# Advanced Communication Systems

# Chapter 2: Characteristic of Propagation in Communication Systems

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Slides with references from HUT Finland, La Hore uni., Mc. Graw Hill Co., A.B. Carlson's "Communication Systems", and Leon W.Couch "Digital and Analog Communication Systems" books

# Signals and Spectra

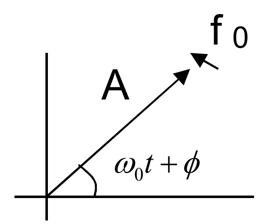
- A generic sinusoidal signal

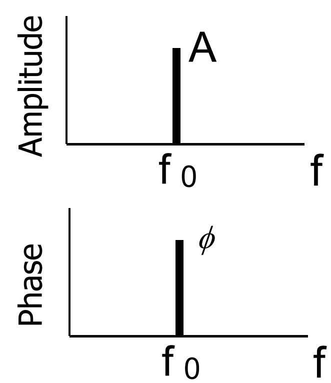
$$v(t) = A\cos(\omega_0 t + \phi); \qquad \omega_0 = 2\pi f_0$$

- Phasor representation

$$A\cos(\omega_0 t + \phi) = A\,\mathrm{Re}\left[e^{j(\omega_0 t + \phi)}\right] = \mathrm{Re}\left[A e^{j\phi} e^{j\omega_0 t}\right]$$

- Frequency domain representation
  - Rotating phasors
  - Frequency plots
    - **Amplitude**
    - **Phase**

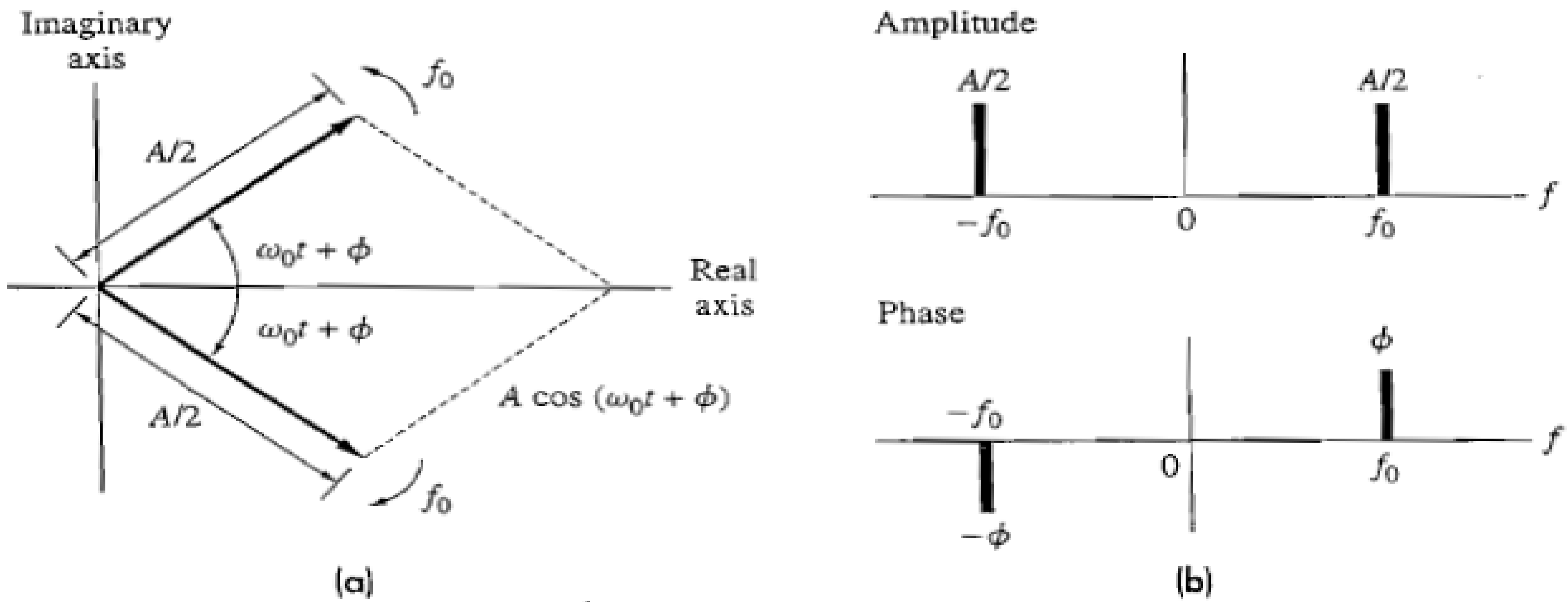


- Two sided spectra can be seen from

$$A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

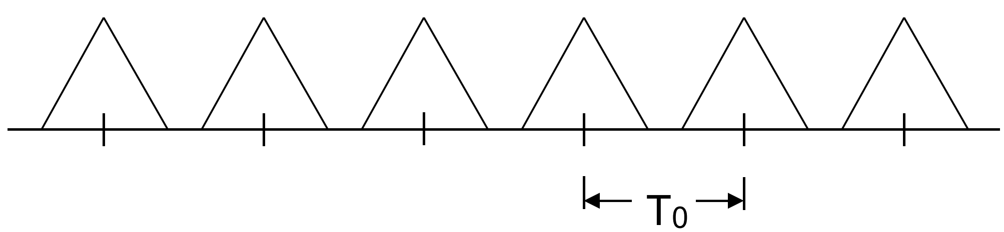
- This represents two rotating phasors
- Amplitude and phase spectrum (two sided)




(a) Conjugate phasors; (b) two-sided spectrum.

# Periodic Signals

- A signal $x_p(t)$ is periodic if there exists $T$ such that $x_p(t) = x_p(t + T)$
- Smallest such $T$ is called fundamental period $T_0$
- Any integer multiple of $T_0$ is also a period



# Average signal and Power

- Average signal

$$\langle v(t) \rangle \triangleq \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)\, dt$$

- For periodic signals

$$\langle v(t) \rangle = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} v(t)\, dt = \frac{1}{T_0} \int_{T_0} v(t)\, dt$$

- Average power

$$P \triangleq \langle |v(t)|^2 \rangle = \frac{1}{T_0} \int_{T_0} |v(t)|^2\, dt$$

# Normalized Power

- In the concept of *normalized* power, R is assumed to be 1Ω, although it may be another value in the actual circuit.
- Another way of expressing this concept is to say that the power is given on a per-ohm basis.
- It can also be realized that *the square root of the normalized power is the rms value.*

