# complexity

March 27, 2024

# 1 Lecture 6: Computational complexity, dynamic programming

- Time complexity: Big O notation
- Recursive functions
- Dynamic programming: Levenshtein distance

## 1.1 Midterm exam

- April 24, 10:15-11:00 (first half of the lecture), AND-3-02/06
- Pen-and-paper, multiple-choice and short text answers (no writing code)
- **Not** allowed: computers, documentation, slides, cheat sheet, any other material or devices
- More information on OLAT (``Exercise & Exam Info'')

## 1.2 Learning objectives

By the end of this lecture, you should:

- Understand what computational complexity is and why it is important
- Be able to determine and reduce the time complexity of simple algorithms
- Know the time complexity of some commonly used operations with `lists`, `sets`, and `dicts`
- Understand how recursion works and be able to write recursive functions
- Know what dynamic programming is and why it is useful
- Understand the dynamic programming algorithm for calculating the Levenshtein distance

### 1.2.1 Imports

[1]:
```
import random
import string
import timeit

import utils
```

## 1.3 How can we measure the efficiency of a program?

### 1.3.1 What resources does a program need?

- Time (seconds)
- Memory (bytes)
- Network data (megabits)
- Power (kilowatt-hours)

- …

#### 1.3.2 How to measure usage of these resources?

- **Benchmarking:** Measure how many resources the program uses in absolute units
  - Requires running the program (many times, maybe under different conditions)
  - Depends on input data, hardware, and other factors
- **Computational complexity:** Determine how quickly runtime increases with increasing input length
  - Based on inherent characteristics of the program
  - Requires theoretical analysis of the code
  - Independent of hardware (can be done with pen and paper)

#### 1.3.3 Two types of computational complexity

- **Time complexity:** How complex is our program in terms of the **time** it takes to run?
- **Space complexity:** How complex is our program in terms of the **memory** it takes to run?

Computational complexity tells us how **scalable** our algorithms are (e.g., with increasing corpus size, document length, vocabulary size, etc.)

### 1.4 Time complexity

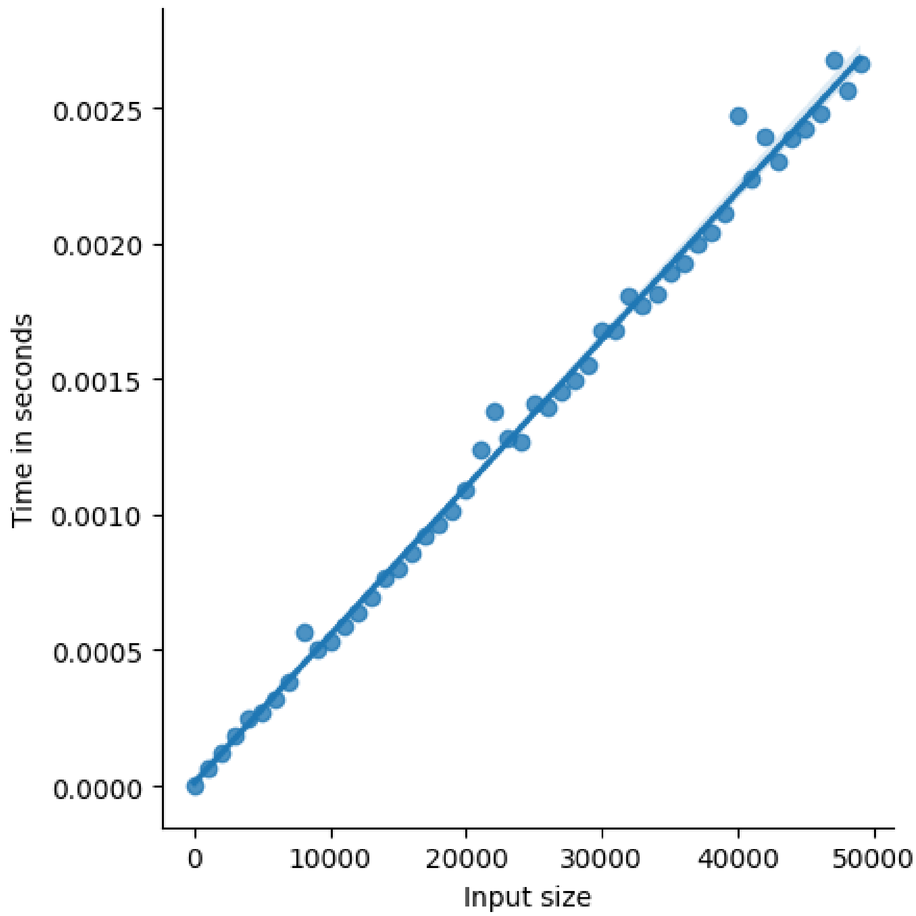
Given an algorithm, how quickly does the **number of operations** grow when we increase the **input length**?

1. For each operation, count how many times it is called
2. Sum up the counts
3. Keep only highest-order terms, ignore constant factors

[2]:
```
def minimum(numbers):
    min_number = float("inf")     # Called 1 time
    for number in numbers:
        if number < min_number:   # Called n times
            min_number = number   # Called n times
    return min_number
```

- Total number of operations: $2n + 1$
- Drop lower-order terms and constant factors $\rightarrow n$
- **Time complexity:** $O(n)$
  $\rightarrow$ Runtime increases **linearly** with length of the input ($n$)

