Post Mid Sem

Normal Random Variable

5 March '18

X is said to be normal random variable with parameters μ and σ^2 if the pdf of X is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\{(x-\mu)^2/2\sigma^2\}}$$

variables

where $-\infty < x < \infty$

If $f(x)$ is pdf then $\int f(x)\,dx = 1$.

Proof $\Rightarrow$ $I = \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\{(x-\mu)^2/2\sigma^2\}} dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\}} dx$$

let $\frac{x-\mu}{\sigma} = z$.

$\Rightarrow dx = \sigma dz$.

$$\therefore I = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \sigma\, dz$$

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz \quad - (1)$$

let $I_1 = \int_{-\infty}^{\infty} e^{-z^2/2} dz \quad - (2)$

$$I_1 I_1 = \int_{-\infty}^{\infty} e^{-z^2/2} dz \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z^2+y^2}{2}} dz\,dy$$

$$I_1^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r\,d\theta\,dr$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/2} dr.$$

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$$= 2\pi$$

$$I_1 = \sqrt{2\pi} \quad - (3)$$

$$\Rightarrow I = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \qquad [\text{By } (1), (2) \text{ and } (3)].$$

$$= 1$$

3.1) Show that the constant in the normal distribution must be $\frac{1}{\sqrt{2\pi}}$

Let the constant be a. $\Rightarrow f(x) = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\int f(x)\, dx = 1.$

We know that $\int f(x) dx = \frac{a}{\sigma} \int e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$. from prev. proof.

$$= \frac{a}{\sigma} \qquad a\sqrt{2\pi} = 1$$

$$\Rightarrow a = \frac{1}{\sqrt{2\pi}}.$$

Prove that if X is normally distributed with parameters μ and σ^2 then $Y = aX + b$ is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$.

Given $X \sim N(\mu, \sigma^2)$

Prove $Y \sim N(a\mu + b, a^2\sigma^2)$.

$$F_Y(x) = P\{Y \le x\}$$

$$= P\{aX + b \le x\}$$

$$= P\left\{X \le \frac{x-b}{a}\right\}$$

$$= F_X\left(\frac{x-b}{a}\right).$$

$$\boxed{F_Y(x) = F_X\left(\frac{x-b}{a}\right)}$$

Expressing CDF of one R.V. in terms of CDF of another R.V.

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pdf of F_Y is derivative of F_Y w.r.t. x.

$$~~f_Y(y) =~~ \quad f_Y(x) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right)$$

$$\Rightarrow f_Y(x) = \frac{1}{a}\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\left\{\left(\frac{x-b}{a}-\mu\right)^2/2\sigma^2\right\}}\right]$$

$$= \frac{1}{\sigma a\sqrt{2\pi}} e^{-(x-(a\mu+b))^2/2(\sigma a)^2}$$

$$= N(a\mu+b,\ a^2\sigma^2)$$

Before solving expectation & variance we consider $z = \frac{x-\mu}{\sigma}$

$\Rightarrow x = \sigma z + \mu$

we get

$a = \sigma$

& $b = \mu$

i.e. $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$.

$$E[z] = \int_{-\infty}^{\infty} z \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} dz = 0$$

$$E[z^2] = \int_{-\infty}^{\infty} \frac{z^2}{\sigma\sqrt{2\pi}} e^{-z^2/2} dz = 1$$

$$Var(z) = 1.$$

$$E[x] = a E[z] + b.$$

$$= \sigma(0) + \mu$$

$$= \mu.$$

$$Var = a^2 Var(z)$$

$$= a^2$$

$$= \sigma^2.$$

$$\int_{-\infty}^{5} e^{-x^2/4} dx, \quad \int_{2}^{5} e^{-x^2} dx.$$

In previous cases we were able to convert these variables into polar forms because of the limits (from $-\infty$ to ∞) but here it isn't possible

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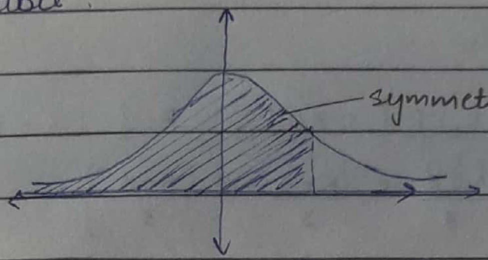
$\phi(x)$ denote the cumulative distribution f^n of a std. random variable.



$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy$$

$$\lim_{x \to \infty} \phi(x) = 1 \quad , \quad \lim_{x \to -\infty} \phi(x) = 0 \quad , \quad \phi(0) = \frac{1}{2}.$$

$$\boxed{\phi(-x) = 1 - \phi(x)}$$

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Normal R.V. for $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad , \quad -\infty < x < \infty$$

take $Z = \frac{x-\mu}{\sigma}$

then let denote the cdf for std. normal r.v. by $\phi(x)$.

$$\phi(x) = P\{Z \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz$$

The value $\phi(x)$ denote the non -ve of x. For negative, we can calculate by following relation

$$\phi(-x) = 1 - \phi(x)$$

$$P\{Z \le x\} = P\left\{\frac{X-\mu}{\sigma} \le x\right\}$$

$\because$ $Z = \frac{X-\mu}{\sigma}$ is the std. normal r.v. where X is normally distributed with parameters μ and σ^2, it follows that

$$F_X(a) = P\{X \le a\} = P\left\{\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right\}$$

$$= P\left\{Z \le \frac{a-\mu}{\sigma}\right\}$$

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$f_x(a) = \phi\left(\frac{a-\mu}{\sigma}\right)$

ex: $X \sim N(3, 9)$.

