MMS

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1. a)

$$\begin{aligned}
E\{aX+bY+c\} &= \iint (ax+by+c)p(x,y)dxdy \\
&= \iint ax\cdot p(x,y) + by\cdot p(x,y) + c\cdot p(x,y)dxdy \\
&= \iint ax\cdot p(x,y)dxdy + \iint by\cdot p(x,y)dxdy + \iint c\cdot p(x,y)dxdy \\
&= \int ax\cdot p(x)dx + \int by\cdot p(y)dy + c \\
&= a\int x\cdot p(x)dx + b\int y\cdot p(y)dy + c \\
&= aE\{X\} + bE\{Y\} + c
\end{aligned}$$

b)

$$\begin{aligned}
E\{(X-\mu_x)^2\} &= E\{X^2 - 2X\mu_x + \mu_x^2\} \\
&= E\{X^2\} - E\{2X\mu_x\} + \mu_x^2 \\
&= E\{X^2\} - 2\mu_x E\{X\} + \mu_x^2 \\
&= E\{X^2\} - 2E\{X\}E\{X\} + E\{X\}^2 \\
&= E\{X^2\} - 2E\{X\}^2 + E\{X\}^2 \\
&= E\{X^2\} - E\{X\}^2 \\
&= E\{X^2\} - \mu_x^2
\end{aligned}$$

c)

$$\begin{aligned}
V\{aX+b\} &= E\{(aX+b)^2\} - E^2\{(aX+b)\} \\
&= E\{(a^2X^2 + 2abX + b^2)\} - E\{(aX+b)\}\cdot E\{(aX+b)\} \\
&= a^2E\{X^2\} + 2abE\{X\} + b^2 - (aE\{X\}+b)\cdot(aE\{X\}+b) \\
&= a^2E\{X^2\} + 2abE\{X\} + b^2 - (a^2E^2\{X\} + 2abE\{X\} + b^2) \\
&= a^2E\{X^2\} - a^2E^2\{X\} \\
&= a^2\cdot(E\{X^2\} - 2E^2\{X\}) \\
&= a^2\cdot V\{X\}
\end{aligned}$$

d)

$$
\begin{aligned}
E\{XY\} &= \iint xy \cdot p(x,y)dxdy \\
&= \iint xy \cdot \underbrace{p(x) \cdot p(y)}_{\text{independence}} dxdy \\
&= \int x \cdot p(x)dx \cdot \int y \cdot p(y)dy \\
&= E\{X\} \cdot E\{Y\}
\end{aligned}
$$

e)

$$
\begin{aligned}
V\{aX+bY\} &= E\{(aX+bY)^2\} - E^2\{(aX+bY)\} \\
&= E\{(a^2X^2 + 2abXY + b^2Y^2)\} - E\{(aX+bY)\} \cdot E\{(aX+bY)\} \\
&= a^2E\{X^2\} + 2abE\{XY\} + b^2E\{Y^2\} - \\
&(aE\{X\} + bE\{Y\}) \cdot (aE\{X\} + bE\{Y\}) \\
&= a^2E\{X^2\} + 2abE\{XY\} + b^2E\{Y^2\} - \\
&(a^2E^2\{X\} + \underbrace{2abE\{XY\}}_{\text{independence}} + b^2E^2\{Y\}) \\
&= a^2E\{X^2\} + E\{Y^2\} - a^2E^2\{X\} - b^2E^2\{Y\} \\
&= a^2 \cdot (E\{X^2\} - 2E^2\{X\}) + b^2 \cdot (E\{Y^2\} - 2E^2\{Y\}) \\
&= a^2 \cdot V\{X\} + b^2 \cdot V\{Y\}
\end{aligned}
$$

2. **a)** We replace the integral's limits with 0 and 1 respectively, because our probability density function is 0 outside of those limits.

$$
\begin{aligned}
\mu_X &= \int_{-\infty}^{\infty} x_n f_{X_n}(x_n)dx \\
&= \int_0^1 x_n 2x_n dx \\
&= \int_0^1 2x_n^2 dx \\
&= \left[\frac{2}{3}x_n^3\right]_0^1 \\
&= \frac{2}{3} - 0 = \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_X^2 &= E(X^2) - \mu_X^2 \\
&= E(X^2) - \frac{2}{3}^2
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 x_n^2 2x_n dx - \frac{4}{9} \\
&= \int_0^1 2x_n^3 dx - \frac{4}{9} \\
&= \left[\frac{2}{4}x_n^4\right]_0^1 - \frac{4}{9} \\
&= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}
\end{aligned}
$$

b)

$$
\begin{aligned}
E(Y_n) &= E(X_n + X_{n-1}) = E(X_n) + E(X_{n+1}) \\
&= \int \int (x_n + x_{n+1}) f_{X_n}(x_n) f_{X_{n+1}}(x_{n+1}) dx_n dx_{n+1} \\
&= \int x_n f_{X_n}(x_n) dx_n + \int x_{n+1} f_{X_{n+1}}(x_{n+1}) dx_{n+1} \\
&= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}
\end{aligned}
$$