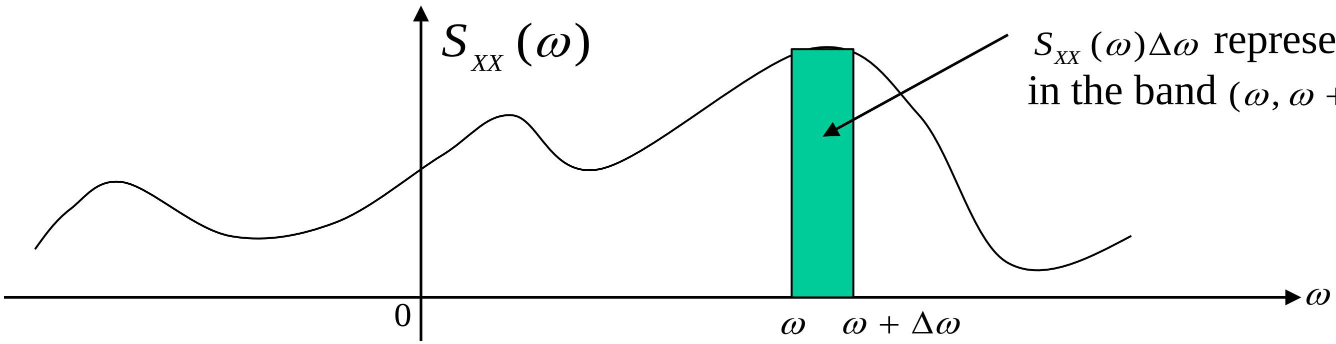
$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega \qquad (3\text{-}9)$$

and in particular for $\tau = 0$, we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \textit{the total power}. \qquad (3\text{-}10)$$

From (3-10), the area under $S_{XX}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{XX}(\omega)$ truly represents the power spectrum.

3. Power Spectrum (5)



The nonnegative-definiteness property of the autocorrelation function translates into the "nonnegative" property for its Fourier transform (power spectrum)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i - t_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega(t_i - t_j)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)\left|\sum_{i=1}^{n} a_i e^{j\omega t_i}\right|^2 d\omega \ge 0. \qquad (3\text{-}11)$$

From (3-11), it follows that

$$R_{XX}(\tau) \textit{ nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \ge 0. \qquad (3\text{-}12)$$

3. Power Spectrum (6)

If $X(t)$ is a real w.s.s process, then $R_{XX}(\tau) = R_{XX}(-\tau)$, so that

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos\omega\tau d\tau \\ &= 2\int_{0}^{\infty} R_{XX}(\tau) \cos\omega\tau d\tau = S_{XX}(-\omega) \geq 0 \end{aligned} \quad (3\text{-}13)$$

so that the power spectrum is an even function, (in addition to being real and nonnegative).

3. Power Spectra and Linear Systems (1)

❑ If a w.s.s process $X(t)$ with autocorrelation function $R_{XX}(\tau) \to S_{XX}(\omega)$ is applied to a linear system with impulse response $h(t)$, then the cross correlation function $R_{XY}(\tau)$ and the output autocorrelation function $R_{YY}(\tau)$ can be determined. From there

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau), \quad R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau). \tag{3-14}$$

If

$$f(t) \leftrightarrow F(\omega), \quad g(t) \leftrightarrow G(\omega) \tag{3-15}$$

then

$$f(t) * g(t) \leftrightarrow F(\omega)G(\omega) \tag{3-16}$$

since

$$\mathbf{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(t) * g(t) e^{-j\omega t} dt$$

$$\mathbf{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau \right\} e^{-j\omega t} dt \tag{3-17}$$

$$= \int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{+\infty} g(t-\tau) e^{-j\omega(t-\tau)} d(t-\tau)$$

$$= F(\omega)G(\omega).$$

3. Power Spectra and Linear Systems (2)

Using (3-15)-(3-17) in (3-14) we get

$$S_{XY}(\omega) = \mathbf{F}\ \{R_{XX}(\omega) * h^*(-\tau)\} = S_{XX}(\omega) H^*(\omega) \qquad (3\text{-}18)$$

since

$$\int_{-\infty}^{+\infty} h^*(-\tau) e^{-j\omega\tau} d\tau = \left(\int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \right)^* = H^*(\omega),$$

where

$$H(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \qquad (3\text{-}19)$$

represents the transfer function of the system, and

$$\begin{aligned} S_{YY}(\omega) &= \mathbf{F}\ \{R_{YY}(\tau)\} = S_{XY}(\omega) H(\omega) \\ &= S_{XX}(\omega) \,|\, H(\omega) \,|^2 . \end{aligned} \qquad (3\text{-}20)$$

From (3-18), the cross spectrum need not be real or nonnegative. However the output power spectrum is real and nonnegative and is related to the input spectrum and the system transfer function as in (3-20). Eq. (3-20) can be used for system identification as well.

<u>Example</u> of Thermal noise (Example 11.1, p. 351, [4])

3. Power Spectra and Linear Systems (3)

❑ **W.S.S White Noise Process**: If $W(t)$ is a w.s.s white noise process, then

$$R_{WW}(\tau) = q\delta(\tau) \quad \Rightarrow \quad S_{WW}(\omega) = q. \qquad (3\text{-}21)$$

Thus the spectrum of a white noise process is flat, thus justifying its name. Notice that a white noise process is unrealizable since its total power is indeterminate.

From (3-20), if the input to an unknown system is a white noise process, then the output spectrum is given by

$$S_{YY}(\omega) = q\,|\,H(\omega)\,|^2 \qquad (3\text{-}22)$$

Notice that the output spectrum captures the system transfer function characteristics entirely, and for rational systems Eq (3-22) may be used to determine the pole/zero locations of the underlying system.

3. Power Spectra and Linear Systems (4)

Example 3.1: A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

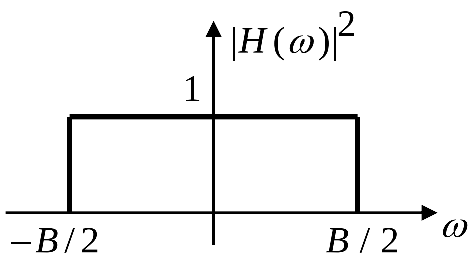
Solution: Let $X(t)$ represent the output of the LPF. Then from (3-22)

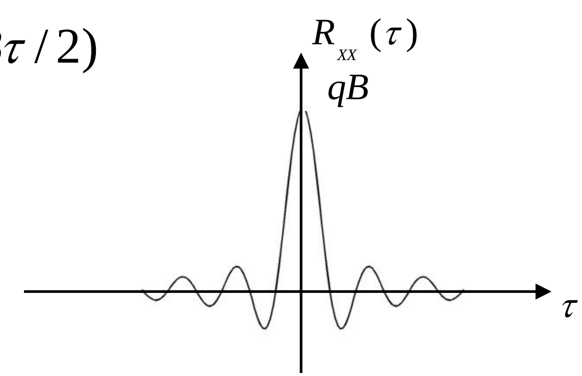
$$S_{XX}(\omega) = q\,|H(\omega)|^2 = \begin{cases} q, & |\omega| \le B/2 \\ 0, & |\omega| > B/2 \end{cases} . \tag{3-23}$$