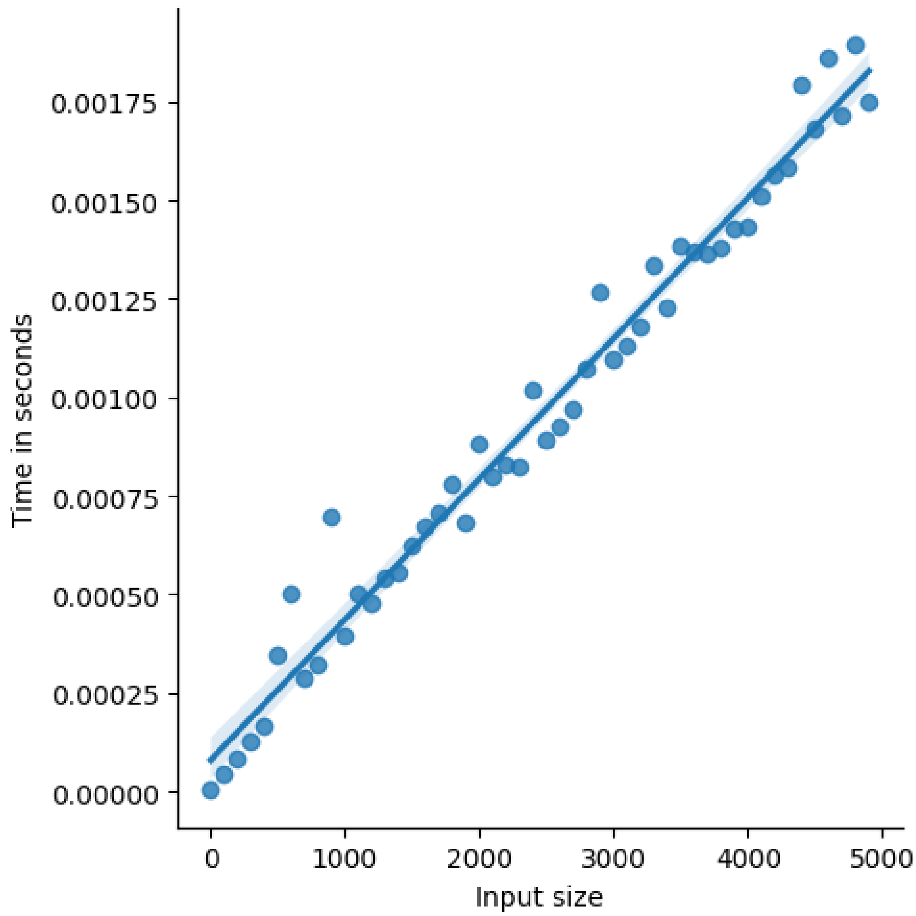
[3]:
```
random_numbers = [random.randint(0, 100) for _ in range(50000)]
utils.plot_time_complexity(minimum, random_numbers, regression_order=1)
```




```
[4]: def optimized_minimum(numbers):
         min_number = float("inf")          # Called 1 time
         for number in numbers:
             if number == -float("inf"):    # Called n times (worst case)
                 return number
             if number < min_number:        # Called n times (worst case)
                 min_number = number        # Called n times (worst case)
         return min_number
```

- In the **best case** (if `numbers[0] == -inf`), we only have 2 operations
- But in the **worst case**, we have $3n + 1$ operations
- Big $O$ notation always assumes the **worst case** scenario
  $\rightarrow$ Time complexity is still $O(n)$

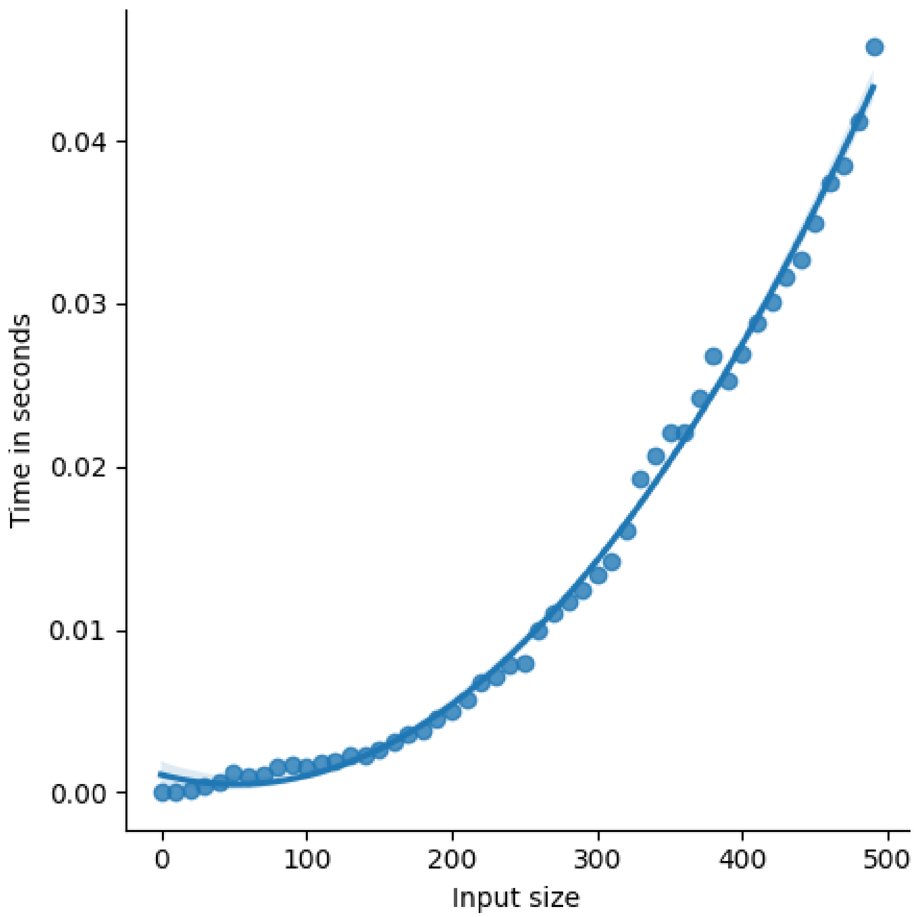
```
[5]: random_numbers = [random.randint(0, 100) for _ in range(5000)]
     utils.plot_time_complexity(optimized_minimum, random_numbers,␣
      ↪regression_order=1)
```




[6]:
```python
def pairwise_sums(numbers):
    """Calculate the sums of all possible pairs of numbers in a list."""
    sums = []                                            # Called 1 time
    for i in range(len(numbers)):
        for j in range(len(numbers)):
            sums.append(numbers[i] + numbers[j])  # Called n² times
    return sums
```

- Total number of operations: $n^2 + 1$
- Drop lower-order terms and constant factors $\rightarrow n^2$
- **Time complexity:** $O(n^2)$
  $\rightarrow$ Runtime increases **quadratically** with length of the input $(n)$