**Definition.** The *average normalized power* is given by:

Where $s(t)$ is the voltage or current waveform

$$P = \langle s^2(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s^2(t)dt$$

# Decibel

- A base 10 logarithmic measure of power ratios.
- The ratio of the power level at the output of a circuit compared with that at the input is often specified by the decibel gain instead of the actual ratio.
- Decibel measure can be defined in 3 ways
  - Decibel Gain
  - Decibel signal-to-noise ratio (SNR in dB)
  - Mili-watt Decibel or dBm
- **Definition:** Decibel Gain

The decibel gain of a circuit is:

$$dB = 10\log\left(\frac{average \quad power \quad out}{average \quad power \quad in}\right) = 10\log\left(\frac{P_{out}}{P_{in}}\right)$$

# Decibel Gain

$$dB = 10\log\left(\frac{average \quad power \quad out}{average \quad power \quad in}\right) = 10\log\left(\frac{P_{out}}{P_{in}}\right)$$

- If resistive loads are involved,

$$P = \frac{\langle v^2(t)\rangle}{R} = \langle i^2(t)\rangle R = \frac{V_{rms}^2}{R} = I_{rms}^2 R = V_{rms} I_{rms}$$

Definition of dB may be reduced to,

$$dB = 20\log\left(\frac{V_{rms \quad out}}{V_{rms \quad in}}\right) + 10\log\left(\frac{R_{in}}{R_{load}}\right)$$

or

$$dB = 20\log\left(\frac{I_{rms \quad out}}{I_{rms \quad in}}\right) + 10\log\left(\frac{R_{load}}{R_{in}}\right)$$

# Decibel signal-to-noise ratio (SNR)

- **Definition.** The *decibel signal-to-noise ratio* (*SNR*) is:

$$(S/N)_{dB} = 10\log\left(\frac{P_{signal}}{P_{noise}}\right) = 10\log\left(\frac{\langle s^2(t)\rangle}{\langle n^2(t)\rangle}\right)$$

Where, Signal Power (S) = $\dfrac{\langle s^2(t)\rangle}{R} = \dfrac{V^2_{rms\quad signal}}{R}$

And, Noise Power (N) = $\dfrac{\langle n^2(t)\rangle}{R} = \dfrac{V^2_{rms\quad noise}}{R}$

$$(S/N)_{dB} = 20\log\left(\frac{V_{rms\quad signal}}{V_{rms\quad noise}}\right)$$

# Decibel with mili watt reference (dBm)

- **Definition.** The decibel power level with respect to 1 mW

$$\boxed{dBm = 10\log\left(\frac{actual \quad power \quad level \quad (watts)}{10^{-3}}\right)}$$

= 30 + 10 log (Actual Power Level (watts)

- Here the "m" in the dBm denotes a milliwatt reference.
- When a 1-W reference level is used, the decibel level is denoted dBW;
- when a 1-kW reference level is used, the decibel level is denoted dBk.

E.g.: If an antenna receives a signal power of 0.3W, what is the received power level in dBm?

$dBm$ = 30 + 10xlog(0.3) = 30 + 10x(-0.523)3 = 24.77 $dBm$

# Fourier Series Representation

- Projection of periodic signals onto basis functions
  - Periodic signal is a weighted sum of these basis functions
- Exponentials are used as basis functions for writing Fourier series
- Any periodic signal can be expressed as a sum of infinite number of exponentials (or sinusoids for real signals)

# Fourier Series

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \qquad n = 0, 1, 2, \ldots$$

$$c_n = \frac{1}{T_0} \int_{T_0} v(t) e^{-j2\pi n f_0 t}\, dt$$

$$c_n = |c_n|\, e^{j \arg c_n} \qquad c_n e^{j2\pi n f_0 t} = |c_n|\, e^{j \arg c_n} e^{j2\pi n f_0 t}$$

DC component: $c(0) = \dfrac{1}{T_0} \displaystyle\int_{T_0} v(t)\, dt = \langle v(t) \rangle$

$f_0 = \frac{1}{T_0}$ (fundamental frequency)

Line spectra at frequencies that are integer multiple of fundamental frequency

# Fourier series example:

$$v(t) = \begin{cases} A & |t| < \tau/2 \\ 0 & |t| > \tau/2 \end{cases}$$

Thus

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} v(t)e^{-j2\pi n f_0 t}\,dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} Ae^{-j2\pi n f_0 t}\,dt$$

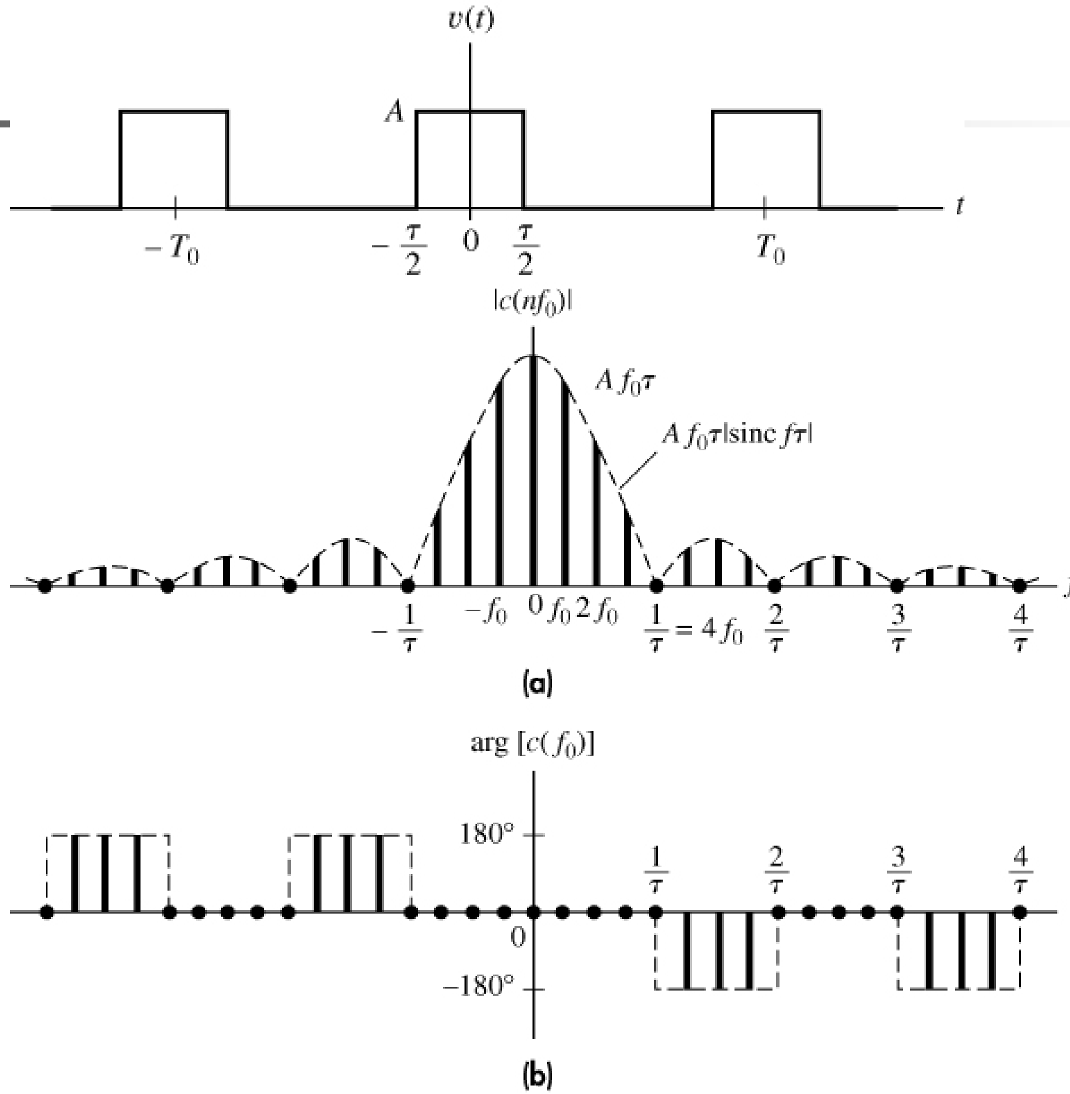
$$= \frac{A}{-j2\pi n f_0 T_0}\left(e^{-j\pi n f_0 \tau} - e^{+j\pi n f_0 \tau}\right)$$