Inverse transform of $S_{XX}(\omega)$ gives the output autocorrelation function to be

$$R_{XX}(\tau) = \int_{-B/2}^{B/2} S_{XX}(\omega) e^{j\omega\tau} d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau} d\omega \tag{3-24}$$

$$= qB\,\frac{\sin(B\tau/2)}{(B\tau/2)} = qB\,\mathrm{sinc}(B\tau/2)$$





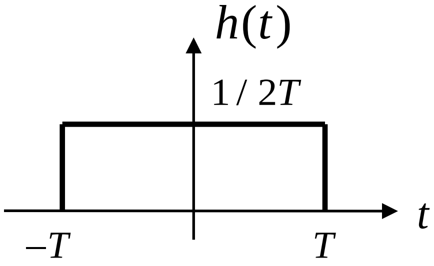
3. Power Spectra and Linear Systems (5)

Eq (3-23) represents colored noise spectrum and (3-24) its autocorrelation function.

Example 3.2: Let

$$Y(t)=\frac{1}{2T}\int_{t-T}^{t+T}X(\tau)d\tau \qquad (3\text{-}25)$$

represent a "smoothing" operation using a moving window on the input process $X(t)$. Find the spectrum of the output $Y(t)$ in term of that of $X(t)$.



Solution: If we define an LTI system with impulse response $h(t)$ as in the figure, then in term of $h(t)$, (3-25) reduces to

$$Y(t)=\int_{-\infty}^{+\infty}h(t-\tau)X(\tau)d\tau=h(t)*X(t) \qquad (3\text{-}26)$$

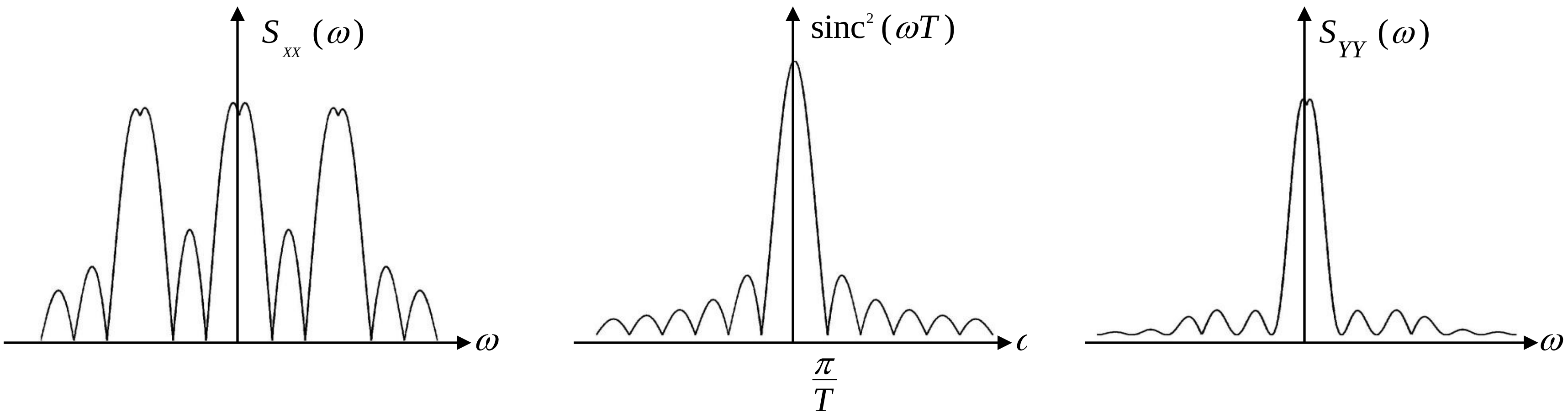
so that $S_{YY}(\omega)=S_{XX}(\omega)\,|\,H(\omega)\,|^2\,.$ (3-27)

where $H(\omega)=\int_{-T}^{+T}\frac{1}{2T}e^{-j\omega t}dt=\text{sinc}(\omega T)$ (3-28)

3. Power Spectra and Linear Systems (6)

so that

$$S_{YY}(\omega) = S_{XX}(\omega)\,\mathrm{sinc}^2(\omega T). \tag{3-29}$$



Notice that the effect of the smoothing operation in (3-25) is to suppress the high frequency components in the input (beyond π / T), and the equivalent linear system acts as a low-pass filter (continuous-time moving average) with bandwidth $2\pi / T$ in this case.

3. Discrete-Time Processes (1)

❑ For discrete-time w.s.s stochastic processes $X(nT)$ with autocorrelation sequence $\{r_k\}_{-\infty}^{+\infty}$, (proceeding as above) or formally defining a continuous time process $X(t)=\sum_n X(nT)\delta(t-nT)$, we get the corresponding autocorrelation function to be

$$R_{XX}(\tau)=\sum_{k=-\infty}^{+\infty} r_k\delta(\tau-kT).$$

Its Fourier transform is given by

$$S_{XX}(\omega)=\sum_{k=-\infty}^{+\infty} r_k e^{-j\omega T}\geq 0, \qquad (3\text{-}30)$$

and it defines the power spectrum of the discrete-time process $X(nT)$. From (3-30),

$$S_{XX}(\omega)=S_{XX}(\omega+2\pi/T) \qquad (3\text{-}31)$$

so that $S_{XX}(\omega)$ is a periodic function with period

$$2B=\frac{2\pi}{T}. \qquad (3\text{-}32)$$

3. Discrete-Time Processes (2)

This gives the inverse relation

$$r_k = \frac{1}{2B}\int_{-B}^{B} S_{xx}(\omega)e^{jk\omega T}d\omega \tag{3-33}$$

and

$$r_0 = E\{|X(nT)|^2\} = \frac{1}{2B}\int_{-B}^{B} S_{xx}(\omega)d\omega \tag{3-34}$$

represents the total power of the discrete-time process $X(nT)$. The input-output relations for discrete-time system $h(nT)$ translate into

$$S_{XY}(\omega) = S_{XX}(\omega)H^*(e^{j\omega}) \tag{3-35}$$

and

$$S_{YY}(\omega) = S_{XX}(\omega)\,|\,H(e^{j\omega})\,|^2 \tag{3-36}$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(nT)\,e^{-j\omega nT} \tag{3-37}$$

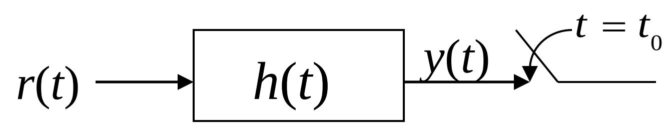
represents the discrete-time system transfer function.