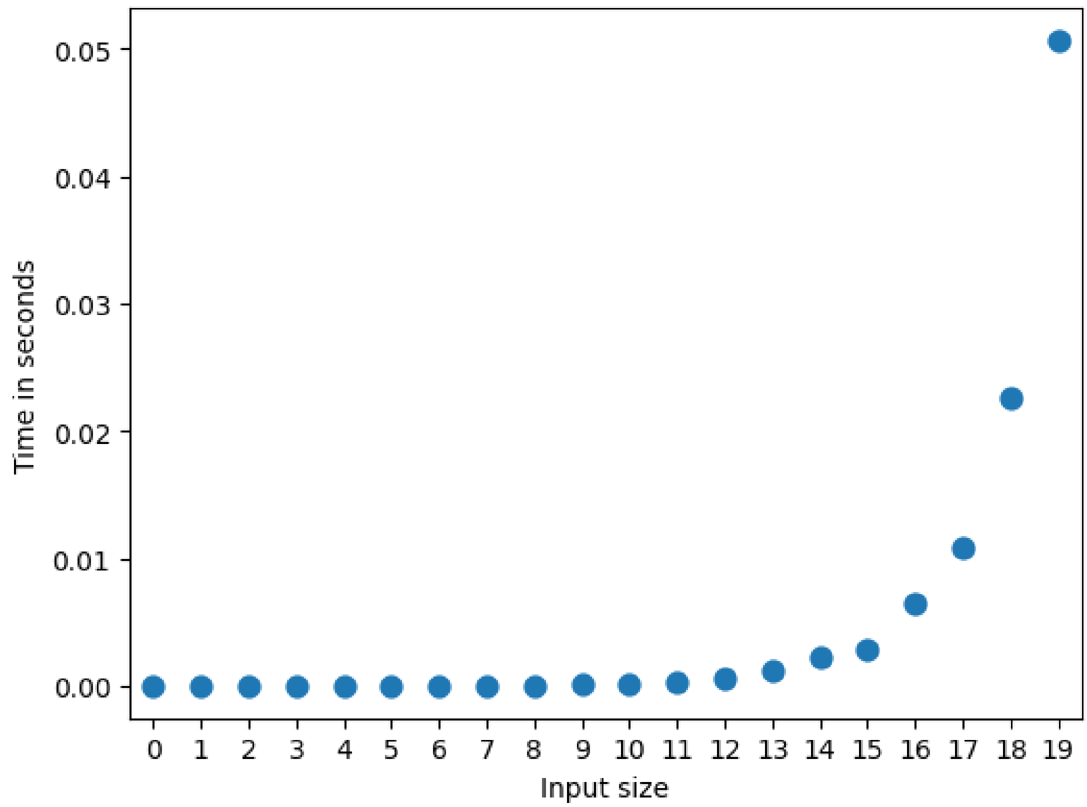
[7]:
```python
utils.plot_time_complexity(pairwise_sums, list(range(500)), regression_order=2)
```




[8]:
```
def subset_sums(numbers):
    """Calculate the sums of all possible subsets of a list."""
    sums = [0]                                  # Called 1 time
    for number in numbers:
        new_sums = []                           # Called n times
        for sum in sums:
            new_sums.append(sum + number)   # Called 2  - 1 times
        sums.extend(new_sums)                   # Called n times
    return sums
```

- Total number of operations: $2^n + 2n$
- Drop lower-order terms and constant factors $\rightarrow 2^n$
- **Time complexity:** $O(2^n)$
  $\rightarrow$ Runtime increases **exponentially** with length of the input ($n$)

[10]:
```
utils.plot_time_complexity(subset_sums, list(range(20)))
```




### 1.4.1 Common time complexity classes

Source: bigocheatsheet.com

### 1.4.2 Remember

- We are not interested in absolute runtime (which depends on hardware)
  → Constant factors are irrelevant
- We are interested in how quickly runtime increases as inputs become very large
  → Lower-order terms become negligible

### 1.4.3 Quiz: Time complexity

pwa.klicker.uzh.ch/join/asaeub

### 1.4.4 Example: Finding duplicate strings

**OpenSubtitles**

- Movie subtitles in many languages
- Available in a cleaner, parallelized version as part of the OPUS corpus
  *The German part can be downloaded as plain text here*
- Commonly used for machine translation

- Subtitles are usually short and contain a lot of duplicates

[11]:
```python
with open('de.txt', 'r') as f:
    lines = f.readlines()
len(lines)
```

[11]: 41612280

[12]:
```python
lines[:10]
```

[12]:
```
['Ich geh lieber wieder an die Arbeit.\n',
 'Verspielt nicht alle Streichhölzer…\n',
 '- Hallo, Mac.\n',
 '- Hallo, Click.\n',
 'Tag, zusammen.\n',
 '- Hallo.\n',
 '- Hallo.\n',
 'Willkommen zu Hause, Mann.\n',
 'Komm, setz dich und spiel uns was vor.\n',
 '- Wir zahlen mit Versprechen.\n']
```

**A naive approach**

[13]:
```python
def get_duplicates_naive(lines):
    duplicates = set()
    for i1, line1 in enumerate(lines):
        for i2, line2 in enumerate(lines):
            if line1 == line2 and i1 != i2:
                duplicates.add(line1)
    return duplicates
```

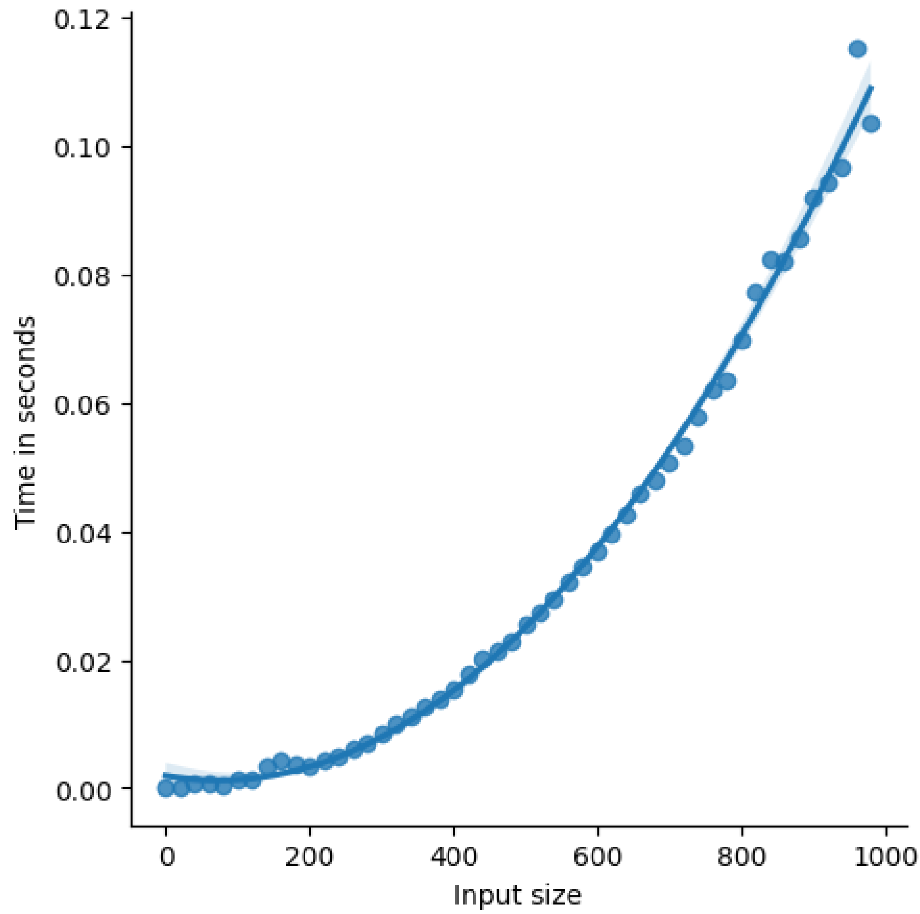
[14]:
```python
get_duplicates_naive(lines[:500])
```