$$= \frac{A}{T_0}\,\frac{\sin \pi n f_0 \tau}{\pi n f_0}$$

Multiplying and dividing by $\tau$ finally gives

$$c_n = \frac{A\tau}{T_0}\,\mathrm{sinc}\, n f_0 \tau$$

# Fourier Series: Example



# Fourier Transform

- Back to the Fourier series:

$$v(t) = \sum_{n=-\infty}^{\infty} c(nf_0) e^{j2\pi n f_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{1}{T_0} \int_{T_0} v(t) e^{-j2\pi n f_0 t} \, dt \right] e^{j2\pi n f_0 t}$$

$$v(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} v(t) e^{-j2\pi f t} \, dt \right] e^{j2\pi f t} \, df \qquad [3]$$

The bracketed term is the **Fourier transform** of $v(t)$ symbolized by $V(f)$ or $\mathcal{F}[v(t)]$ and defined as

$$V(f) = \mathcal{F}[v(t)] \triangleq \int_{-\infty}^{\infty} v(t) e^{-j2\pi f t} \, dt \qquad [4]$$

# Three major properties of V(f)

1. The Fourier transform is a complex function, so $|V(f)|$ is the amplitude spectrum of $v(t)$ and $\arg V(f)$ is the phase spectrum.
2. The value of $V(f)$ at $f = 0$ equals the **net area** of $v(t)$, since

$$V(0) = \int_{-\infty}^{\infty} v(t)\, dt \qquad [6]$$

   which compares with the periodic case where $c(0)$ equals the average value of $v(t)$.
3. If $v(t)$ is **real**, then