3. Matched Filter (1)

- Let $r(t)$ represent a deterministic signal $s(t)$ corrupted by noise. Thus

$$r(t) = s(t) + w(t), \qquad 0 < t < t_0 \tag{3-38}$$

where $r(t)$ represents the observed data, and it is passed through a receiver with impulse response $h(t)$. The output $y(t)$ is given by



$$y(t) \triangleq y_s(t) + n(t) \tag{3-39}$$

where

$$y_s(t) = s(t) * h(t), \qquad n(t) = w(t) * h(t),$$

and it can be used to make a decision about the presence of absence of $s(t)$ in $r(t)$. Towards this, one approach is to require that the receiver output signal to noise ratio $(SNR)_0$ at time instant t_0 be maximized. Notice that

3. Matched Filter (2)

$$(SNR)_0 \triangleq \frac{\text{Output signal power at } t = t_0}{\text{Average output noise power}} = \frac{|y_s(t_0)|^2}{E\{|n(t)|^2\}} \qquad (3\text{-}41)$$

$$= \frac{|y_s(t_0)|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{nn}(\omega)d\omega} = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{WW}(\omega)|H(\omega)|^2\, d\omega}$$

represents the output SNR, where we have made use of (3-20) to determine the average output noise power, and the problem is to maximize $(\text{SNR})_0$ by optimally choosing the receiver filter $H(\omega)$.

❑ **Optimum Receiver for White Noise Input:** The simplest input noise model assumes $w(t)$ to be white noise in (3-38) with spectral density N_0, so that (3-41) simplifies to

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{2\pi N_0\int_{-\infty}^{+\infty}|H(\omega)|^2\, d\omega} \qquad (3\text{-}42)$$

3. Matched Filter (3)

Direct application of Cauchy-Schwarz' inequality in (3-42) gives

$$(SNR)_0 \le \frac{1}{2\pi N_0}\int_{-\infty}^{+\infty} |S(\omega)|^2 \, d\omega = \frac{\int_0^{+\infty} s(t)^2 dt}{N_0} = \frac{E_s}{N_0} \qquad (3\text{-}43)$$

and equality in (3-43) is guaranteed if and only if

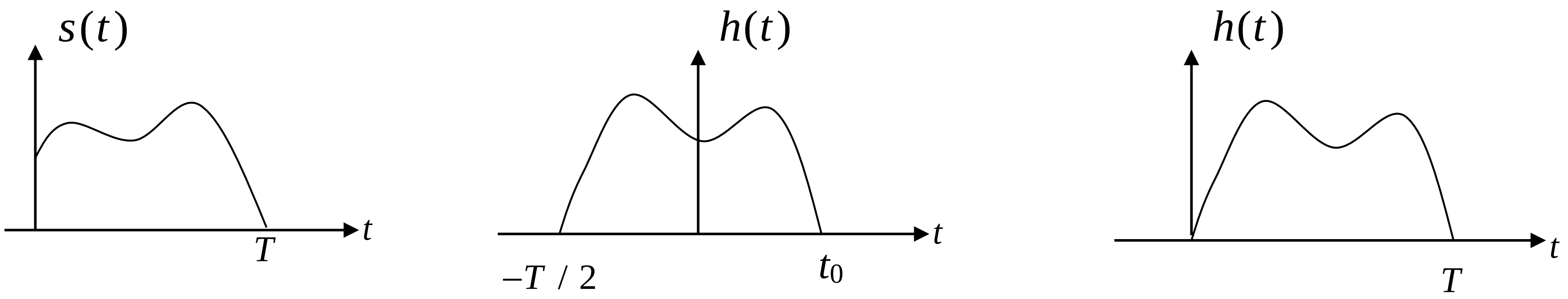
$$H(\omega) = S^*(\omega) e^{-j\omega t_0} \qquad (3\text{-}44)$$

or

$$h(t) = s(t_0 - t). \qquad (3\text{-}45)$$

From (3-45), the optimum receiver that maximizes the output SNR at $t = t_0$ is given by (3-44)-(3-45). Notice that (3-45) need not be causal, and the corresponding SNR is given by (3-43).

3. Matched Filter (4)



(a) (b) $t_0=T/2$ (c) $t_0=T$

The figure shows the optimum $h(t)$ for two different values of t_0. In figure(b), the receiver is noncausal, whereas in figure (c) the receiver represents a causal waveform.

If the receiver is causal, the optimum causal receiver can be shown to be

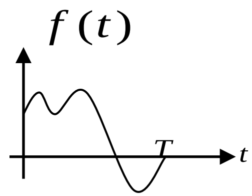
$$h_{opt}(t) = s(t_0 - t)u(t) \tag{3-46}$$

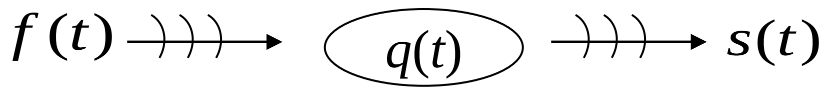
and the corresponding maximum $(SNR)_0$ in that case is given by

$$(SNR_0) = \frac{1}{N_0}\int_0^{t_0} s^2(t)dt \tag{3-47}$$

3. Matched Filter (5)

❑ **Optimum Transmit Signal:** In practice, the signal $s(t)$ in (3-38) may be the output of a target that has been illuminated by a transmit signal $f(t)$ of finite duration T. In that case





$$s(t) = f(t) * q(t) = \int_0^T f(\tau) q(t-\tau) d\tau, \qquad (3\text{-}48)$$

where $q(t)$ represents the target impulse response. One interesting question in this context is to determine the optimum transmit signal $f(t)$ with normalized energy that maximizes the receiver output SNR at $t = t_0$ in the Matched filter. Notice that for a given $s(t)$, (3-45) represents the optimum receiver, and (3-43) gives the corresponding maximum (SNR)0. To maximize $(\text{SNR})_0$ in (3-43), we may substitute (3-48) into (3-43). This gives