[14]:
```
{'- Gut.\n',
 '- Hallo.\n',
 '- Ja.\n',
 '- Morgen.\n',
 '- Nein.\n',
 '- Und wenn?\n',
 'Danke.\n',
 'Grant.\n',
 'Hier.\n',
 'Ja.\n',
 'Lass mich los!\n',
 'Nein.\n',
 'Weit reisen kannst du nur auf Gleisen\n',
 'Wieso?\n',
 'Wirklich?\n'}
```

[16]:
```
timeit.timeit(lambda: get_duplicates_naive(lines[:10000]), number=1)
```

[16]: 2.4275523179999254

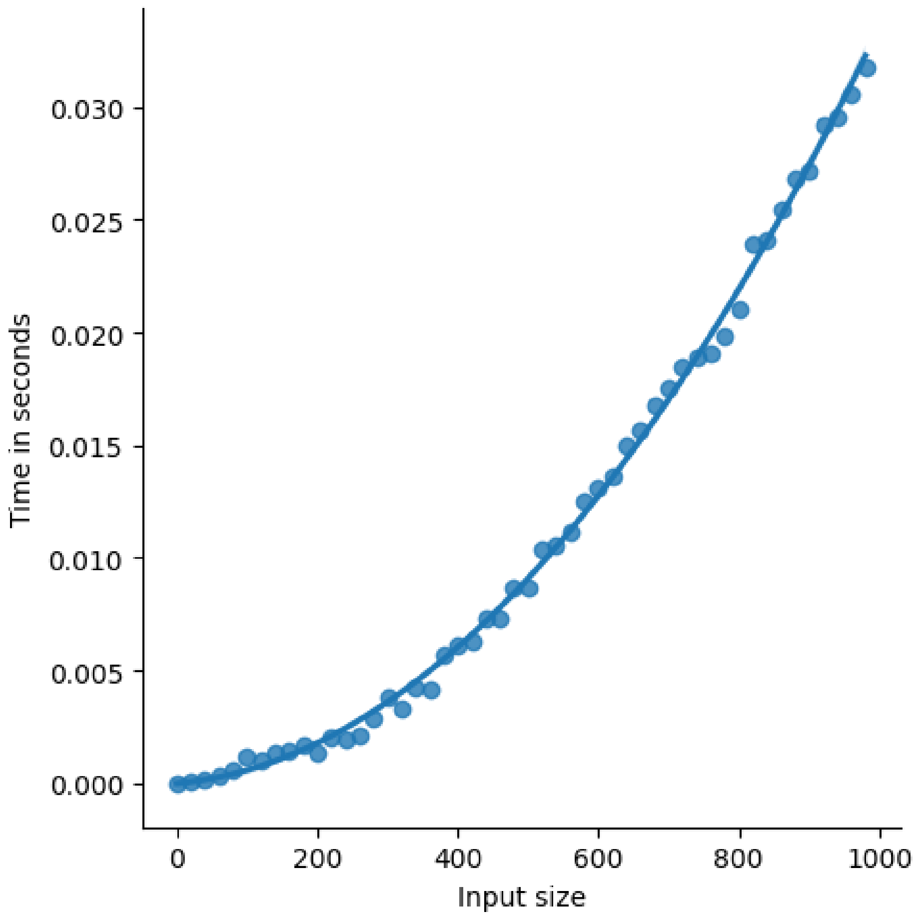
[17]:
```
utils.plot_time_complexity(get_duplicates_naive, lines[:1000],␣
↪regression_order=2)
```




**A better approach?**

[18]:
```
def get_duplicates_maybe_better(lines):
    duplicates = set()
    for line in lines:
        count = lines.count(line)
        if count > 1:
            duplicates.add(line)
    return duplicates
```

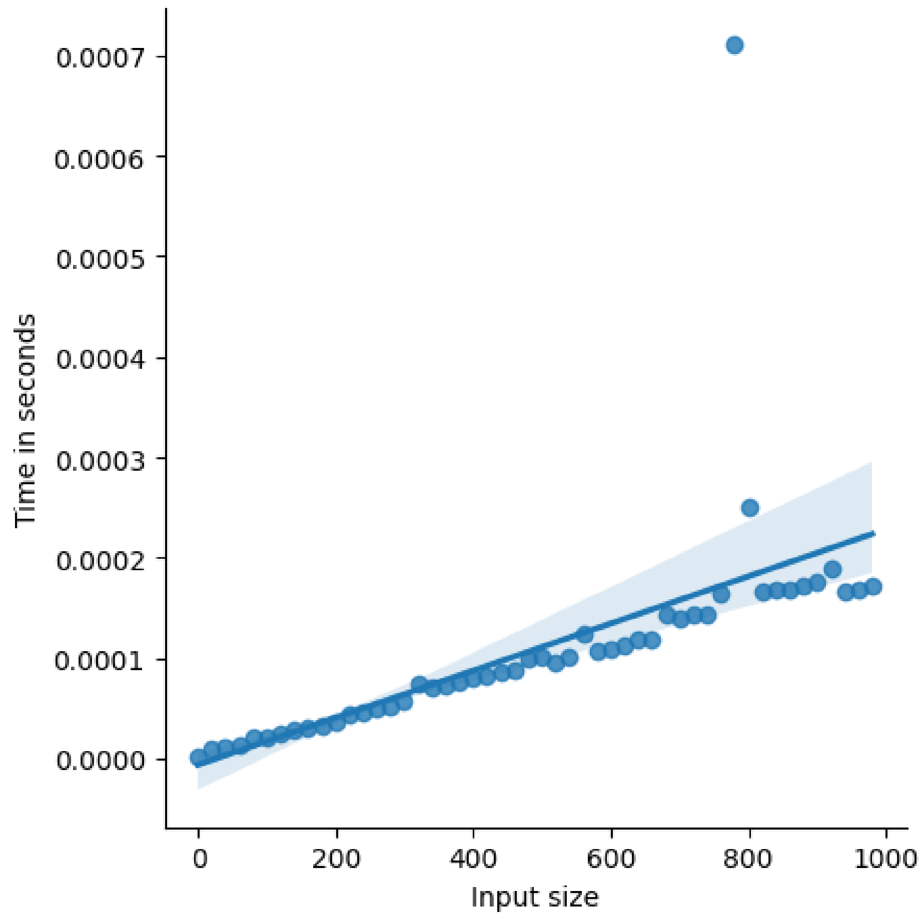
[19]:
```
utils.plot_time_complexity(get_duplicates_maybe_better, lines[:1000],␣
↪regression_order=2)
```