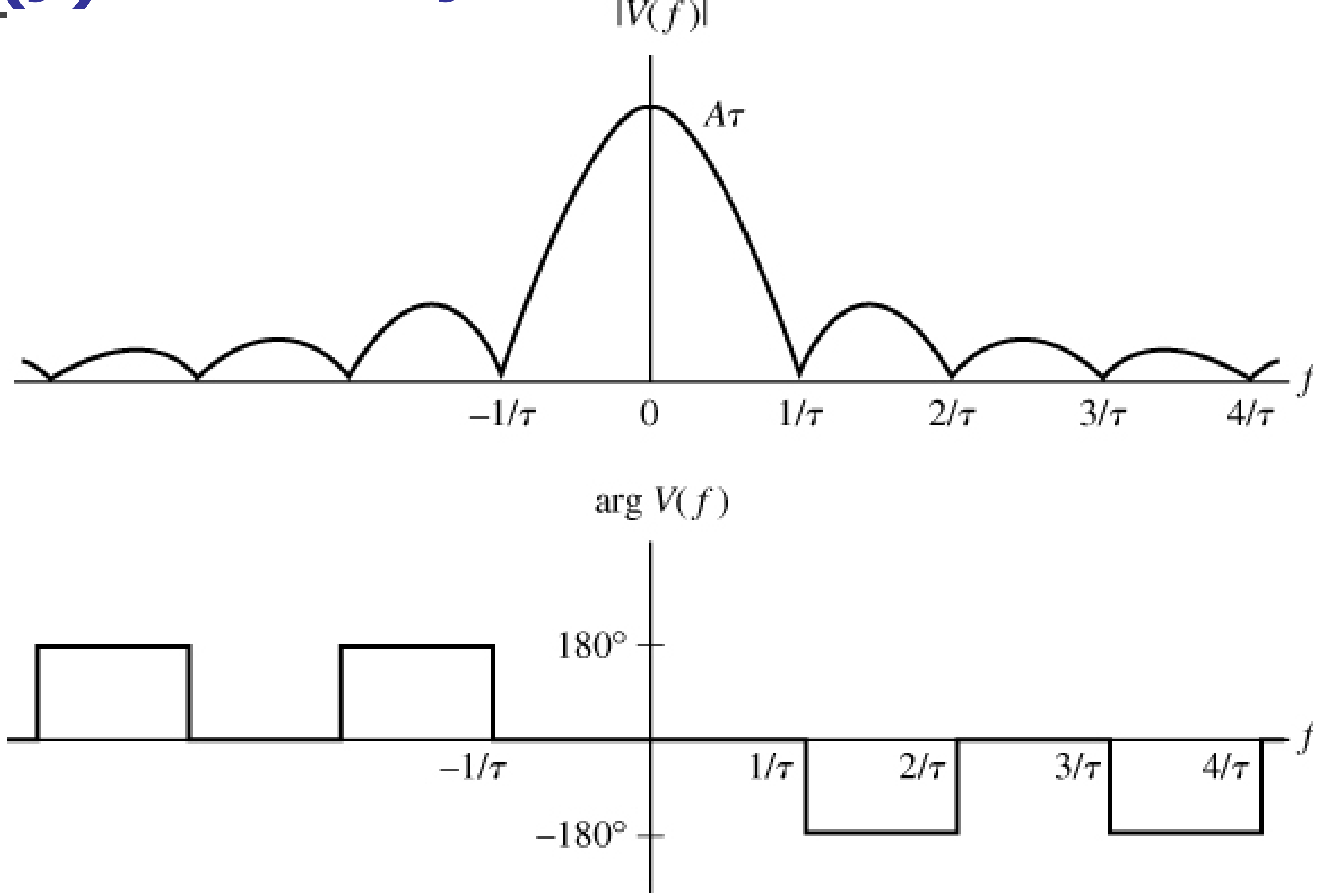
$$V(-f) = V^*(f) \qquad [7a]$$

   and

$$|V(-f)| = |V(f)| \qquad \arg V(-f) = -\arg V(f) \qquad [7b]$$

   so again we have even amplitude symmetry and odd phase symmetry. The term **hermitian symmetry** describes complex functions that obey Eq. (7).

# Rectangular pulse spectrum
# V(*f*) = Aτ sinc *f*τ



| Operation | Function | Fourier Transform |
|---|---|---|
| Linearity | $a_1 w_1(t) + a_2 w_2(t)$ | $a_1 W_1(f) + a_2 W_2(f)$ |
| Time delay | $w(t - T_d)$ | $W(f)\, e^{-j\omega T_d}$ |
| Scale change | $w(at)$ | $\frac{1}{\lvert a \rvert} W\left(\frac{f}{a}\right)$ |
| Conjugation | $w^*(t)$ | $W^*(-f)$ |
| Duality | $W(t)$ | $w(-f)$ |
| Real signal frequency translation [$w(t)$ is real] | $w(t)\cos(w_c t + \theta)$ | $\frac{1}{2}[e^{j\theta} W(f - f_c) + e^{-j\theta} W(f + f_c)]$ |
| Complex signal frequency translation | $w(t)\, e^{j\omega_c t}$ | $W(f - f_c)$ |
| Bandpass signal | $\mathrm{Re}\{g(t)\, e^{j\omega_c t}\}$ | $\frac{1}{2}[G(f - f_c) + G^*(-f - f_c)]$ |
| Differentiation | $\frac{d^n w(t)}{dt^n}$ | $(j2\pi f)^n W(f)$ |
| Integration | $\int_{-\infty}^{t} w(\lambda) d\lambda$ | $(j2\pi f)^{-1} W(f) + \frac{1}{2} W(0)\, \delta(f)$ |
| Convolution | $w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\lambda) \cdot w_2(t - \lambda)\, d\lambda$ | $W_1(f) W_2(f)$ |
| Multiplication[b] | $w_1(t) w_2(t)$ | $W_1(f) * W_2(f) = \int_{-\infty}^{\infty} W_1(\lambda)\, W_2(f - \lambda)\, d\lambda$ |
| Multiplication by $t^n$ | $t^n w(t)$ | $(-j2\pi)^{-n} \frac{d^n W(f)}{df^n}$ |

| Function | Time Waveform $w(t)$ | Spectrum $W(f)$ |
|---|---|---|
| Rectangular | $\Pi\left(\frac{t}{T}\right)$ | $T[\mathrm{Sa}(\pi f T)]$ |
| Triangular | $\Lambda\left(\frac{t}{T}\right)$ | $T[\mathrm{Sa}(\pi f T)]^2$ |
| Unit step | $u(t) \triangleq \begin{cases} +1, & t > 0 \\ 0, & t < 0 \end{cases}$ | $\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$ |
| Signum | $\mathrm{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$ | $\frac{1}{j\pi f}$ |
| Constant | $1$ | $\delta(f)$ |
| Impulse at $t = t_0$ | $\delta(t - t_0)$ | $e^{-j2\pi f t_0}$ |
| Sinc | $\mathrm{Sa}(2\pi W t)$ | $\frac{1}{2W}\,\Pi\left(\frac{f}{2W}\right)$ |
| Phasor | $e^{j(\omega_0 t + \varphi)}$ | $e^{j\varphi}\,\delta(f - f_0)$ |
| Sinusoid | $\cos(\omega_c t + \varphi)$ | $\frac{1}{2}e^{j\varphi}\,\delta(f - f_c) + \frac{1}{2}e^{-j\varphi}\,\delta(f + f_c)$ |
| Gaussian | $e^{-\pi(t/t_0)^2}$ | $t_0 e^{-\pi(f t_0)^2}$ |
| Exponential, one-sided | $\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$ | $\frac{T}{1 + j2\pi f T}$ |
| Exponential, two-sided | $e^{-\|t\|/T}$ | $\frac{2T}{1 + (2\pi f T)^2}$ |
| Impulse train | $\sum_{k=-\infty}^{k=\infty} \delta(t - kT)$ | $f_0 \sum_{n=-\infty}^{n=\infty} \delta(f - n f_0)$, where $f_0 = 1/T$ |

# Convolution

- The *convolution* of a waveform $w_1(t)$ with a waveform $w_2(t)$ to produce a third waveform $w_3(t)$ which is

$$\boxed{w_3(t) = w_1(t) * w_2(t) \equiv \int_{-\infty}^{\infty} w_1(\lambda) w_2(t-\lambda) d\lambda}$$

Evaluation of the integral involves 3 steps.

- Time reversal of $w_2$ to obtain $w_2(-\lambda)$,
- Time shifting of $w_2$ by $t$ seconds to obtain $w_2(-(\lambda-t))$, and
- Multiplying this result by $w_1$ to form the integrand $w_1(\lambda)w_2(-(\lambda-t))$.

