MMS (winter term 2018 / 2019)
— Exercise 10 —

Timo Gerkmann, Robert Rehr, David Ditter

Themen: Statistische Momente und Autokorrelation

Abgabefrist: 15.01.2019 – 8:30 Uhr

Abgabe: Textaufgaben als PDF-Dateien, Praktische Programmieraufgaben bitte als py-Dateien abgeben. Diese Dateien gepackt im Anhang per E-Mail an robert.rehr@uni-hamburg.de senden!

Theoretical Considerations (10 Pkt.)

1) Properties of the Expected Value (5 Pkt.)

Prove that the following statements hold.

a) Show that the expectation operator $\mathbb{E}\{\cdot\}$ is linear, i.e., $\mathbb{E}\{aX + bY + c\} = a\mathbb{E}\{X\} + b\mathbb{E}\{Y\} + c$.

b) Show that $\mathbb{E}\{(X - \mu_X)^2\} = \mathbb{E}\{X^2\} - \mu_X^2$, where $\mathbb{E}\{X\} = \mu_X$.

c) Show that $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$, where $\mathrm{Var}(X) = \mathbb{E}\{(X - \mu_X)^2\}$

d) Show that if X and Y are independent, then $\mathbb{E}\{XY\} = \mathbb{E}\{X\}\mathbb{E}\{Y\}$.

e) Show that if X and Y are independent, then $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.

Hint: For a) and d), start from $\mathbb{E}\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p(x,y)dxdy$, where $p(x,y)$ is the joint probability density function of X and Y. Then use the properties for joint probability density functions, e.g., for marginalization or for independence.

2) Analysis of a Random Process (5 Pkt.)

Assume a stationary random process $\ldots, X_{n-1}, X_n, X_{n+1}, \ldots$ with realizations $\ldots, x_{n-1}, x_n, x_{n+1}, \ldots$. The random variables X_n follow the probability density function

$$f_{X_n}(x_n) = \begin{cases} 2x_n, & \text{if } 0 \leq x_n \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The samples of the random process are independent, meaning that $f_{X_i,X_j}(x_i, x_j) = f_{X_i}(x_i)f_{X_j}(x_j)$ if $i \neq j$.

a) Compute the mean $\mu_X = \mathbb{E}\{X_n\}$ and the variance $\sigma_X^2 = \mathrm{Var}(X_n)$.

b) Let $Y_n = X_n + X_{n-1}$. Compute the mean $\mathbb{E}\{Y_n\}$ and the variance $\mathrm{Var}(Y_n)$.

c) Let $S_n^{(M)} = \sum_{m=0}^{M-1} X_{n-m}$ with $M \in \mathbb{N}$. Compute the mean $\mathbb{E}\{S_n^{(M)}\}$ and the variance $\mathrm{Var}(S_n^{(M)})$.

d) Let $\bar{X}_n^{(M)} = \frac{1}{M}S_n^{(M)}$. Compute the mean $\mathbb{E}\{\bar{X}_n^{(M)}\}$ and the variance $\mathrm{Var}(\bar{X}_n^{(M)})$.

e) Let $T_n^{(M)} = S_n^{(M)} - M\mu_X$. Compute the mean $\mathbb{E}\{T_n^{(M)}\}$ and the variance $\mathrm{Var}(T_n^{(M)})$.

f) Let $Z_n^{(M)} = \frac{1}{\sqrt{M\sigma_X^2}}T_n^{(M)}$. Compute the mean $\mathbb{E}\{Z_n^{(M)}\}$ and the variance $\mathrm{Var}(Z_n^{(M)})$.

g) The autocorrelation function $\varphi_{XX}[\lambda] = \mathbb{E}\{X_n X_{n+\lambda}\}$.

Hint: Use the properties given in 1) to solve the questions b) – g).

Practical Considerations (10 Pkt.)

1) Distance estimation (10 Pkt.)

Load the wave file `echosig.wav` which is included in the zip-file. For this, use the `read` function from the `pysoundfile` module, which you have to install separately. The wave file contains a speech signal without any background noise. However, a single reflection of the sound waves from a wall has been artificially added to the signal. As shown in Figure 1, it is assumed that the sound source and the microphone are positioned so that they stand perpendicular to the wall surface.

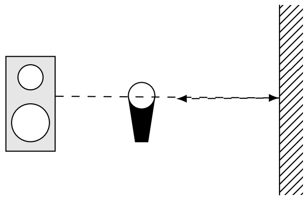

Figure 1: Schematic of the sound wave reflection

Questions

a) Estimate and plot the autocorrelation function of the input signal. Use the `correlate` function from `scipy.signal` to compute the autocorrelation function. Plot the correlation function against the lag $\lambda \in [-200, 200]$.

- Which element in the output vector of the `correlate` function does correspond to $\lambda = 0$? What is its index?
- Which elements of the autocorrelation function can be associated with echoed signal?

b) Using the autocorrelation function, estimate the distance between the microphone and the wall from which the sound was reflected. The speed of sound is given as $c = 343$ m/s. Describe how you calculated the distance.