**Actually a better approach**

[20]:
```
def get_duplicates_really_better(lines):
    lines_set = set()
    duplicates = set()
    for line in lines:
        if line in lines_set:
            duplicates.add(line)
        else:
            lines_set.add(line)
    return duplicates
```

[23]:
```
utils.plot_time_complexity(get_duplicates_really_better, lines[:1000],␣
 ↪regression_order=1)
```




#### 1.4.5 Time complexity in `lists`

Due to the way `list` is implemented in Python, the following methods need to iterate over all elements (in the worst case):

- `list.count()`
- `list.index()`
- `list.__contains__()`

Their time complexity is $O(n)$ (= linear).

**Overview: Time complexity in `lists`**

| Method | Time complexity |
|---|---|
| `append(x)` | $O(1)$ |
| `__getitem__(i)` | $O(1)$ |
| `__len__()` | $O(1)$ |
| `pop()` | $O(1)$ |
| `pop(0)` | $O(n)$ |

| Method | Time complexity |
|---|---|
| `remove(x)` | $O(n)$ |
| `insert(i, x)` | $O(n)$ |
| `__contains__(x)` | $O(n)$ |
| `count(x)` | $O(n)$ |
| `reverse()` | $O(n)$ |
| `sort()` | $O(n \log n)$ |

More details: wiki.python.org/moin/TimeComplexity

### 1.4.6 Time complexity in `dicts` and `sets`

`dict` and `set` are implemented using **hash tables**. These are very efficient for looking up values:

- `set.__contains__()`
- `dict.__getitem__()`

These methods have time complexity $O(1)$ (= constant).

**Overview: Time complexity in `sets`**

| Method | Time complexity |
|---|---|
| `add(x)` | $O(1)$* |
| `pop()` | $O(1)$ |
| `__len__()` | $O(1)$ |
| `__contains__()` | $O(1)$ |

**Overview: Time complexity in `dicts`**

| Method | Time complexity |
|---|---|
| `__setitem__(x)` | $O(1)$* |
| `__getitem__(x)`, `get(x)` | $O(1)$ |
| `pop()` | $O(1)$ |
| `__len__()` | $O(1)$ |
| `__contains__()` | $O(1)$ |

* assuming no hash collisions

More details: wiki.python.org/moin/TimeComplexity

## 1.5 Space complexity

- Big $O$ notation can also be used for **memory usage**
- Same principle: we look at the implementation of the algorithm and figure out how much memory is used in the **worst case** (not by running the code)

### 1.5.1 Example: Finding the $k$ longest strings

[ ]:
```python
def longest_naive(strings, k=3):
    return sorted(strings, key=len)[-k:]

longest_naive(['a', 'ab', 'abc', 'abcd', 'abcde'])
```

- `sorted()` creates a new list of size $n$
- The return value is a list of size $k$
- **Space complexity:** $O(n + k)$
- Time complexity: $O(n \log n)$

[ ]:
```python
def longest_better(strings, k=3):
    longest = []
    for string in strings:
        if len(longest) < k:
            longest.append(string)
        else:
            shortest_longest = min(longest, key=len)
            if len(string) > len(shortest_longest):
                longest.remove(shortest_longest)
                longest.append(string)
    return longest

longest_better(['a', 'ab', 'abc', 'abcd', 'abcde'])
```

- The auxiliary list `longest` has size $k$
- The return value has size $k$
- Everything else requires only constant space
- **Space complexity:** $O(k)$
- Time complexity: ?

## 1.6 Recursive functions

**Problem:** Calculate the sum of numbers in arbitrarily nested data structures like this:

[24]:
```python
data = [1, 2, [3, 4], 5, [6, [7, 8]]]
```

This won't work:

[25]:
```python
sum(data)
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call last)
Cell In[25], line 1
----> 1 sum(data)

TypeError: unsupported operand type(s) for +: 'int' and 'list'
```

**Solution:** Recursively sum up elements of nested lists:

[26]:
```
def deepsum(data):
    total = 0
    for item in data:
        if isinstance(item, list):
            total += deepsum(item)  # Recursive call
        else:
            total += item           # Termination
    return total
```

[27]:
```
deepsum([1, 2, [3, 4], 5, [6, [7, 8]]])
```

[27]: 36

How many times was `deepsum` called?

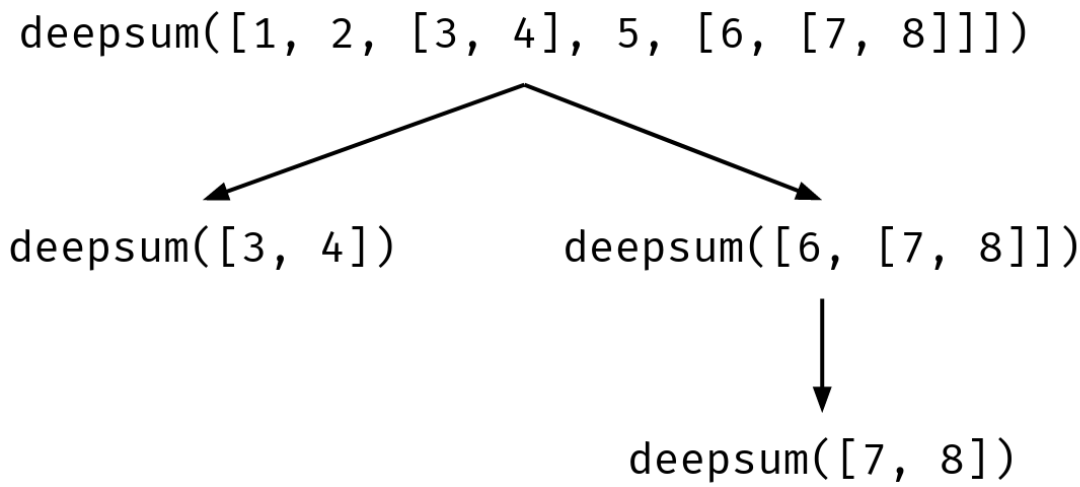
[28]:
```
call_counter = utils.CallCounter()

@call_counter.register
def deepsum(data):
    total = 0
    for item in data:
        if isinstance(item, list):
            total += deepsum(item)
        else:
            total += item
    return total

deepsum([1, 2, [3, 4], 5, [6, [7, 8]]])
call_counter.print_most_common()
```

```
1       deepsum([1, 2, [3, 4], 5, [6, [7, 8]]])
1       deepsum([3, 4])
1       deepsum([6, [7, 8]])
1       deepsum([7, 8])
```

### 1.6.1 Recursion tree




What about the time complexity of `deepsum`?