Note: we denote a signal s(t) as a waveform w(t)

# Example for Convolution

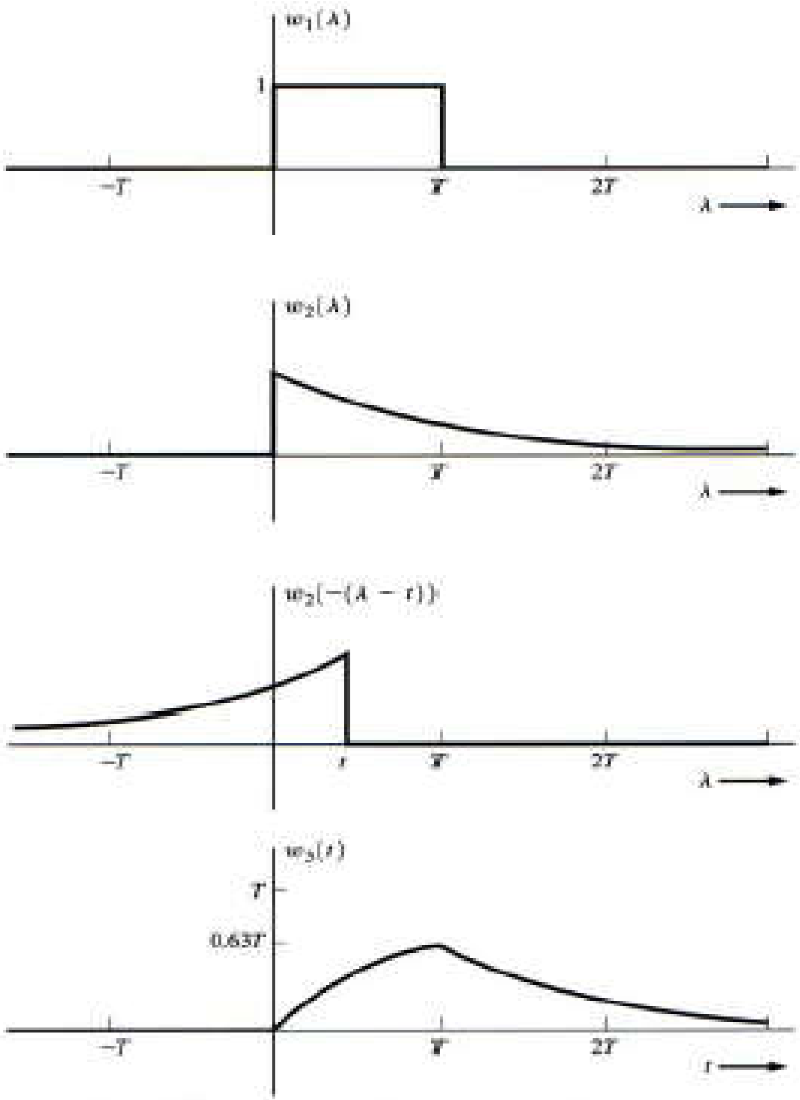



**Figure 2–7** Convolution of a rectangle and an exponential.

$$w_1(t) = \Pi\left(\frac{t - \frac{T}{2}}{T}\right)$$

$$w_2(t) = e^{-\frac{t}{T}}u(t)$$

For $0 < t < T$

$$w_3(t) = \int_0^t 1e^{+(\lambda - t)/T}\,d\lambda = T(1 - e^{-t/T})$$

For $t > T$

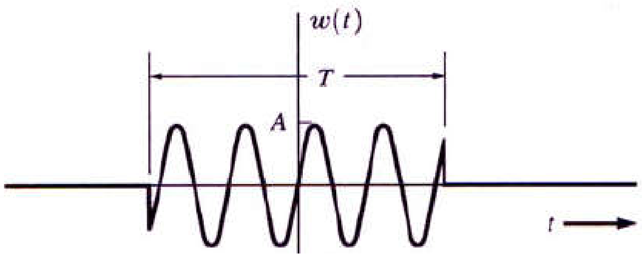
$$w_3(t) = \int_0^T 1e^{+(\lambda - t)/T}\,d\lambda = T(e - 1)\,e^{-t/T}$$

$$w_3(t) = \begin{cases} 0, & t < 0 \\ T(1 - e^{-t/T}), & 0 < t < T \\ T(e - 1)e^{-t/T}, & t > T \end{cases}$$

# Power Spectral Density (PSD)

- We define the truncated version of the waveform by:

$$w_T(t) = \begin{cases} w(t) & -T/2 < t < T/2 \\ 0 & t \quad elsewhere \end{cases} = w(t)$$




- The average normalized power:

$$\boxed{P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} w^2(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} w_T^2(t)dt}$$

- Using Parseval's theorem to calculate power from the frequency domain

$$\boxed{P = \lim_{f\to\infty} \frac{1}{f} \int_{-\infty}^{\infty} \left|W_f(f)\right|^2 dt = \int_{-\infty}^{\infty} \left( \lim_{f\to\infty} \frac{\left|W_f(f)\right|^2}{f} \right) df}$$

- **Definition:** The *Power Spectral Density* (PSD) for a deterministic power waveform is

$$\boxed{P_W(\mathrm{f}) = \lim_{f\to\infty} \frac{\left|W_f(f)\right|^2}{f}}$$

- where $w_T(t) \leftrightarrow W_T(f)$ and $P_w(f)$ has units of watts per hertz.
- The PSD is always a real nonnegative function of frequency.
- PSD is not sensitive to the phase spectrum of w(t)
- The normalized average power is

$$\boxed{P = \left\langle w^2(t) \right\rangle = \int_{-\infty}^{\infty} P_w(f)\,df}$$

- This means the area under the PSD function is the normalized average power.

# Autocorrelation Function

- **Definition:** The autocorrelation of a real (physical) waveform is

$$\boxed{R_w(\tau) \equiv \langle w(t)w(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w(t)w(t+\tau)dt}$$

- Wiener-Khintchine Theorem: PSD and the autocorrelation function are Fourier transform pairs;

$$\boxed{R_w(\tau) \leftrightarrow P_w(f)}$$

The PSD can be evaluated by either of the following two methods:

1. *Direct method:* by using the definition,
2. *Indirect method*: by first evaluating the autocorrelation function and then taking the Fourier transform:

$$P_w(f) = \Im\,[R_w(\tau)\,]$$

- The average power can be obtained by any of the four techniques.

$$\boxed{P = \langle w^2(t) \rangle = W_{rms}^2 = \int_{-\infty}^{\infty} P_w(f)df = R_w(0)}$$

# Normalized Power

**Theorem:** For a periodic waveform *w*(*t*), the normalized power is given by:

$$\boxed{P_w = \langle w^2(t) \rangle = \sum_{n=-\infty}^{n=\infty} |c_n|^2}$$

where the $\{c_n\}$ are the complex Fourier coefficients for the waveform.

**Proof:** For periodic $w(t)$, the Fourier series representation is valid over all time and one may evaluate the normalized power:

$$P_w = \left\langle \left( \sum_n c_n e^{jn\omega_0 t} \right)^2 \right\rangle = \left\langle \sum_n \sum_m c_n c_m^* e^{jn\omega_0 t} e^{-jm\omega_0 t} \right\rangle$$

$$= \sum_n \sum_m c_n c_m^* \left\langle e^{j(n-m)\omega_0 t} \right\rangle = \sum_n \sum_m c_n c_m^* \delta_{nm} = \sum_n c_n c_n^*$$

$$\boxed{P_w = \sum_n |c_n|^2}$$

# Power Spectral Density for Periodic Waveforms

**Theorem:** For a periodic waveform, the power spectral density (PSD) is given by

$$\boxed{P(f) = \sum_{n=-\infty}^{n=\infty} |c_n|^2 \delta(f - nf_0)}$$

where $T_0 = 1/f_0$ is the period of the waveform and $\{c_n\}$ are the corresponding Fourier coefficients for the waveform.