Calculate $P\{2 < X < 5\}$, $P\{x > 0\}$, $P\{|x-3| > 6\}$

→ (i) $P\{2 < X < 5\} = P\left\{\frac{2-3}{3} < \frac{x-3}{3} < \frac{5-3}{3}\right\}$

$= P\left\{-\frac{1}{3} < \frac{x-3}{3} < \frac{2}{3}\right\}$

$= P\left\{-\frac{1}{3} < z < \frac{2}{3}\right\}$

$= \phi\left(\frac{2}{3}\right) - \phi\left(-\frac{1}{3}\right)$

$= \phi(0.66) - \{1 - \phi(0.33)\}$

$= \phi(0.66) + \phi(0.33) - 1.$

(ii) $P\{x > 0\} = P\left\{\frac{x-3}{3} > \frac{-3}{3}\right\}$

$= P\{z > -1\}$.

$= \phi(\infty) - \phi(-1)$.

$= 1 - \{1 - \phi(1)\}$

$= \phi(1)$.

$= 0.8413$

(iii) $P\{|X-3| > 6\} = P\{x < -3\} + P\{x > 9\}$

$\cancel{= P\left\{\frac{x-3}{3} < -2\right.}$

Q: Let $X \sim N(-5, 4)$. Calculate $P(X > -3 \mid X > -5)$

→ $P(X > -3 \mid X > -5) = \dfrac{P(X > -3 \cap X > -5)}{P(X > -5)}$

$= \dfrac{P(X > -3)}{P(X > -5)}$.

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Celebrations

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Q: If $Y = 3 - 2X$

Find $P\{X > 2 \mid Y < 1\}$.

→ $P\{X > 2 \mid 3 - 2X < 1\}$

$P\{X > 2 \mid X > 1\}$

$$= \frac{P\{X > 2 \cap X > 1\}}{P\{X > 1\}}$$

$$= \frac{P\{X > 2\}}{P\{X > 1\}}$$

Solve.

(X, Y) $X, Y \rightarrow$ both are R.V.

X & Y can be both discrete

" continuous

one discrete & other continuous

★ **Jointly Distributed Random Variable**

If X and Y are r.v. then joint probability mass function of two discrete r.v. X and Y is defined as:

$$P_{XY}(x, y) = P\{X = x, Y = y\}$$

Now, define the range for X and Y

$$R_{XY} = \{(x, y) : P_{XY}(x, y) > 0\}$$

If $R_X = \{x_1, x_2, \ldots x_n\}$ and $R_Y = \{y_1, y_2, \ldots y_n\}$

then $R_{XY} \subset R_X \times R_Y$

As P_{XY} is a PMF, we can write

$$\sum_{(x_i, y_i) \in R_{XY}} P_{XY}(x_i, y_i) = 1$$

we can also find joint PMF of $P((x, y) \in A)$ for any $A \subset R^2$

The event $X = x$ can be written as $\{x_i, y_i) : x_i = x, y_i \in R_Y\}$

and $Y = y$ can be

$$\{(x_i, y_i) ; x_i \in R, Y = y\}$$

$$P_{XY}(x, y) = P(X = x, Y = y) = P\{(X = x) \cap (Y = y)\}$$

$$P_{XY}(x, y) = P\{X = x, Y = y\}$$

$$= P\{(X = x) \text{ and } (Y = y)\}$$

$$R_{XY} = \{(x, y) : P_{XY}(x, y) > 0\}$$

$$R_X = \{x_1, x_2, x_3, \ldots\}$$

$$R_Y = \{y_1, y_2, \ldots\}.$$

$$R_{XY} \subset R_X \times R_Y$$

$$\sum_{x_i, y_i} P_{XY}(x_i, y_i) = 1$$

$$\sum_{x_i} \sum_{y_i} P_{XY}(x_i, y_i) = 1$$

The event $X = x$ can be written as $\{(x_i, y_i) : x_i = x, y_i \in R_Y\}$

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Similarly, for event $Y = y$, we have $\{(x_i, y_i) : x_i \in R_x, y_i = y\}$

* **Marginal Point**

Joint pmf contains the information regarding the distribution of X and Y. Thus, we can obtain pmf of X from its joint from its joint PMF with Y as:

$$P_X(x) = P(X = x) = \sum_{y_i \in R_Y} P\{X = x, Y = y_i\}.$$

$$= \sum_{y_i} P_{XY}(x, y_i) \text{ for any } x \in R_X$$

$$P_Y(y) = P(Y = y) = \sum_{x_i \in R_X} P\{X = x_i, Y = y\}$$

$$= \sum_{x_i} P_{XY}(x_i, y) \text{ for any } y \in R_Y.$$

ex: Consider 2 random variable X and Y with joint pmf as

X \ Y	Y=0	Y=1	Y=2
X=0	1/6	1/4	1/8
X=1	1/8	1/6	1/6

(a) find $P(X = 0, Y \leq 1)$.

(b) find marginal pmf of X and Y

(c) find $P(Y = 1 \mid X = 0)$

(d) Are X and Y independent.

→ $R_X = \{0, 1\}$, $R_Y = \{0, 1, 2\}$

$R_{XY} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$

(a) $P(X=0, Y \leq 1) = P(X=0, Y=0) + P(X=0, Y=1)$

$= 1/6 + 1/4$

$= 5/12$

(b) Marginal pmf of x

$P_x(0) = \sum_{y_i \in R_y} P(X=0, Y=y)$

$= P(X=0, Y \leq 2)$

$= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$

$= 1/6 + 1/4 + 1/8$

$= 13/24$

$P_x(1) = \sum_{y_i \in R_y} P(1, y_i)$

$= P(X=1, Y \leq 2)$

$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=2, Y=2)$

$= 1/8 + 1/6 + 1/6$

$= 11/24$

$\therefore P_x(x) = \begin{cases} 13/24 & \text{when } x=0 \\ 11/24 & \text{when } x=1 \\ 0 & \text{o.w.} \end{cases}$