3. Matched Filter (6)

$$(SNR)_0 = \int_0^{\infty} |\{\int_0^T q(t-\tau_1) f(\tau_1) d\tau_1\}|^2 \, dt$$

$$= \frac{1}{N_0} \int_0^T \int_0^T \underbrace{\int_0^{\infty} q(t-\tau_1) q^*(t-\tau_2) dt}_{\Omega(\tau_1,\tau_2)} \, f(\tau_2) d\tau_2 \, f(\tau_1) d\tau_1 \qquad (3\text{-}49)$$

$$= \frac{1}{N_0} \int_0^T \{\int_0^T \Omega(\tau_1,\tau_2) f(\tau_2) d\tau_2\} f(\tau_1) d\tau_1 \le \lambda_{\max} / N_0$$

where $\Omega(\tau_1,\tau_2)$ is given by

$$\Omega(\tau_1,\tau_2) = \int_0^{\infty} q(t-\tau_1) q^*(t-\tau_2) dt \qquad (3\text{-}50)$$

and $\lambda_{\max}$ is the largest eigenvalue of the integral equation

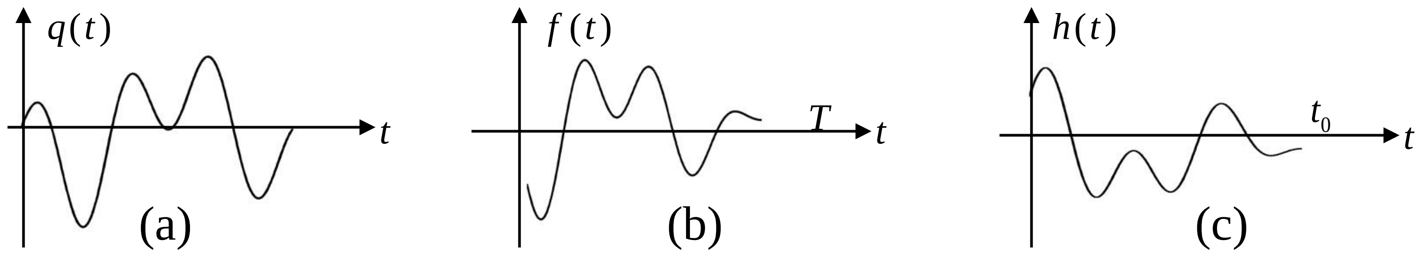
$$\int_0^T \Omega(\tau_1,\tau_2) f(\tau_2) d\tau_2 = \lambda_{\max} f(\tau_1), \qquad 0 < \tau_1 < T. \qquad (3\text{-}51)$$

and

$$\int_0^T f^2(t) dt = 1. \qquad (3\text{-}52)$$

3. Matched Filter (7)

Observe that the kernel $\Omega(\tau_1,\tau_2)$ in (3-50) captures the target characteristics so as to maximize the output SNR at the observation instant, and the optimum transmit signal is the solution of the integral equation in (3-51) subject to the energy constraint in (3-52). The figure below shows the optimum transmit signal and the companion receiver pair for a specific target with impulse response $q(t)$ as shown there.



If the causal solution in (3-46)-(3-47) is chosen, in that case the kernel in (3-50) simplifies to

$$\Omega(\tau_1,\tau_2)=\int_0^{t_0} q(t-\tau_1)q^*(t-\tau_2)dt. \qquad \text{(3-53)}$$

and the optimum transmit signal is given by (3-51). Notice that in the causal case, information beyond $t=t_0$ is not used.

3. Matched Filter (8)

❑ What if the additive noise in (3-38) is not white?

Let $S_{WW}(\omega)$ represent a (non-flat) power spectral density. In that case, what is the optimum matched filter?

If the noise is *not* white, one approach is to *whiten* the input noise first by passing it through a whitening filter, and then proceed with the whitened output as before.

$$r(t) = s(t) + \underbrace{w(t)}_{\text{colored noise}} \longrightarrow \boxed{\begin{array}{c}\text{Whitening Filter}\\ g(t)\end{array}} \longrightarrow s_g(t) + \underbrace{n(t)}_{\text{white noise}}$$

Notice that the signal part of the whitened output $s_g(t)$ equals

$$s_g(t) = s(t) * g(t) \tag{3-54}$$

where $g(t)$ represents the whitening filter, and the output noise $n(t)$ is white with unit spectral density.

3. Matched Filter (9)

Whitening Filter: What is a whitening filter? From the discussion above, the output spectral density of the whitened noise process $S_{nn}(\omega)$ equals unity, since it represents the normalized white noise by design. But from (3-20)

$$1 = S_{nn}(\omega) = S_{ww}(\omega)\,|\,G(\omega)\,|^2,$$

which gives

$$|\,G(\omega)\,|^2 = \frac{1}{S_{ww}(\omega)}. \qquad (3\text{-}55)$$

i.e., the whitening filter transfer function $G(\omega)$ satisfies the magnitude relationship in (3-55). To be useful in practice, it is desirable to have the whitening filter to be *stable* and *causal* as well. Moreover, at times its inverse transfer function also needs to be implementable so that it needs to be stable as well. How does one obtain such a filter (if any)?

3. Matched Filter (10)

From there, any spectral density that satisfies the finite power constraint

$$\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega < \infty \qquad (3\text{-}56)$$

and the Paley-Wiener constraint (see Eq. (12-4), p. 402, [4])

$$\int_{-\infty}^{+\infty} \frac{|\log S_{XX}(\omega)|}{1+\omega^2} d\omega < \infty \qquad (3\text{-}57)$$

can be factorized as

$$S_{XX}(\omega) = |H(j\omega)|^2 = H(s)H(-s)|_{s=j\omega} \qquad (3\text{-}58)$$

where $H(s)$ together with its inverse function $1/H(s)$ represent two filters that are both analytic in Re $s > 0$. Thus $H(s)$ and its inverse $1/H(s)$ can be chosen to be *stable* and *causal* in (3-58). Such a filter is known as the *Wiener factor*, and since it has all its poles and zeros in the left half plane, it represents a *minimum phase factor*. In the rational case, if $X(t)$ represents a real process, then $S_{XX}(\omega)$ is even and hence (3-58) reads

3. Matched Filter (11)

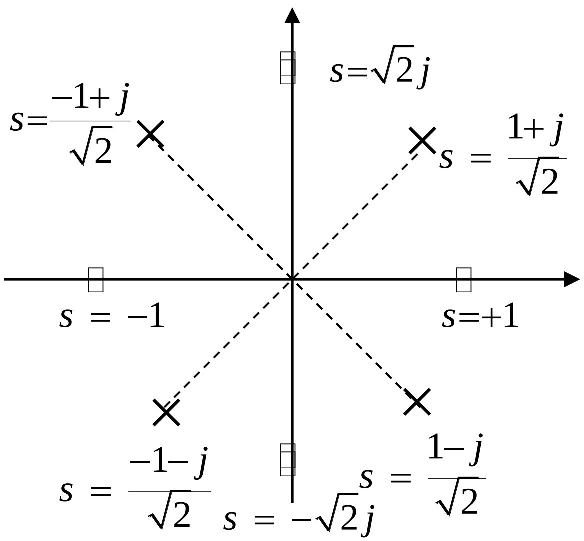
$$0 \le S_{XX}(\omega^2) = \tilde{S}_{XX}(-s^2)\,|_{s=j\omega} = H(s)H(-s)\,|_{s=j\omega}\,. \qquad (3\text{-}59)$$