***Proof.*** Let $w(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$. Then the autocorrelation function of $w(t)$ is

$$R(\tau) = \langle w^*(t) w(t+\tau) \rangle = \left\langle \sum_{n=-\infty}^{\infty} c_n^* e^{-jn\omega_0 t} \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_0 (t+\tau)} \right\rangle$$

$$R(\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n^* c_m e^{jm\omega_0 \tau} \left\langle e^{j\omega_0 (m-n)t} \right\rangle$$

But $\left\langle e^{j\omega_0 (n-m)t} \right\rangle = \delta_{nm}$, so $\quad R(\tau) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jn\omega_0 \tau}$

$$P(f) = \Im[R(\tau)] = \Im\left[ \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jn\omega_0 \tau} \right]$$

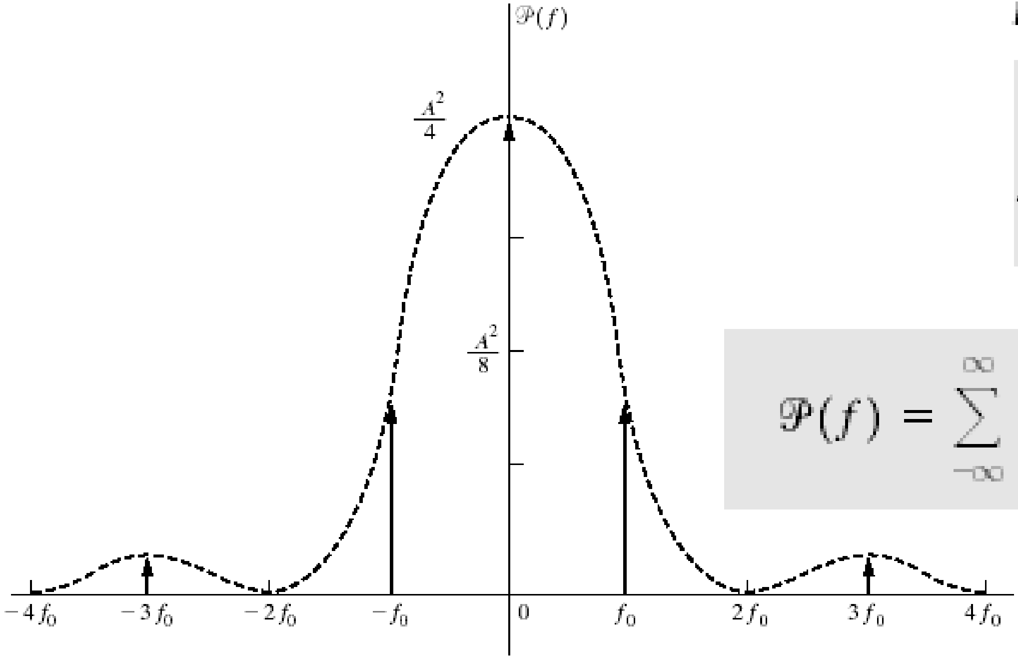
PSD is the FT of the Autocorrelation function

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \Im[e^{jn\omega_0 \tau}] = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

# Power Spectral Density for a Square Wave

- The PSD for the periodic square wave will be found.
- Because the waveform is periodic, FS coefficients can be used to evaluate the PSD. Consequently this problem becomes one of evaluating the FS coeffic

$$\mathcal{P}(f) = \sum_{n=-\infty}^{n=\infty} |c_n|^2 \, \delta(f - nf_0)$$



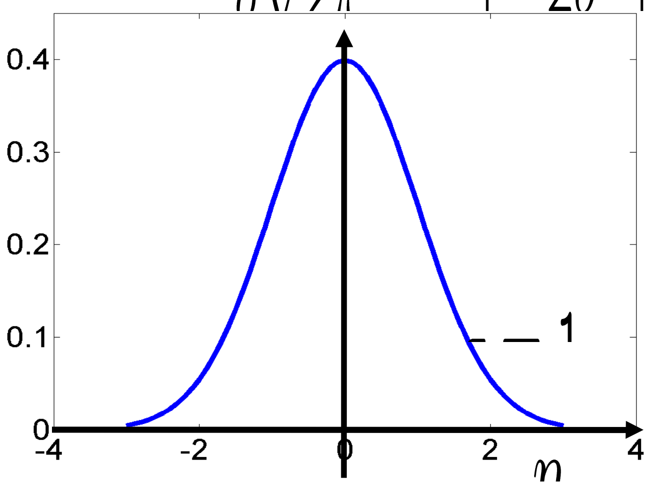

$$f_0) = \frac{A}{2} \, e^{-jn\pi/2} \, \frac{\sin(n\pi/2)}{n\pi/2}$$

$$\mathcal{P}(f) = \sum_{-\infty}^{\infty} \left(\frac{A}{2}\right)^2 \left(\frac{\sin(n\pi/2)}{n\pi/2}\right)^2 \delta(f - nf_0)$$

# Noise in communication systems

- Thermal noise is described by a zero-mean Gaussian random process, n(t).
- Its PSD is flat, hence, it is called white noise.

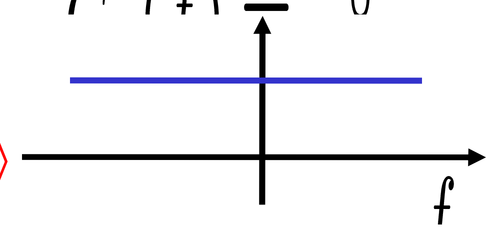
$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma^2}\right]$$




**Probability density function**

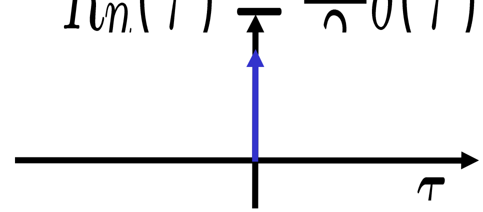
$$G_n(f) = \frac{N_0}{2} \text{ [w/Hz]}$$