Marginal pmf of Y

$P_y(0) = \sum_{x_i \in R_x} P(X=x, Y=0)$

$= P(X \leq 1, Y=0)$

$= 1/6 + 1/8$

$= 7/24$

$P_y(1) = P(X \leq 1, Y=1)$

$= 1/4 + 1/6$

$= 5/12$

$P_y(2) = P(X \leq 1, Y=2)$

$= 1/8 + 1/6$

$= 7/24$

(b) Marginal pmf of X

$$P_X(0) = \sum_{y_i \in R_Y} P(X=0, Y=y_i)$$

$$= P(X=0, Y \leq 2)$$

$$= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$$

$$= 1/6 + 1/4 + 1/8$$

$$= 13/24$$

$$P_X(1) = \sum_{y_i \in R_Y} P(1, y_i)$$

$$= P(X=1, Y \leq 2)$$

$$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$$

$$= 1/8 + 1/6 + 1/6$$

$$= 11/24$$

$$\therefore P_X(x) = \begin{cases} 13/24 & \text{when } x=0 \\ 11/24 & \text{when } x=1 \\ 0 & \text{o.w.} \end{cases}$$

Marginal pmf of Y

$$P_Y(0) = \sum_{x_i \in R_X} P(X=x, Y=0)$$

$$= P(X \leq 1, Y=0)$$

$$= 1/6 + 1/8$$

$$= 7/24.$$

$$P_Y(1) = P(X \leq 1, Y=1)$$

$$= 1/4 + 1/6$$

$$= 5/12$$

$$P_Y(2) = P(X \leq 1, Y=2)$$

$$= 1/8 + 1/6$$

$$= 7/24$$

$$\therefore P_Y(y) = \begin{cases} 7/24 & \text{when } y=0 \\ 5/12 & \text{when } y=1 \\ 7/24 & \text{when } y=2 \\ 0 & \text{o.w.} \end{cases}$$

(c) $P(Y=1 \mid X=0) = \dfrac{P(Y=1 \cap X=0)}{P(X=0)}$

$= \dfrac{P_{XY}(0,1)}{P(X=0)}$

$= \dfrac{1/4}{13/24}$

$= \dfrac{6}{13}$

{ If A and B are independent
$P(A \cap B) = P(A) \cdot P(B)$
$\Rightarrow P(A|B) = P(A)$ }

(d) Not independent as. $P(Y=1 \mid X=0) \neq P(Y=1)$

Q:1) Suppose a car showroom has 10 cars of a brand out of which 5 are good, 2 are having defective transmission and 3 have defective steering. If 2 cars are selected at random then construct its PMF.

$X \rightarrow$ no. of cars having defective transmission

$Y \rightarrow$ " " " " " steering.

$\rightarrow$

$X = \{0, 1, 2\}$

$Y = \{0, 1, 2, 3\}$ ~~XX~~ instead $Y = \{0, 1, 2\}$.

$P_{XY}(0,0) = P(X=0, Y=0)$

$= \dfrac{{}^5C_2}{{}^{10}C_2} = \dfrac{10}{45}$

$P_{XY}(0,1) = \dfrac{{}^5C_1 \times {}^3C_1}{{}^{10}C_2} = \dfrac{15}{45}$

$P_{XY}(1,2) = P_{XY}(2,2) = P_{XY}(2,1) = 0$

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$$P_{XY}(0,2) = \frac{{}^3C_2}{{}^{10}C_2} = \frac{3}{45}$$

$$P_{XY}(1,0) = \frac{{}^5C_1 \times {}^2C_1}{{}^{10}C_2} = \frac{10}{45}$$

$$P_{XY}(2,0) = \frac{{}^2C_2}{{}^{10}C_2} = \frac{1}{45}$$

$$P_{XY}(1,1) = \frac{{}^2C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{6}{45}.$$

X \ Y	Y = 0	Y = 1	Y = 2
X = 0	10/45	15/45	3/45
X = 1	10/45	6/45	0
X = 2	1/45	0	0

$P(X<0, Y\le 1) = 0$ [$X<0$ isn't defined].

$F_X(a) = P\{X \le a\}$

for two random variable X and Y the join cdf is as

$$F(a,b) = P\{X \le a, Y \le b\}$$
$$-\infty < a, b < \infty$$

$F(0,0)$, $F(1,1) = P\{X \le 1, Y \le 1\}$

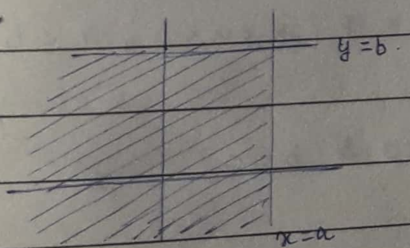
9 March $F_X(a) = F(a) = P\{x \le a\}$

Joint C.D.F. for 2 random variable X and Y

$$F_{XY}(a,b) = F(a,b)$$
$$= P\{X \le a, Y \le b\}$$

Marginal C.D.F.

$$F_X(a) = P\{X \le a\} = \lim_{b\to\infty} P\{X \le a, Y \le b\} = \lim_{b\to\infty} F(a,b)$$



$$F_Y(b) = \lim_{a\to\infty} F(a,b)$$
$$= F(\infty, b).$$

$P\{x_1 \le X \le x_2, \; y_1 < Y < y_2\} = F(x_2, y_2) + F(x_1, y_1) - F(x_2, y_1) - F(x_1, y_2)$

$F(\infty, \infty) = 1, \; F(-\infty, b) = 0, \; F(a, -\infty) = 0.$

Q: Let $X \sim$ Bernoulli (P) and $Y \sim$ Bernoulli (q) be independent, $0 < p, q < 1$, then calculate joint pmf and CDF of X and Y

→ $R_X = \{0, 1\}, \; R_Y = \{0, 1\}$

$R_{XY} = \{(0,0), (0,1), (1,0), (1,1)\}.$

$P_{XY}(x, y) = P\{X = x, Y = y\}$

$P_{XY}(0, 0) = P\{X = 0, Y = 0\}.$

$= P\{X = 0\} \cdot P\{Y = 0\}.$

$= (1-p)(1-q)$

$P_{XY}(1, 0) = P(1) P(0)$

$= p(1-q)$

$P_{XY}(0, 1) = P(0) P(1)$

$= \cancel{q(1-p)} \; (1-p)q$

$P_{XY}(1, 1) = P(1) P(1)$

$= pq$

PMF of X and Y

$$f_{XY}(x, y) = \begin{cases} (1-p)(1-q) & \text{when } X = 0, Y = 0 \\ p(1-q) & \text{when } X = 1, Y = 0 \\ (1-p)q & \text{when } X = 0, Y = 1 \\ pq & \text{when } X = 1, Y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$F(a, b)$ and a, b be any real no. $a < 0$ and $b < 0$

$F(a, b) = P\{X \le a, Y \le b\} = P\{X \le a < 0, \; Y \le b < 0\}$

$= 0$

$(a \le 0 \;\&\; b < 0)$ or $(a < 0 \;\&\; b \le 0)$.