<u>Example 18.3:</u> Consider the spectrum

$$S_{XX}(\omega) = \frac{(\omega^2+1)(\omega^2-2)^2}{(\omega^4+1)}$$

which translates into

$$\tilde{S}_{XX}(-s^2) = \frac{(1-s^2)(2+s^2)^2}{1+s^4}.$$



The poles (×) and zeros (□) of this function are shown in the figure. From there to maintain the symmetry condition in (3-59), we may group together the left half factors as

$$H(s) = \frac{(s+1)(s-\sqrt{2}j)(s+\sqrt{2}j)}{\left(s+\frac{1+j}{\sqrt{2}}\right)\left(s+\frac{1-j}{\sqrt{2}}\right)} = \frac{(s+1)(s^2+2)}{s^2+\sqrt{2}s+1}$$

and it represents the Wiener factor for the spectrum $S_{XX}(\omega)$ above. Observe that the poles and zeros (if any) on the $j\omega$-axis appear in even multiples in $S_{XX}(\omega)$ and hence half of them may be paired with H(s) (and the other half with $H(-s)$) to preserve the factorization condition in (3-58). Notice that $H(s)$ is stable, and so is its inverse.

More generally, if $H(s)$ is minimum phase, then ln $H(s)$ is analytic on the right half plane so that

$$H(\omega) = A(\omega)e^{-j\varphi(\omega)} \tag{3-60}$$

gives

$$\ln H(\omega) = \ln A(\omega) - j\varphi(\omega) = \int_0^{+\infty} b(t)e^{-j\omega t}dt.$$

Thus

$$\ln A(\omega) = \int_0^t b(t)\cos\omega t\ dt$$

$$\varphi(\omega) = \int_0^t b(t)\sin\omega t\ dt$$

3. Matched Filter (13)

Since $\cos\omega t$ and $\sin\omega t$ are Hilbert transform pairs, it follows that the phase function $\varphi(\omega)$ in (3-60) is given by the Hilbert transform of $\ln A(\omega)$. Thus

$$\varphi(\omega) = \mathcal{H}\{\ln A(\omega)\}. \tag{3-61}$$

Eq. (3-60) may be used to generate the unknown phase function of a minimum phase factor from its magnitude.

For discrete-time processes, the factorization conditions take the form

$$\int_{-\pi}^{\pi} S_{XX}(\omega)d\omega \ < \infty \tag{3-62}$$

and

$$\int_{-\pi}^{\pi} \ln S_{XX}(\omega)d\omega \ > -\infty. \tag{3-63}$$

In that case

$$S_{XX}(\omega) = |H(e^{j\omega})|^2$$

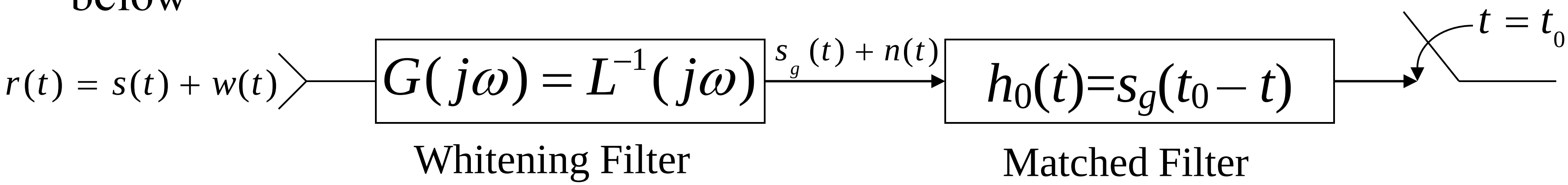
where the discrete-time system

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k}$$

3. Matched Filter (14)

is analytic together with its inverse in $|z| > 1$. This unique minimum phase function represents the Wiener factor in the discrete-case.

❑ **Matched Filter in Colored Noise:** Returning back to the matched filter problem in colored noise, the design can be completed as shown in figure below



where $G(j\omega)$ represents the whitening filter associated with the noise spectral density $S_{WW}(\omega)$ as in (3-55)-(3-58). Notice that $G(s)$ is the inverse of the Wiener factor $L(s)$ corresponding to the spectrum $S_{WW}(\omega)$ i.e.,

$$L(s)L(-s)\,|_{s=j\omega} = |\,L(j\omega)\,|^2 = S_{ww}(\omega). \qquad (3\text{-}64)$$

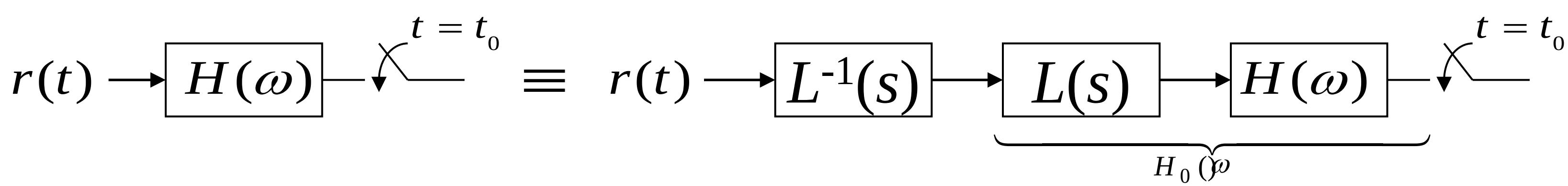
The whitened output $s_g(\mathrm{t}) + n(t)$ in the figure is similar to (3-38), and from (3-45) the optimum receiver is given by $h_0(t) = s_g(t_0 - t)$

3. Matched Filter (15)

where

$$s_g(t) \leftrightarrow S_g(\omega) = G(j\omega)S(\omega) = L^{-1}(j\omega)S(\omega).$$

If we insist on obtaining the receiver transfer function $H(\omega)$ for the original colored noise problem, we can deduce it easily from the below figure



where

$$H(\omega) = L^{-1}(j\omega)H_0(\omega) = L^{-1}(\omega)S_g^*(\omega)e^{-j\omega t_0}$$
$$= L^{-1}(\omega)\{L^{-1}(\omega)S(\omega)\}^* e^{-j\omega t_0} \qquad (3\text{-}65)$$

3. AM/FM Noise Analysis (1)

- Consider the noisy AM signal

$$X(t) = m(t)\cos(\omega_0 t + \theta) + n(t), \tag{3-66}$$

and the noisy FM signal

$$X(t) = A\cos(\omega_0 t + \varphi(t) + \theta) + n(t), \tag{3-67}$$

where

$$\varphi(t) = \begin{cases} c\int_0^t m(\tau)d\tau & \text{FM} \\ cm(t) & \text{PM.} \end{cases} \tag{3-68}$$

Here $m(t)$ represents the message signal and θ a random phase jitter in the received signal. In the case of FM, $\omega(t) = \varphi'(t) = c\, m(t)$ so that the instantaneous frequency is proportional to the message signal. We will assume that both the message process $m(t)$ and the noise process $n(t)$ are w.s.s with power spectra $S_{mm}(\omega)$ and $S_{nn}(\omega)$ respectively. We wish to determine whether the AM and FM signals are w.s.s, and if so their respective power spectral densities.