The deeper the data structure, the longer the runtime: - `deepsum([[1], [[2], [[3]]]])` takes longer than `deepsum([1, 2, 3])`

The broader the data structure, the longer the runtime: - `deepsum([1, 2, 3, 4, 5, 6])` takes longer than `deepsum([1, 2, 3])`

→ Runtime depends on number of elements and depth: $O(n \times d)$

## 1.7 Levenshtein distance

How to turn `zebra` into `amoeba`?

- **Edit operations**: we can *insert*, *delete*, or *replace* letters
- Every edit operation comes with a **cost**
- The **edit distance** is the smallest possible cost to get from word A to word B
- The most common variant is the **Levenshtein distance** and defines:

| Edit operation | Cost |
|---|---|
| Insertion | 1 |
| Deletion | 1 |
| Substitution | 1 |

→ Levenshtein distance = number of edit operations

### 1.7.1 `zebra` → `amoeba`: naive approach

1. Replace `z` with `a` → costs 1
2. Replace `e` with `m` → costs 1
3. Replace `b` with `o` → costs 1
4. Replace `r` with `e` → costs 1

5. Replace `a` with `b` → costs 1
6. Insert `a` → costs 1

**Total cost: 6** → Can we do better?

### 1.7.2 `zebra` → `amoeba`: optimal solution

1. Replace `z` with `a` → costs 1
2. Insert `m` → costs 1
3. Insert `o` → costs 1
4. Keep `e`
5. Keep `b`
6. Delete `r` → costs 1
7. Keep `a`

**Total cost: 4** (= Levenshtein distance)

### 1.7.3 Quiz: Levenshtein distance

[pwa.klicker.uzh.ch/join/asaeub](pwa.klicker.uzh.ch/join/asaeub)

### 1.7.4 A convenient property of the Levenshtein distance problem

We can derive the Levenshtein distance of the **full strings** from the Levenshtein distance between some **substrings**.

For example, if we already know the following:

- **levenshtein(`zebra` → `amoeb`) = 5**
- **levenshtein(`zebr` → `amoeba`) = 4**
- **levenshtein(`zebr` → `amoeb`) = 4**

Then we can easily get **levenshtein(`zebra` → `amoeba`)**.

1. Suppose we already know that **levenshtein(`zebra` → `amoeb`) = 5**
   → Turning `zebra` into `amoeba` is possible with **1 additional edit operation** (inserting `a`)
   → Total cost: **6**

2. Suppose we already know that **levenshtein(`zebr` → `amoeba`) = 4**
   → Turning `zebra` into `amoeba` is possible with **1 additional edit operation** (deleting `a`)
   → Total cost: **5**

3. Suppose we already know that **levenshtein(`zebr` → `amoeb`) = 4**
   → Turning `zebra` into `amoeba` is possible **without additional edit operations** (keeping `a`)
   → Total cost: **4**

Solution 3 is the cheapest, and there are no other solutions.

Therefore, **levenshtein(`zebra` → `amoeba`) = 4**

### 1.7.5 Recursive definition of Levenshtein distance

$$\text{levenshtein}(a,b) = \begin{cases} |a| & \text{if } |b| = 0, \\ |b| & \text{if } |a| = 0, \\ \text{levenshtein}\left(a[:-1], b[:-1]\right) & \text{if } a[-1] = b[-1], \\ 1 + \min \begin{cases} \text{levenshtein}\left(a, b[:-1]\right) \\ \text{levenshtein}\left(a[:-1], b\right) \\ \text{levenshtein}\left(a[:-1], b[:-1]\right) \end{cases} & \text{otherwise} \end{cases}$$

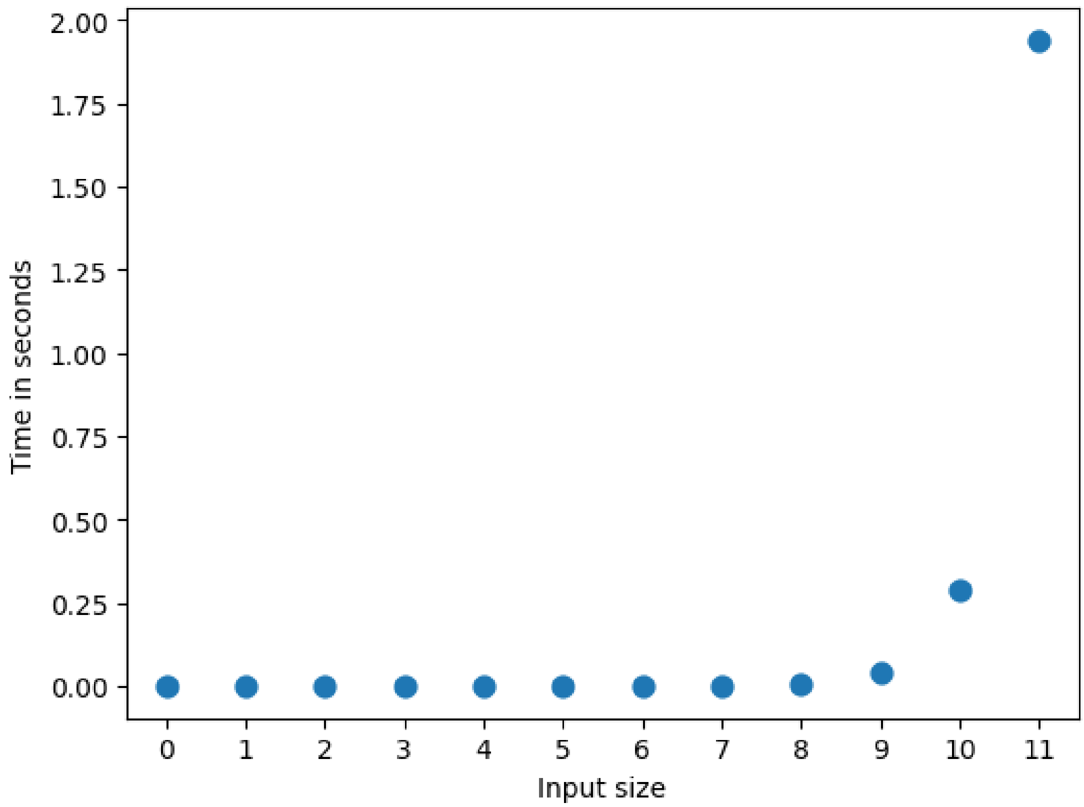
[29]:
```
def levenshtein(a: str, b: str) -> int:
    if a == "":
        return len(b)                            # Termination
    if b == "":
        return len(a)                            # Termination
    if a[-1] == b[-1]:
        return levenshtein(a[:-1], b[:-1])  # Recursive call
    return 1 + min(
        levenshtein(a, b[:-1]),                 # Recursive call
        levenshtein(a[:-1], b),                 # Recursive call
        levenshtein(a[:-1], b[:-1]),            # Recursive call
    )

levenshtein("zebra", "amoeba")
```

[29]: 4

What is the time complexity of the recursive Levenshtein distance algorithm?