$F(a, b) = 0$

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when $a \le 1$ & $b \le 1$

$F(a,b) = P\{X \le 1, Y < 1\} = P\{0,0\}$
$= (1-p)(1-q)$

if $a < 1$ & $b \ge 1$

$F(a,b) = P\{X \le 1, Y < 1\}$
$= P\{(0,0)\} + P\{(0,1)\}$
$= (1-p)(1-q) + (1-p)q$

when $a > 1$ & $b > 1$

$F(a,b) = P\{X \le a, Y \le b\}$
$= P(0,0) + P(1,0) + P(0,1) + P(1,1)$
$= 1$

Conditioning and Independence.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(X \in C \mid Y \in D) = \frac{P(X \in C, Y \in D)}{P(Y \in D)}$$

For discrete random variable X and event A the conditional PMF of X given A is defined as:

$$P_{X|A}\, x_i = P(X = x_i \mid A) = \frac{P(X = x_i \cap A)}{P(A)}$$

A = output will be less than 5.

For discrete random variable pmf of X ~~and~~ given Y is given by:

$$P_{X|Y}(x_i|y_i) = \frac{P_{XY}(x_i, y_i)}{P_Y(y_i)}.$$

Similarly

$$P_{Y|X}(y_i|x_i) = \frac{P_{XY}(x_i, y_i)}{P_X(x_i)}$$

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Independent Random Variable

for discrete random variable X and Y are independent if

$$P_{XY}(x_i, y_i) = P(x_i) \cdot P(y_i) \text{ for all } x_i, y_i$$

Similarly $F_{XY}(x_i, y_i) = F_X(x_i) \cdot F_Y(y_i)$.

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Continuous Random Variable.

2 random variables X and Y, are jointly continuous if there exist a non -ve f^n $f_{XY}: R^2 \longrightarrow R$ s.t. for any set $A \subseteq R^2$ we have $P((X, Y) \in A) = \iint f_{XY}(x, y)\,dx\,dy$. The f^n $f_{XY}(x, y)$ is called joint probability density f^n (PDF) of X and Y. Range of (X, Y) is $R_{XY} = \{(x, y) : f_{XY}(x, y) > 0\}$ as f_{XY} in PDF, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = 1.$$

Marginal PDF's

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx.$$

Q:1) Let X and Y are two jointly continuous random variable with PDF

$$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \le x \le 1 \;\&\; 0 \le y \le 1. \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate c

(b) Find $P\left(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2}\right)$

(c) Find marginal PDF's $f_X(x)$, $f_Y(y)$.

(d) Find $P(Y \le X)$

(e) Find $P(X = Y)$

(f) Find $P(X = 2Y)$

(g) $P\left(X \le \frac{X}{4}, Y \le \frac{X}{2}\right)$

(h) $P(X \le Y)$

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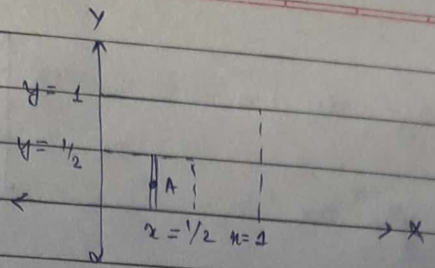


(a) Range $R_{XY} = \{(x, y) : 0 \le x \le 1 \ \& \ 0 \le y \le 1\}$

Now by the property of PDF, then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = 1$

$\Rightarrow \int_0^1\int_0^1 (x + cy^2)\,dx\,dy = 1.$

$\Rightarrow \int_0^1 \left[\frac{x^2}{2} + cy^2 x\right]_0^1 dy = 1.$

$\Rightarrow \int_0^1 \left(\frac{1}{2} + cy^2\right) dy = 1.$

$\Rightarrow \left[\frac{y}{2} + \frac{cy^3}{3}\right]_0^1 = 1$

$\Rightarrow \frac{1}{2} + \frac{c}{3} = 1$

$\Rightarrow c = 3/2.$

then $f(x, y) = \begin{cases} x + (3/2)y^2 & \\ 0 & \text{o.w.} \end{cases}$

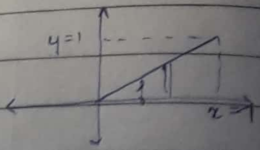
(b) $P\left(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx\,dy = \frac{3}{32}$

(c) Marginal PDF

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy = \int_0^1 \left(x + \frac{3y^2}{2}\right) dy = x + \frac{1}{2}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx = \int_0^1 \left(x + \frac{3y^2}{2}\right) dx = \frac{1}{2} + \frac{3y^2}{2}.$

(d) $P\{Y \le X\} = \int_0^1 \int_0^x \left(x + \frac{3y^2}{2}\right) dy\, dx$



$= \int_0^1 \left[\frac{xy}{1} + \frac{y^3}{2}\right]_0^x dx$

$= \int_0^1 \left(\frac{x^2}{1} + \frac{x^3}{2}\right) dx$

$= \left[\frac{x^3}{3} + \frac{x^4}{8}\right]_0^1$

$= \frac{1}{3} + \frac{1}{8}$

$= \frac{11}{24}$.