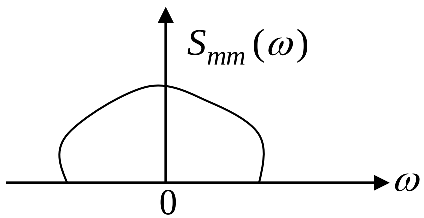
3. AM/FM Noise Analysis (2)

<u>Solution: AM signal:</u> In this case from (3-66), if we assume $\theta \sim U(0, 2\pi)$, then

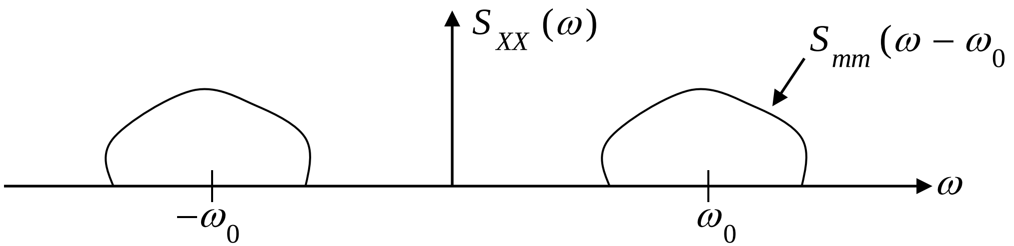
$$R_{XX}(\tau) = \frac{1}{2} R_{mm}(\tau) \cos \omega_0 \tau + R_{nn}(\tau) \qquad (3\text{-}69)$$

so that (see figure below)

$$S_{XX}(\omega) = \frac{S_{XX}(\omega - \omega_0) + S_{XX}(\omega + \omega_0)}{2} + S_{nn}(\omega). \qquad (3\text{-}70)$$



(a)



(b)

Thus AM represents a stationary process under the above conditions.

3. AM/FM Noise Analysis (3)

<u>FM signal:</u> In this case (suppressing the additive noise component in (3-67), we obtain

$$
\begin{aligned}
R_{XX}(t+\tau/2, t-\tau/2) &= A^2 E\{\cos(\omega_0(t+\tau/2)+\varphi(t+\tau/2)+\theta)\times \\
&\quad \cos(\omega_0(t-\tau/2)+\varphi(t-\tau/2)+\theta)\} \\
&= \frac{A^2}{2} E\{\cos[\omega_0\tau+\varphi(t+\tau/2)-\varphi(t-\tau/2)] \\
&\quad +\cos[2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2)+2\theta]\} \\
&= \frac{A^2}{2}[E\{\cos(\varphi(t+\tau/2)-\varphi(t-\tau/2))\}\cos\omega_0\tau \\
&\quad -E\{\sin(\varphi(t+\tau/2)-\varphi(t-\tau/2))\}\sin\omega_0\tau]
\end{aligned}
\qquad (3\text{-}71)
$$

since

$$
\begin{aligned}
&E\{\cos(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2)+2\theta)\} \\
&= E\{\cos(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2))\}E\{\cancelto{0}{\cos 2\theta}\} \\
&-E\{\sin(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2))\}E\{\cancelto{0}{\sin 2\theta}\}=0.
\end{aligned}
$$

3. AM/FM Noise Analysis (4)

Eq (3-71) can be rewritten as

$$R_{xx}(t+\tau/2, t-\tau/2) = \frac{A^2}{2}[a(t,\tau)\cos\omega_0\tau - b(t,\tau)\sin\omega_0\tau] \qquad (3\text{-}72)$$

where

$$a(t,\tau) \triangleq E\{\cos(\varphi(t+\tau/2)-\varphi(t-\tau/2))\} \qquad (3\text{-}73)$$

and

$$b(t,\tau) \triangleq E\{\sin(\varphi(t+\tau/2)-\varphi(t-\tau/2))\} \qquad (3\text{-}74)$$

In general, $a(t,\tau)$ and $b(t,\tau)$ depend on both t and τ so that noisy FM is *not* w.s.s in general, even if the message process $m(t)$ is w.s.s. In the special case when $m(t)$ is a stationary Gaussian process, from (3-68), $\varphi(t)$ is also a stationary Gaussian process with autocorrelation function

$$R_{\varphi'\varphi'}(\tau) = c^2 R_{mm}(\tau) = \frac{-d^2 R_{\varphi\varphi}(\tau)}{d\tau^2} \qquad (3\text{-}75)$$

for the FM case.

3. AM/FM Noise Analysis (5)

In that case the random variable

$$Y \stackrel{\Delta}{=} \varphi(t+\tau/2)-\varphi(t-\tau/2) \sim N(0,\sigma_Y^2) \tag{3-76}$$

where

$$\sigma_Y^2 = 2(R_{\varphi\varphi}(0)-R_{\varphi\varphi}(\tau)). \tag{3-77}$$

Hence its characteristic function is given by

$$E\{e^{j\omega Y}\} = e^{-\omega^2\sigma_Y^2/2} = e^{-(R_{\varphi\varphi}(0)-R_{\varphi\varphi}(\tau))\omega^2} \tag{3-78}$$

which for $\omega = 1$ gives

$$E\{e^{jY}\} = E\{\cos Y\} + jE\{\sin Y\} = a(t,\tau) + jb(t,\tau), \tag{3-79}$$

where we have made use of (3-76) and (3-73)-(3-74). On comparing (3-79) with (3-78), we get

$$a(t,\tau) = e^{-(R_{\varphi\varphi}(0)-R_{\varphi\varphi}(\tau))} \tag{3-80}$$

and

$$b(t,\tau) \equiv 0 \tag{3-81}$$

3. AM/FM Noise Analysis (6)

so that the FM autocorrelation function in (3-72) simplifies into

$$R_{XX}(\tau) = \frac{A^2}{2} e^{-(R_{\varphi\varphi}(0) - R_{\varphi\varphi}(\tau))} \cos \omega_0 \tau. \tag{3-82}$$

Notice that for stationary Gaussian message input $m(t)$ (or $\varphi(t)$), the nonlinear output $X(t)$ is indeed strict sense stationary with autocorrelation function as in (3-82).

Find $R_{XX}(\tau)$ for narrowband FM and wideband FM?