[30]:
```
random_string = "".join(random.choices(string.ascii_letters, k=12))
utils.plot_time_complexity(lambda x: levenshtein(x, "".join(reversed(x))),␣
 ↪random_string, number=1)
```

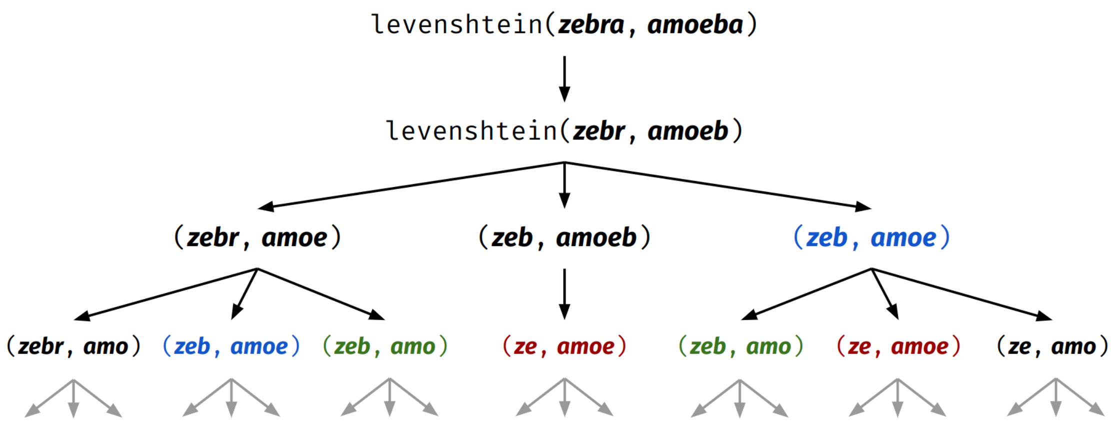



Why is it so bad?

[31]:
```
call_counter = utils.CallCounter()
levenshtein = call_counter.register(levenshtein)
levenshtein("zebra", "amoeba")
call_counter.print_most_common()
```

```
108     levenshtein('', 'a')
107     levenshtein('z', '')
77      levenshtein('z', 'a')
77      levenshtein('', '')
40      levenshtein('', 'am')
38      levenshtein('ze', '')
31      levenshtein('z', 'am')
30      levenshtein('ze', 'a')
16      levenshtein('ze', 'am')
9       levenshtein('zeb', '')
9       levenshtein('z', 'amo')
9       levenshtein('', 'amo')
8       levenshtein('zeb', 'a')
6       levenshtein('zeb', 'am')
6       levenshtein('ze', 'amo')
```

```
4       levenshtein('zeb', 'amo')
3       levenshtein('ze', 'amoe')
2       levenshtein('zeb', 'amoe')
1       levenshtein('zebra', 'amoeba')
1       levenshtein('zebr', 'amoeb')
1       levenshtein('zebr', 'amoe')
1       levenshtein('zebr', 'amo')
1       levenshtein('zebr', 'am')
1       levenshtein('zebr', 'a')
1       levenshtein('zebr', '')
1       levenshtein('zeb', 'amoeb')
```




## 1.8 More efficient implementation of Levenshtein distance

*(See `levenshtein.pdf` or video on OLAT)*

Good online demo: https://phiresky.github.io/levenshtein-demo/

[32]:
```python
def levenshtein_recursive(a: str, b: str) -> int:
    """Return the Levenshtein distance between two strings using recursion."""
    if a == "":
        return len(b)
    if b == "":
        return len(a)
    if a[-1] == b[-1]:
        return levenshtein_recursive(a[:-1], b[:-1])
    return 1 + min(
        levenshtein_recursive(a, b[:-1]),
        levenshtein_recursive(a[:-1], b),
        levenshtein_recursive(a[:-1], b[:-1]),
    )

levenshtein_recursive("zebra", "amoeba")
```

[32]: 4

[33]:
```python
def levenshtein_dynamic(a: str, b: str) -> int:
    """Return the Levenshtein distance between two strings using dynamic␣
↪programming."""
    # Initialize table
    table = [[0] * (len(b) + 1) for _ in range(len(a) + 1)]
    for i in range(len(a) + 1):
        table[i][0] = i
    for j in range(len(b) + 1):
        table[0][j] = j
    # Fill table
    for i in range(1, len(a) + 1):
        for j in range(1, len(b) + 1):
            if a[i - 1] == b[j - 1]:
                table[i][j] = table[i - 1][j - 1]  # Keep
            else:
                table[i][j] = 1 + min(
                    table[i][j - 1],      # Insert
                    table[i - 1][j],      # Delete
                    table[i - 1][j - 1],  # Replace
                )
    # Solution in the bottom right corner
    return table[-1][-1]

levenshtein_dynamic("zebra", "amoeba")
```