(e), (f) $P(X=Y)=0$, $P(X=2Y)=0$, the probability of line in R^2 is zero

(g) $P\left(Y \le \frac{X}{4} \mid Y \le \frac{X}{2}\right) = \frac{P(Y \le X/4)}{P(Y \le X/2)} =$

So, 1st calculate $P\left(Y \le \frac{X}{4}\right) = \int_0^1 \left(\int_0^{x/4} \left(x + \frac{3y^2}{2}\right) dy\right) dx$.

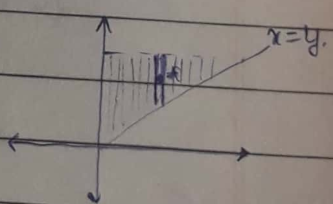
$= \int_0^1 \left[xy + \frac{y^3}{2}\right]_0^{x/4} dx$

$= \int_0^1 \left(\frac{x^2}{4} + \frac{x^3}{128}\right) dx$

$= \left[\frac{x^3}{12} + \frac{x^4}{512}\right]_0^1$

$=$

(h) $P(X \le Y) = \int_0^1 \int_x^1 \left(x + \frac{3}{2}y^2\right) dy\, dx$.



Teacher's Signature..........

$$= \int_0^1 \left[xy + \frac{1}{2}y^3\right]_x^1 dx$$

$$= \int_0^1 \left(x + \frac{1}{2} - x^2 - \frac{x^3}{2}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x}{2} - \frac{x^3}{3} - \frac{x^4}{8}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{8}$$

$$= \frac{13}{24}$$

Q:2) Joint PDF of X and Y is $f(x,y) = \begin{cases} 2 & \text{if } x>0,\ y>0,\ 0<x+y<1 \\ 0 & \text{otherwise} \end{cases}$

then find $P\left(X+Y<\frac{1}{2}\right)$.

→ $A = \left\{(x,y) : x+y < \frac{1}{2}\right\}$

$$P((x,y) \in A) = \iint_A f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^{1/2} \left(\int_0^{\frac{1}{2}-x} 2\, dy\right) dx$$

$$= \int_0^{1/2} 2\left(\frac{1}{2} - x\right) dx$$

$$= 2\left[\frac{x}{2} - \frac{x^2}{2}\right]_0^{1/2}$$

$$= \left[x - x^2\right]_0^{1/2}$$

$$= 1/4$$

Q: $f(x,y) = \begin{cases} \frac{21x^2y}{4} & \text{if } x^2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Calculate (a) $P(X \geq Y)$ (b) $P(X=Y)$.

$\rightarrow$ $R_{XY} = \{(x, y)$
$x^2 \le y \le 1$ $\qquad x^2 = y \Rightarrow$ eq^n of parabola.

$\Rightarrow x^2 \le y$.

$\Rightarrow x^2 \le 1$

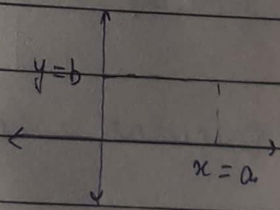
$$P(X \ge Y) = P\{(x, y) \in A\}$$
$$= \iint_A f_{XY}(x, y)\,dx\,dy$$
$$= \int_0^1 \left(\int_{x^2}^{x} \frac{21}{4} x^2 y\,dy \right) dx$$
$$= \int_0^1 \left(\frac{21 x^2 y^2}{8} \right)_{x^2}^{x} dx$$
$$= \int_0^1 \frac{21}{8} [x^4 - x^6]\,dx$$
$$= \frac{21}{8} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1$$
$$= \frac{21}{8} \left[\frac{1}{5} - \frac{1}{7} \right]$$
$$= \frac{\cancel{21}^{3}}{\cancel{8}_{4}} \times \frac{\cancel{2}^{1}}{\cancel{35}_{5}}$$
$$= \frac{3}{20}$$

13th March 18

Joint Cumulative Distribution Function.

For 2 random variables X and Y it is defined as $F_{XY}(x, y) = P(X \le x, Y \le y)$. $F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v)\,du\,dv$

$$f_{XY} = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$



Teacher's Signature..........

$F(\infty, \infty) = 1$

$F(-\infty, y) = 0$

$F(x, -\infty) = 0$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \quad \forall y \in R_Y$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy, \quad \forall x \in R_X$$

$F_X(x) = F_{X\cdot Y}(x, \infty)$ for any x.

$F_Y(y) = F_{XY}(\infty, y)$ for any y

Q:1) Let X and Y be 2 independent uniform $(0,1)$ random variable. Calculate F_{XY} (CDF) and PDF f_{XY}

→

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

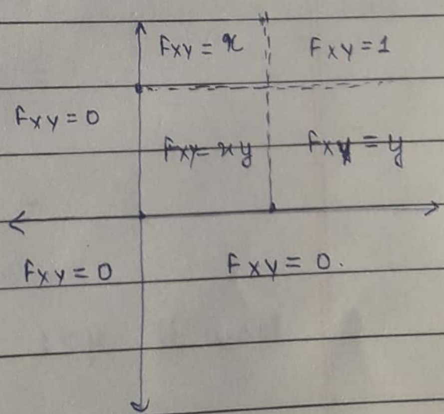
$$f_Y(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{otherwise} \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{otherwise} \\ y & y \in [0,1] \\ 1 & y > 1. \end{cases}$$

Given that X and Y are independent then $F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$

$$F_{XY} = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ xy & (x, y) \in [0,1]^2 \\ x & y > 1 \;\; x \in [0,1] \\ y & x > 1 \;\; y \in [0,1] \\ 1 & x > 1, \; y > 1. \end{cases}$$



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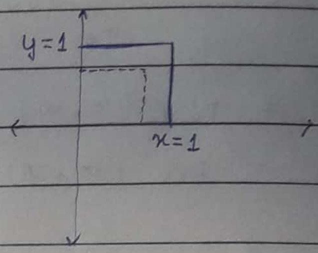
Dt. / /

$$f_{XY}(x,y) = \begin{cases} 1 & (x,y) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

Q:2) $$f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

→ first, let

$x < 0$ or $y < 0$



As no ~~part~~ point will be part of range R_{XY} so $F_{XY}(x,y) = 0$ for $x=0$ or $y=0$.