Chapter 4: Eigenanalysis

4. Eigenanalysis: Definitions (1)

- Let $\mathbf{R}$ denote $(M\times M)$-correlation matrix of wide-sense DTStP represented by $(M\times 1)$-observation vector $u(n)$. Finding $(M\times 1)$-vector $\mathbf{q}$ such that:

$$\mathbf{R}\mathbf{q} = \lambda\mathbf{q} \tag{4.1}$$

for some constant λ, rewriting in form:

$$(\mathbf{R} - \lambda\mathbf{I})\mathbf{q} = \mathbf{0} \tag{4.2}$$

(4.2) has nonzero solution in vector $\mathbf{q}$ if:

$$\det(\mathbf{R} - \lambda\mathbf{I}) = 0 \tag{4.3}$$

Equation (4.3) called *characteristic equation*, whose roots $\lambda_1, \lambda_2, \ldots, \lambda_M$ called *eigenvalues*. General, (4.3) has M distinct roots $\rightarrow$ M solutions in the vector $\mathbf{q}$.

4. Eigenanalysis: Definitions (2)

- Let λ_I denote i-th eigenvalue of matrix $\mathbf{R}$, $\mathbf{q}_i$ be a nonzero vector such that:

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i \tag{4.4}$$

- Vector $\mathbf{q}_i$ called *eigenvector* associated with λ_i

  Example: see pp. 161-162, [1]

4. Eigenanalysis: Properties (1)

- $\lambda_1, \lambda_2, \ldots, \lambda_M$ are eigenvalues of correlation matrix $\mathbf{R}$, then eigenvalues of correlation matrix $\mathbf{R}^k$ equal $\lambda_1^k, \lambda_2^k, \ldots, \lambda_M^k$ for any $k > 0$.
- Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_M$ be eigenvector corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of $(M \times M)$-correlation matrix $\mathbf{R}$. Then, $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_M$ are linearly independent.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be eigenvalues of $(M \times M)$-correlation matrix $\mathbf{R}$. Then all these eigenvalues are real and nonnegative.
- Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_M$ be eigenvector corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of $(M \times M)$-correlation matrix $\mathbf{R}$. Then, $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_M$ are orthogonal to each other.

4. Eigenanalysis: Properties (2)

- Let $\mathbf{q}_1$, $\mathbf{q}_2$,..., $\mathbf{q}_M$ be eigenvector corresponding to distinct eigenvalues λ_1, λ_2,..., λ_M of $(M\times M)$-correlation matrix $\mathbf{R}$. Define $(M\times M)$-matrix:

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]$$

where

$$\mathbf{q}_i^H \mathbf{q}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Define $(M\times M)$-diagonal matrix:

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_M)$$

Then, original matrix $\mathbf{R}$ may be diagonalized as:

$$\boxed{\mathbf{Q}^H \mathbf{R} \mathbf{Q} = \mathbf{\Lambda}} \qquad (4.5)$$

where $\mathbf{Q}$ is nonsingular with $\mathbf{Q}^{-1}=\mathbf{Q}^H \rightarrow \mathbf{Q}$ is *unitary* matrix.

4. Eigenanalysis: Properties (3)

(4.5) can be written:

$$\mathbf{R} = \mathbf{Q}\Lambda\mathbf{Q}^H = \sum_{i=1}^{M} \lambda_i \mathbf{q}_i \mathbf{q}_i^H \qquad (4.6)$$

- Let $\lambda_1, \lambda_2, \ldots, \lambda_M$ be eigenvalues of $(M\times M)$-correlation matrix $\mathbf{R}$. Then sum of these eigenvalues equals the trace of $\mathbf{R}$:

$$\mathrm{tr}\left[\mathbf{R}\right] = \sum_{i=1}^{M} \lambda_i \qquad (4.7)$$

trace of a square matrix defined as sum of diagonal elements of the matrix.

4. Eigenanalysis: Properties (4)

- Condition number = eigenvalue spread = eigenvalue factor:

$$\chi(\mathbf{R}) = \frac{\lambda_{\max}}{\lambda_{\min}} \tag{4.8}$$

- Eigenvalues of correlation matrix of a DTStP are bounded by min and max values of power spectral density of the process:

$$S_{\min} \le \lambda_i \le S_{\max}\,, \quad i = 1,2,\ldots,M \tag{4.9}$$

Condition number is bounded by:

$$\chi(\mathbf{R}) = \frac{\lambda_{\max}}{\lambda_{\min}} \le \frac{S_{\max}}{S_{\min}}$$

4. Eigenanalysis: Properties (5)

- **Minimax Theorem**: Let (M×M)-correlation matrix $\mathbf{R}$ have eigenvalues λ_1, λ_2,..., λ_M that are arranged in decreasing order as:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$$

Then:

$$\lambda_k = \min_{\dim(C)=k} \max_{\substack{\mathbf{x}\in C \\ \mathbf{x}\neq \mathbf{0}}} \frac{\mathbf{x}^H \mathbf{R}\mathbf{x}}{\mathbf{x}^H \mathbf{x}}, \quad k = 1,2,\cdots,M \tag{4.10}$$

where C is *subspace* of vector space of all (M×1)-complex vector; dim(C): dimension of subspace C.

Subspace of dimension k defined as set of complex vectors that can be written as a linear combination of $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_k$

4. Eigenanalysis: Properties (6)

- **Karhunen-Loeve expansion:** Let $(M\times 1)$-vector $\boldsymbol{u}(n)$ drawn from a wide-sense stationary process of zero mean and correlation matrix $\mathbf{R}$. Let $\mathbf{q}_1, \mathbf{q}_2,..., \mathbf{q}_M$ be eigenvector corresponding to M eigenvalues of $\mathbf{R}$. Vector $\mathbf{u}(n)$ may be expanded as linear combination of these eigenvector:

$$\mathbf{u}(n) = \sum_{i=1}^{M} c_i(n)\mathbf{q}_i \qquad (4.11)$$

Coefficients of the expansion are zero-mean, uncorrelated random variables defined by:

$$c_i(n) = \mathbf{q}_i^H \mathbf{u}(n) \qquad (4.12)$$

4. Eigenanalysis: Properties (7)

where $$E[c_i(n)] = 0, \quad i = 1,2,\cdots,M$$

$$E[c_i(n)c_j^*(n)] = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases} \qquad (4.13)$$

<u>Physical interpretation</u>: viewing eigenvectors $\mathbf{q}_1$, $\mathbf{q}_2, \ldots, \mathbf{q}_M$ as coordinates of an M-dimensional space, thus representing random vector $\mathbf{u}(n)$ by set of its projections $c_1(n)$, $c_2(n)$, ..., $c_M(n)$ on these axes.