**Power spectral density**

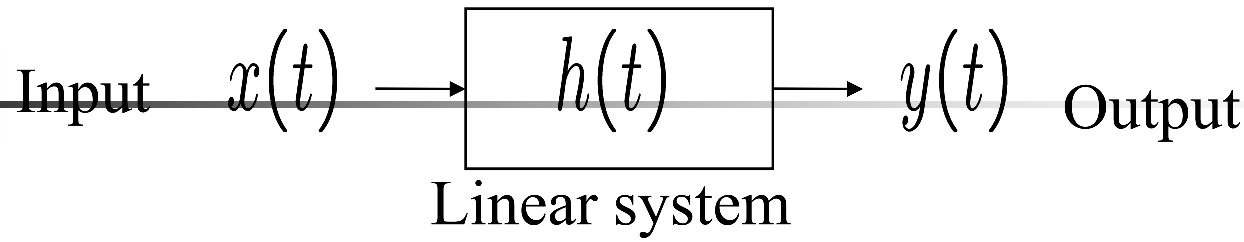



$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

**Autocorrelation function**



# Signal transmission through linear systems

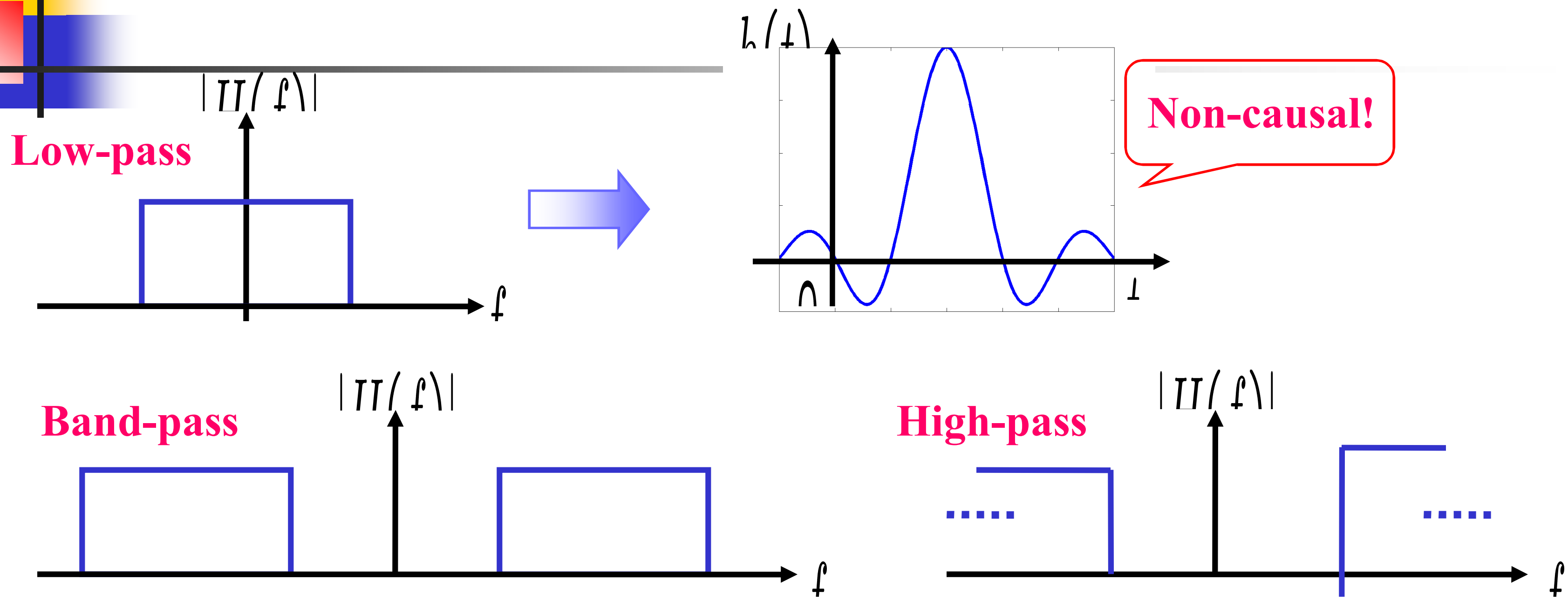



- Deterministic signals: $$Y(f) = X(f)H(f)$$
- Random signals: $$G_Y(f) = G_X(f)|H(f)|^2$$
- Ideal distortion less transmission:

All the frequency components of the signal not only arrive with an identical time delay, but also are amplified or attenuated equally.

$$y(t) = Kx(t - t_0) \text{ or } H(f) = Ke^{-j2\pi f t_0}$$

- Ideal filters:




- Realizable filters:
  - RC filters

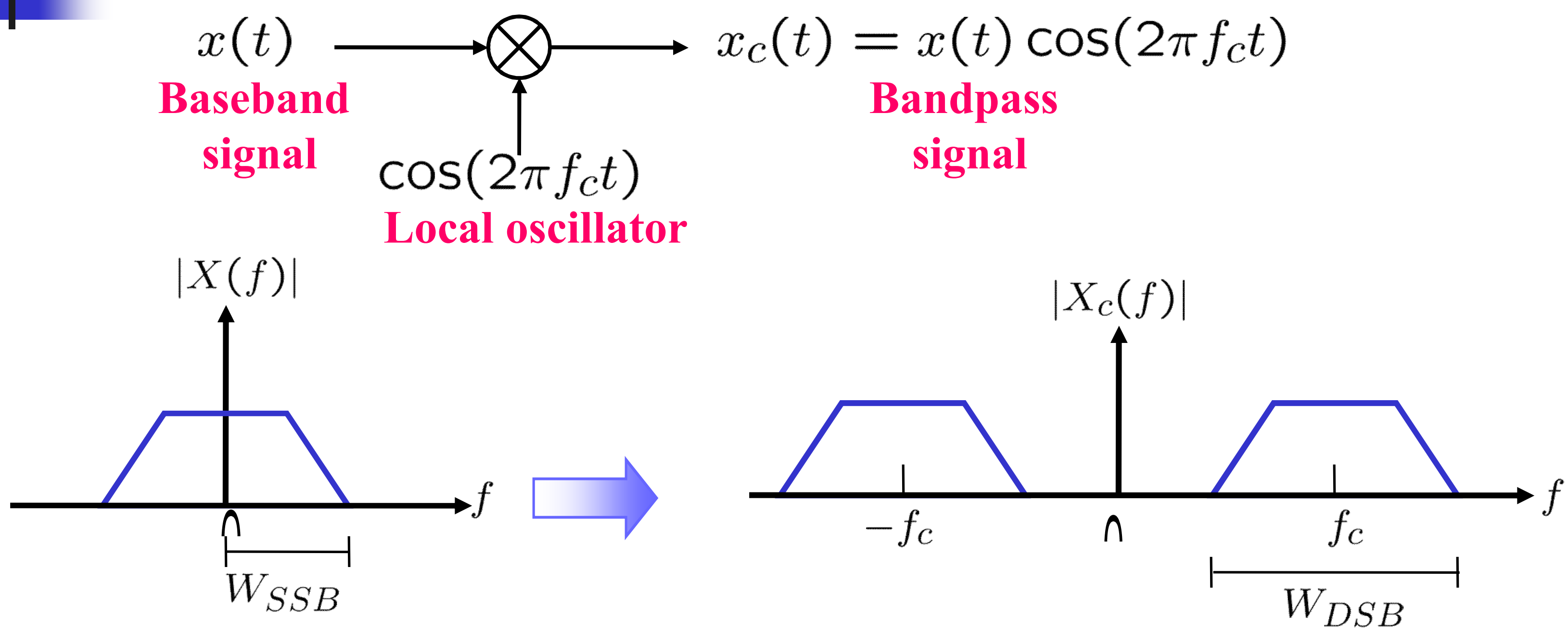
$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Butterworth filter