Now, $(x,y) \in [0,1]^2$

$$= \int_{-\infty}^{x}\left(\int_{-\infty}^{y} f(x,y)\,dy\right)dx$$

$$= \int_{0}^{x}\int_{0}^{y}\left(x + \frac{3}{2}y^2\right)dy\,dx$$

$$= \int_{0}^{x}\left(xy + \frac{y^3}{2}\right)dx$$

$$= \frac{x^2y}{2} + \frac{xy^3}{2}.$$

Now if $x > 1$ & $y \in [0,1]$

$$F_{XY}(x,y) = \int_{0}^{1}\left(\int_{0}^{y} f_{XY}\,dy\right)dx.$$

$$= \int_{0}^{1}\left(xy + \frac{y^3}{2}\right)dx.$$

$$= \left[\frac{x^2y}{2} + \frac{xy^3}{2}\right]_0^1$$

$$= \frac{y}{2} + \frac{y^3}{2}$$

Now if $y > 1$ & $x \in [0,1]$

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$$F_{XY}(x,y) = \int_0^x \int_0^1 f_{xy}(x,y)\,dy\,dx$$

$$= \int_0^x \left[xy + \frac{y^3}{2}\right]_0^1 dx$$

$$= \int_0^x \left(x + \frac{1}{2}\right) dx$$

$$= \frac{x^2}{2} + \frac{x}{2}$$

$$F_{xy}(x,y) = \begin{cases} 0 & \text{otherwise } x < 0 \text{ or } y < 0 \\ \frac{x^2 y}{2} + \frac{xy^3}{2} & (x,y) \in [0,1]^2 \\ \frac{y}{2} + \frac{y^3}{2} & y > 1,\ x \in [0,1] \\ \frac{x^2}{2} + \frac{x}{2}. & x > 1,\ y \in [0,1] \\ 1 & \text{if } x > 1, y > 1. \end{cases}$$

Marginal pdf

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 \left(x + \frac{3y^2}{2}\right) dx = \frac{1}{2} + \frac{3y^2}{2}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 \left(x + \frac{3}{2}y^2\right) dy = x + \frac{1}{2}$$

$$F_X(x) = F(x,\infty) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = \int_0^x \int_0^1 \left(x + \frac{3y^2}{2}\right) dy\right.$$

$$=$$

$$F_Y(y) = F(\infty, y) = \int_{-\infty}^{\infty} \left(\int_{\infty}^{y} f_{XY}(x,y)\,dy \right) dx =$$

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19 March

Let $X : \Omega \to \mathbb{R}$

$Y : \Omega \to \mathbb{R}$

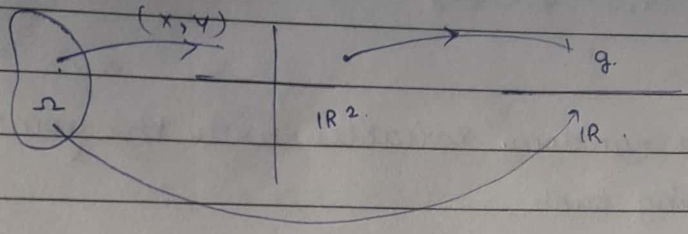
$(X, Y) \Rightarrow$ random vector

Functions of random vectors :- Let X & Y be 2 discrete random variables $\{R_X = \text{Range of } X,\ R_Y = \text{Range of } Y\}$. $(R_X \subseteq \mathbb{R},\ R_Y \subseteq \mathbb{R})$ are at most countable.

Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ be a function.

$$Z = g(X, Y)$$

$$Z : \Omega \to \mathbb{R}$$



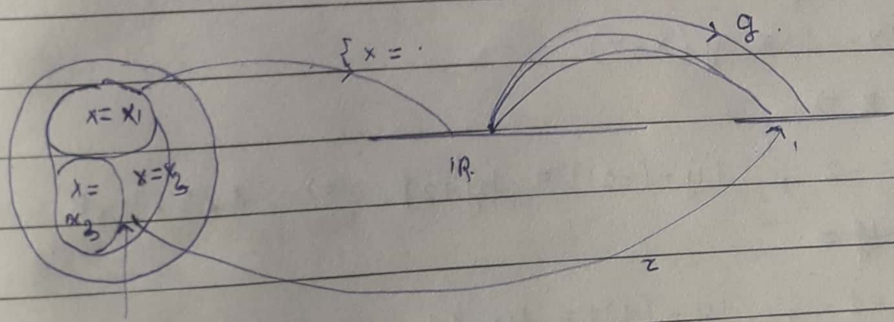
R_Z = countable set

Q: What is the pmf of r.v. Z ?

→ Function of a single r.v.