From (4.11): $$\sum_{i=1}^{M} \left|c_i(n)\right|^2 = \left\|\mathbf{u}(n)\right\|^2 \qquad (4.14)$$

From (4.12), (4.13):

$$E\left[\left|c_i(n)\right|^2\right] = \lambda_i, \quad i = 1,2,\cdots,M \qquad (4.15)$$

4. Eigenanalysis: Low-Rank Modeling (1)

- **Dimensionality reduction**: Transformation in a way that a *data vector* (in *data space*) can be represented by a reduced number of effective feature (in *feature space*) and yet retain most of intrinsic content of input data.

- Consider M-dimensional data vector $\mathbf{u}(n)$ (representing realization of a wide sense stationary process) that may be expanded as a linear combination of eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_M$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ of $(M \times M)$-correlation matrix $\mathbf{R}$ of $\mathbf{u}(n)$. See (4.11). If knowing *prior knowledge* that $(M\text{-}p)$ eigenvalues $\lambda_{p+1}, \ldots, \lambda_M$ are all very small, then truncating (4.11) at term $i=p$. Then, an *approximate reconstruction* of data vector $\mathbf{u}(n)$ can be defined:

4. Eigenanalysis: Low-Rank Modeling (2)

$$\mathbf{u}'(n) = \sum_{i=1}^{p} c_i(n)\mathbf{q}_i \qquad p < M \tag{4.16}$$

- Vector $\mathbf{u}'(n)$ has rank $p < M$ of original data vector $\mathbf{u}(n)$. Model in (4.16): **low-rank model**.

- Meaning: approximation $\mathbf{u}'(n)$ can be reconstructed by using a set of p numbers $c_1(n)$, $c_2(n)$, ..., $c_p(n)$. In other words, new vector $\mathbf{c}(n)=[c_1(n), c_2(n), ..., c_p(n)]$ is viewed as a *reduced-rank representation* for $\mathbf{u}(n)$.

- *Transformation*: M-dimensional data space to p-dimensional feature space $\rightarrow$ *subspace decomposition*

4. Eigenanalysis: Low-Rank Modeling (3)

❑ **Reconstruction error vector**:

$$\mathbf{e}(n) = \mathbf{u}(n) - \mathbf{u}'(n) = \sum_{i=p+1}^{M} c_i(n)\mathbf{q}_i \qquad (4.17)$$

Mean-square error:

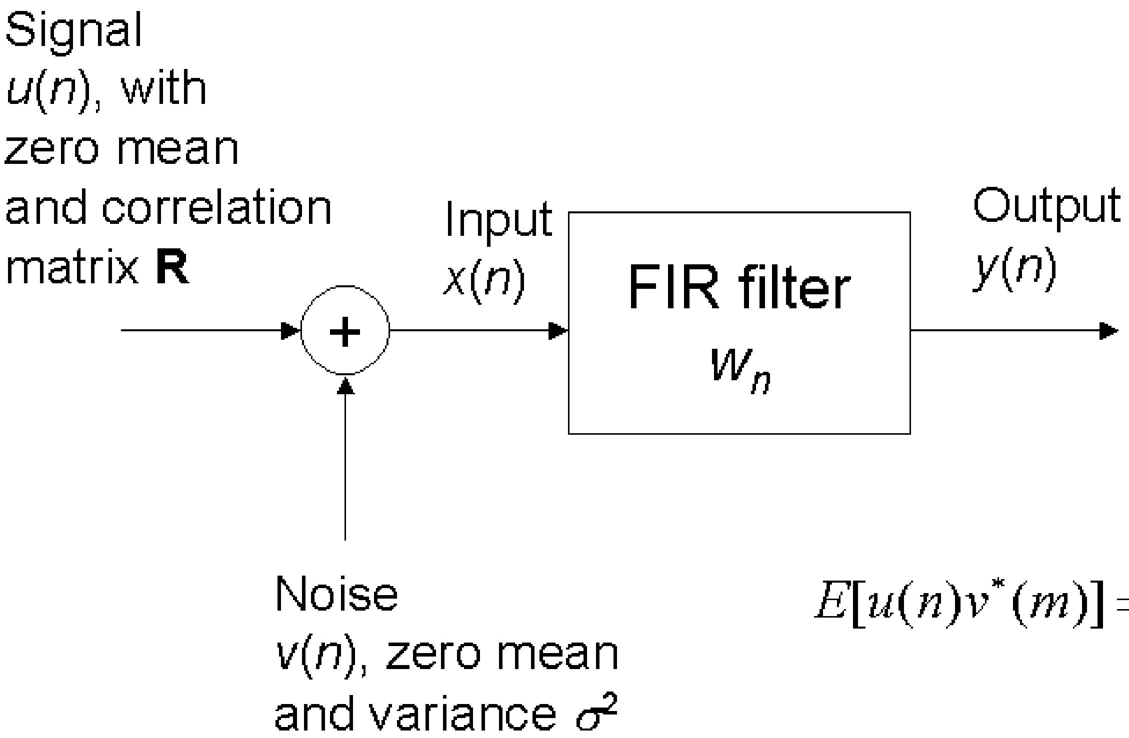
$$\varepsilon = E\left[\|\mathbf{e}(n)\|^2\right] = \sum_{i=p+1}^{M} \lambda_i \qquad (4.18)$$

→ Error is small → good data reconstruction if eigenvalues $\lambda_{p+1},\ldots,\lambda_M$ are all very small.

Example: Application of Low-Rank Modeling (see pp.179-180, [1])

4. Eigenanalysis: Eigenfilters (1)

- **Design optimum FIR filter with optimization criterion of maximizing output signal-to-noise ratio**



$$E[u(n)v^*(m)] = 0, \quad \forall (n,m)$$

4. Eigenanalysis: Eigenfilters (2)

- Effects of signal only: Average power of signal at filter output:

$$P_o = \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{4.18}$$

- Effects of noise only: Average power of noise at filter output:

$$N_o = \sigma^2 \mathbf{w}^H \mathbf{w} \tag{4.19}$$

- Output signal-to-noise ratio:

$$(SNR)_o = \frac{P_o}{N_o} = \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma^2 \mathbf{w}^H \mathbf{w}} \tag{4.20}$$

4. Eigenanalysis: Eigenfilters (3)

- Optimum problem: max $(SNR)_o$ subject to $\mathbf{w}^H\mathbf{w}=1$. Applying minimax theorem (4.10):

$$(SNR)_{o,\max} = \frac{\lambda_{\max}}{\sigma^2} \qquad (4.21)$$

Optimum FIR filter that produces (4.21) having *coefficient vector* $\mathbf{w}_o=\mathbf{q}_{\max}$ is called *eigenfilter*, where $\mathbf{q}_{\max}$: eigenvector associated with $\lambda_{\max}$.

Note: Eigen filter maximizes the output SNR for a **random signal** (w.s.s) in additive white noise,
while a **matched filter** maximizes the output SNR for a **known signal** in additive white noise.