[33]: 4

### 1.9 Dynamic programming

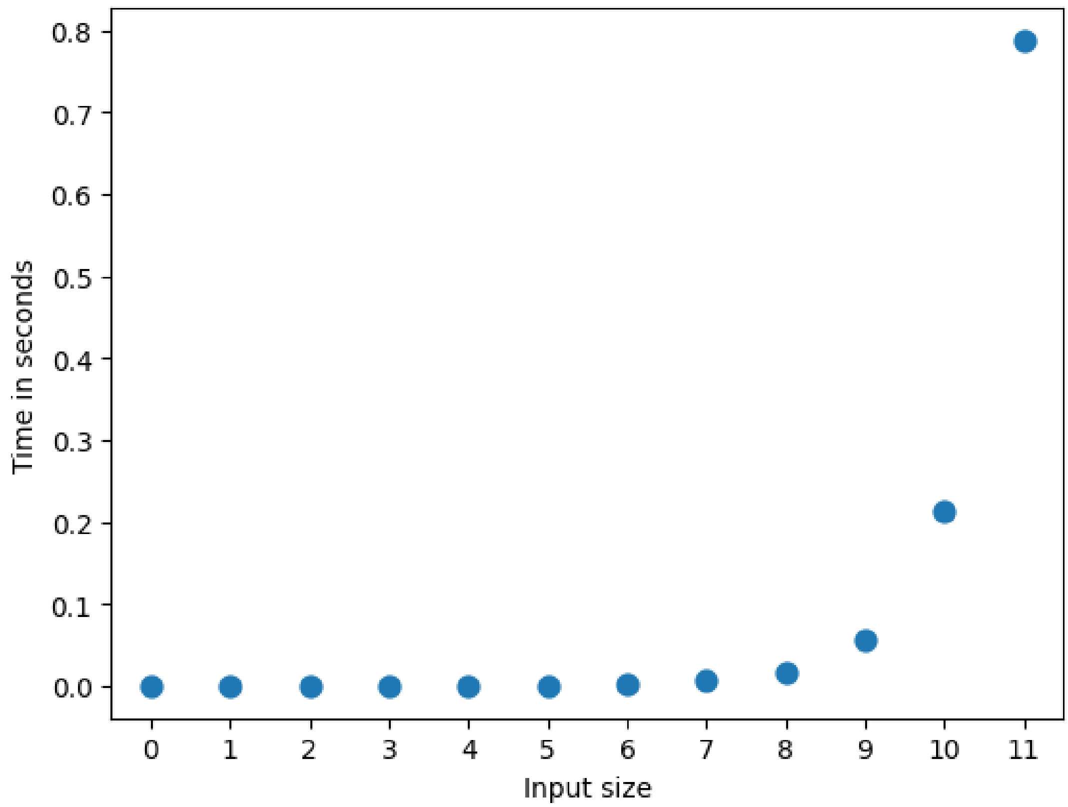
- This tabular approach of finding the edit distance is an example of **dynamic programming**
- **Some recursive problems** can be solved more efficiently using dynamic programming

Requirements for applying dynamic programming:

- The problem can be divided into **subproblems**
  *Example: Edit distance between strings > edit distance between substrings*
- The optimal solution for the problem can be **derived from optimal solutions for the subproblems**
  *Example: If we know the edit distance between all substrings, we know the edit distance between the full strings*

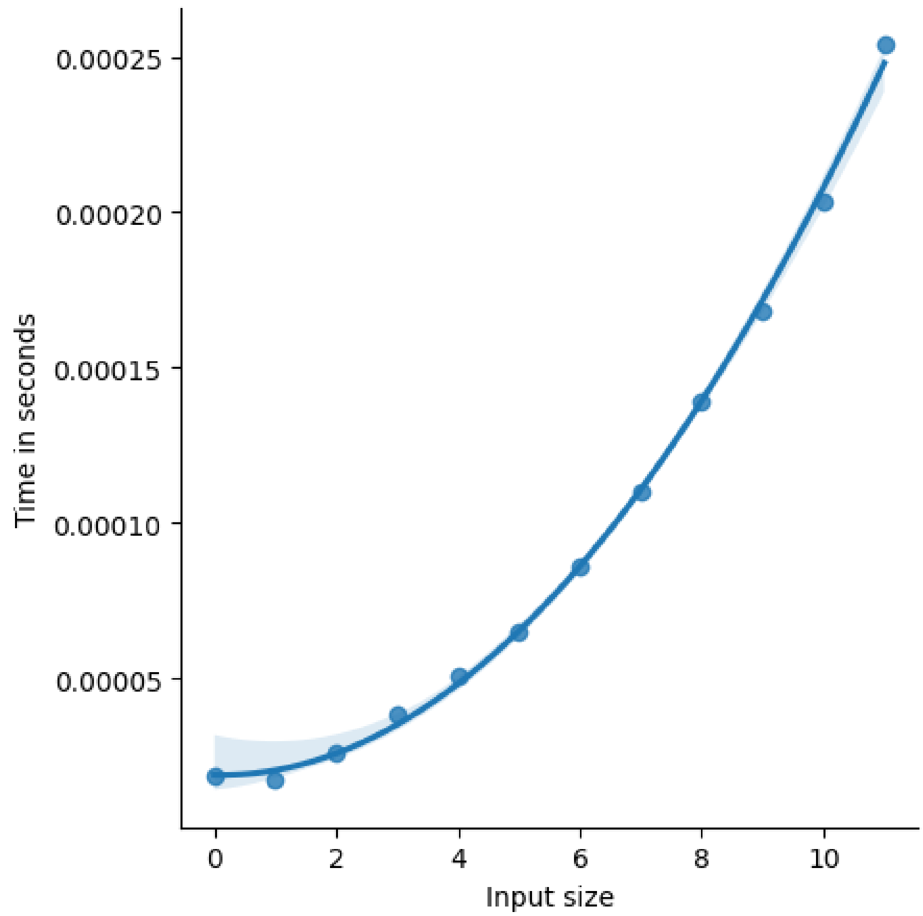
#### 1.9.1 Complexity without dynamic programming

[34]:
```python
random_string = "".join(random.choices(string.ascii_letters, k=12))
utils.plot_time_complexity(lambda x: levenshtein_recursive(x, "".
↪join(reversed(x))), random_string, number=1)
```




### 1.9.2 Time complexity with dynamic programming

[35]:
```
random_string = "".join(random.choices(string.ascii_letters, k=12))
utils.plot_time_complexity(lambda x: levenshtein_dynamic(x, "".
 ↪join(reversed(x))), random_string, number=10, regression_order=2)
```




### 1.9.3 More examples of dynamic programming

- Text-to-speech: Viterbi algorithm for finding the best speech samples in context
- Syntax parsing: CYK algorithm for context-free grammar parsing
- Graphs (e.g., WordNet): Dijkstra's algorithm for finding the shortest path between two nodes
- Sequence alignment (similar to edit distance!): matching DNA or protein sequences

## 1.10 Take-home messages

- **Computational complexity** measures how efficient an algorithm is as the size of its input increases
  - **Time complexity** and **space complexity**
  - $O(1) < O(n) < O(n \log n) < O(n^2) < O(2^n)$
  - Complexity is a theoretical concept -- it doesn't tell us anything about how many seconds or bytes the algorithm will take to run!
- Checking if a specific value exists in a `list` is slow! Use `set` or `dict` instead
- **Recursive functions** are functions that call themselves
- **Dynamic programming** is a technique to reduce time complexity by dividing the problem into subproblems and storing the results of those subproblems

- **Levenshtein distance** is the lowest number of edit operations (insertions, deletions, substitutions) required to turn one string into another

## 1.11 Enjoy your spring break! :)