Let $X : \Omega \to \mathbb{R}$ be discrete r.v. & $g : \mathbb{R} \to \mathbb{R}$ be function. Then $Z = g(X)$ is again discrete & its pmf is given by

$$P(Z = z) = \sum_{x = g(x) = z} f_X(x) = P(X = x).$$



$$P\{Z = z\} = P\{w \in \Omega \mid Z(w) = z\}$$

$$= P\left[\bigcup_{x \in R_X : g(x) = z} \{w \in \Omega \mid X(w) = x\}\right]$$

$$= \sum_{x \in R_X : g(x) = z} P(X = x).$$

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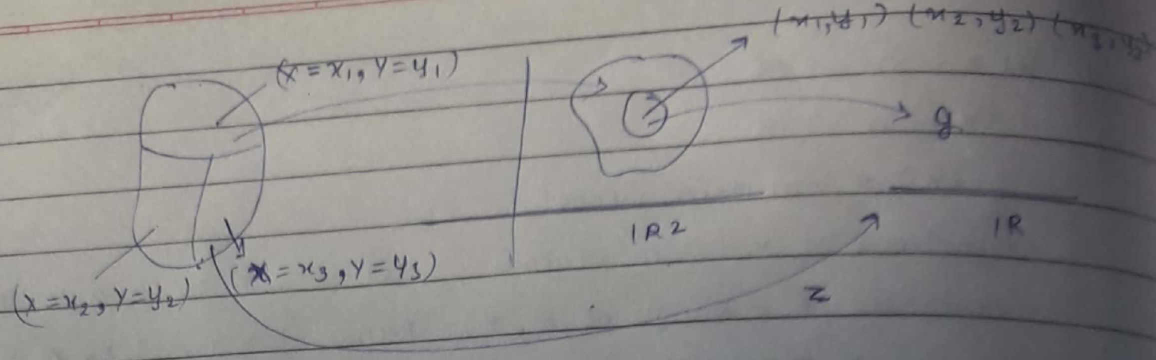

Q: What is the pmf of r.v. Z ?

→ $\{Z = z\} = \bigcup_{\substack{(x,y):\, g(x,y) = z \\ \in R_{XY}}} \{X = x,\ Y = y\}$

$P(Z = z) = \sum_{(x,y):\, g(x,y) = z} f(x,y) = P(X = x,\ Y = y)$.

ex: Let X & Y be 2 random variables with the joint pmf given by the following table.

X \ Y	-1	0	2	6
-2	1/9	1/27	1/27	1/3
+1	2/3	0	1/9	1/9
3	0	0	-1/9	4/27

Find the PMF of $|Y - X|$.

→ $g(x,y) = |y - x|$.

$Z = |Y - X| = g(X, Y)$

Range of Z ⇒

Fix $x = -2$, $|y - (-2)| = |y + 2|$ ⇒ 1, 2, 4, 8
Putting values of y.

Fix $x = 1$, $|y - (1)| = |y - 1|$ ⇒ 2, 1, 1, 5

" $x = 3$, $|y - 3|$ ⇒ 4, 3, 1, 3

$R_Z = \{1, 2, 3, 4, 5, 8\}$.

$P(Z = 1) = \sum_{(x,y):\, |y - x| = 1} f(x,y)$

$= f(-2,-1) + f(1,0) + f(1,2) + f(3,2)$

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$= \frac{3}{9} = \frac{1}{3}$

$P(Z=5) = f(1,6) = \frac{1}{9} = P(Z=8)$.

$P(Z=2) = \frac{7}{27}$, $P(Z=3) = \frac{4}{27}$, $P(Z=4) = \frac{1}{27}$

To verify check the sum of all the probabilities.

$\Rightarrow \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{7}{27} + \frac{4}{27} + \frac{1}{27}$

$= \frac{9+3+3+7+4+1}{27}$

$= 1$

$\Rightarrow$ Correct.

ex: Let X & Y be 2 random variables with the joint pmf given by the following table:

X \ Y	1	2	. . . .	N
1	$1/N^2$	$1/N^2$	- - - -	$1/N^2$
2	⋮	⋮		
⋮				
N	$1/N^2$			

Find the pmf of $\min\{X, Y\}$

$\rightarrow$ $g : R^2 \rightarrow \mathbb{R}$

$g(x,y) = \min(x,y)$.

$Z = \min\{X, Y\}$.

$x = 1, 2, \ldots N$, $y = 1, 2, \ldots N$

$R_Z = \{1, 2, \ldots N\}$.

Fix $i \in \{1, 2, \ldots N\}$.

$$P(z=i) = \sum_{(x,y):\, g(x,y)=\min(x,y)=i} f(x,y)$$

$$= \sum_{y=i}^{N} f(i,y) + \sum_{x=i+1}^{N} f(x,i).$$

$$= \frac{N-(i-1)}{N^2} + \frac{N-i}{N^2}.$$

$$= \frac{2(N-i)+1}{N^2}.$$

$$\sum_{i=1}^{N} P(z=i) = \frac{1}{N^2}\sum_{i=1}^{N}(2N-2i+1)$$

$$= \frac{1}{N^2}\left(2N^2 - \frac{2N(N+1)}{2} + N\right)$$

$$= \frac{1}{N^2}\left(2N^2 - N^2 - N + N\right)$$

$$= 1.$$

Discrete R.V.

16 Mar 18 ex: Show that sum of 2 independent poisson random variable with parameter λ & μ is a poisson random variable with parameter $\lambda+\mu$.

$X \sim \text{Poisson}(\lambda)$

$Y \sim \text{Poisson}(\mu)$.

→ $P(X=k) = \dfrac{\mu^k e^{-\mu}}{k!}$ $\quad k = 0, 1, 2, \ldots$

$Z = X + Y.$

$g(x,y) = x+y$

Range of Z

Fix $x=0$, $z = 0+y$, $z = 0, 1, 2, 3 \ldots$

Fix $x=1$, $z = 1+y$, $z = 1, 2, \ldots$

Fix $x=2$, $z = 2+y$. $z = 2, 3 \ldots$

Teacher's Signature.........

$R_Z = \{0, 1, 2 \ldots\ldots\ldots\}$

$$P(Z=k) = \sum_{(x,y):\, x+y=k} f(x,y)$$ but here we don't know $f(x,y)$

$$= \sum_{(x,y):\, x+y=k} f_X(x) f_Y(y)$$ ($\because$ X & Y are ind.)

if $x = i$, then $y = k-i$.