$$|H_n(f)| = \frac{1}{\sqrt{1 + (f/f_u)^{2n}}}$$

# Bandwidth of signal

## Baseband versus bandpass:



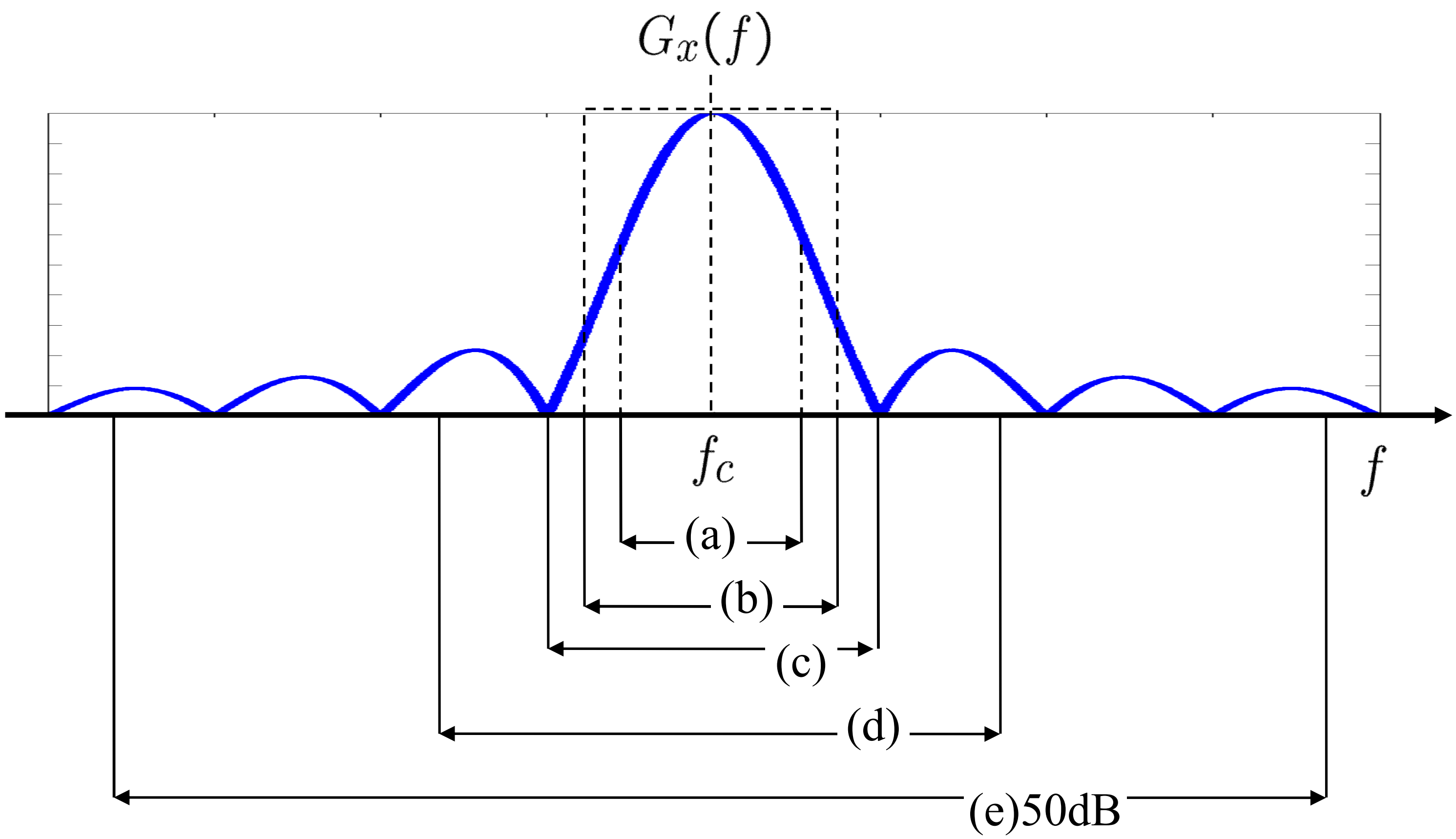

- Bandwidth dilemma:
- Bandlimited signals are not realizable!
- Realizable signals have infinite bandwidth!

# Bandwidth of signal …

- Different definition of bandwidth:

a) Half-power bandwidth
b) Noise equivalent bandwidth
c) Null-to-null bandwidth

a) Fractional power containment bandwidth
b) Bounded power spectral density
c) Absolute bandwidth



# Power Transfer Function

- Derive the relationship between the power spectral density (PSD) at the input, $P_x(f)$, and that at the output, $P_y(f)$ , of a linear time-invariant network.

Using the definition of PSD

$$\boxed{P_W(f) = \lim_{f\to\infty} \frac{|W_T(f)|^2}{f}}$$

PSD of the output is

$$\boxed{P_y(f) = \lim_{f\to\infty} \frac{|W_y(f)|^2}{f}}$$

Using transfer function

in a formal sense, we obtain

$$\boxed{P_y(f) = |H(f)|^2 \lim_{f\to\infty} \frac{|W_x(f)|^2}{f}}$$

$$P_y(f) = |H(f)|^2 P_x(f)$$

Thus, the power transfer function of the network is

$$\boxed{G_h(f) = \frac{P_y(f)}{P_x(f)} = |H(f)|^2}$$