$i \in \{0, 1, 2, \ldots\ldots k\}$

$$\Rightarrow P(Z=k) = \sum_{i=0}^{k} f_X(i) f_Y(k-i)$$

$$= \sum_{i=0}^{k} \frac{\mu^i e^{-\mu}}{i!} * \frac{\lambda^{(k-i)} e^{-\lambda}}{(k-i)!}$$

$$\cancel{= e^{-(\mu+\lambda)} \sum_{i=0}^{k} e^{-(\mu+\lambda)}}$$

$$= e^{-(\mu+\lambda)} \sum_{i=0}^{k} \frac{k!\, \mu^i \lambda^{k-i}}{k!\, i!\, (k-i)}$$

$$= \frac{e^{-(\mu+\lambda)}}{k!} \sum_{i=0}^{k} \frac{\mu^i \lambda^{k-i}\, k!}{i!\, (k-i)!}$$

$$= \frac{e^{-(\mu+\lambda)}}{k!} (\mu+\lambda)^k.$$

$\Rightarrow Z \sim$ Poisson $(\lambda + \mu)$

One Dimensional Case

If X is discrete & $g: \mathbb{R} \to \mathbb{R}$ a f^n then $g(X)$ is also discrete.

Let X be a continuous r.v. & $g: \mathbb{R} \to \mathbb{R}$ be a function. Is $g(X)$ a continuous r.v.?

ex: Let $Y \sim \exp(5)$ & consider $g: \mathbb{R} \to \mathbb{R}$.

$$g(x) = \min\{x, 10\}$$
$$= \begin{cases} x, & x \leq 10 \\ 10 & x > 10 \end{cases}$$

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$Z = g(X)$

$F_Z(z) = P\{Z \le z\} \quad \forall\, z \in \mathbb{R}$

$\{Z \le z\} = \{w \in \Omega \mid Z(w) \le z\}$

$Z(w) \le z \quad = \quad \text{Min}\{X(W), 10\}$

$\hookrightarrow \Leftrightarrow$ either $X(w) \le Z$ or $10 < Z$ or both

$\{Z \le z\} = \{X \le z\} \cup \{W : 10 \le z\}$

$$\{w \in \Omega \mid 10 \le z\} = \begin{cases} \Omega & ; z \ge 10 \\ \phi & ; z < 10 \end{cases}$$

$$\{Z \le z\} = \begin{cases} \{X \le z\} & ; z < 10 \\ \Omega & ; z \ge 10 \end{cases}$$

$$F_Z(z) = \begin{cases} P\{X \le z\} = F_X(z) & ; z < 10 \\ P(\Omega) = 1 & ; z \ge 10 \end{cases}$$

$$F_{\cancel{X}}(z) = \begin{cases} 0 & ; z < 0 \\ 1 - e^{-5z} & ; z \ge 0 \end{cases}$$

$$F_Z(z) = \begin{cases} 0 & ; z < 0 \\ 1 - e^{-5z} & 0 \le z < 10 \\ 1 & z \ge 10 \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & ; z \le 0 \\ 5e^{-5z} & ; 0 < z < 10 \\ 0 & ; z \ge 10 \end{cases}$$

*** Alert

$$\int_0^{10} f_Z(z)\,dz = \int_0^{10} 5e^{-5z}\,dz$$

$$= \frac{5}{-5}\left[e^{-5z}\right]_0^{10}$$

$$= 1 - e^{-50} < 1.$$

Something has been done wrong as pdf summation/integration comes = 1

Teacher's Signature.........

If a random variable has pdf then it's distribution f^n is continuous on $\mathbb{R}$

What is the range of z

$R_X = [0, \infty)$ $\quad Z = \min\{X, 10\}$.

$R_Z = [0, 10]$ .....

17 Mar ~~ex:~~ Suppose (X, Y) has joint pdf $f(x, y)$ & $g : \mathbb{R}^2 \to \mathbb{R}$. How to compute pdf of r.v. $Z = (X, Y)$ if it exists.

$Z : \Omega \to \mathbb{R}$

ex: Let X and Y be r.v.'s with joint pdf $f(x, y)$. Find the density of $X + Y$ (if it $\in x_i, y_i$).

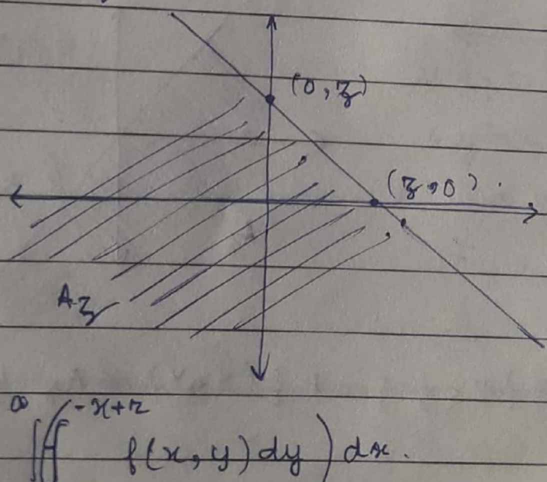
→ If the distribution function of r.v. Z can be written as

$$F_Z(z) = \int_{-\infty}^{z} g(t)\,dt \quad \forall\ z \in \mathbb{R}$$ then g is a pdf for r.v. Z.

$$F_Z(z) = P\{Z \le z\}$$
$$= P\{(X, Y) \in A_z\}$$

where $A_z = \{(x, y) \in \mathbb{R}^2 \mid x + y \le z\}$

$$\therefore F_Z(z) = \iint_{A_z} f(x, y)\,dx\,dy$$



$x + y = z$

$y = -x + z$

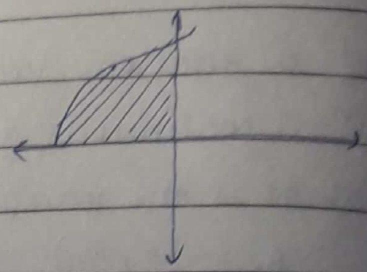
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{-x+z} f(x, y)\,dy\right)dx.$$

Substitute $y = s - x$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} f(x, s - x)\,ds\right)dx$$

Teacher's Signature..........

$$= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} f(x, s-x)\, dx\right) ds$$



$$g(s) = \underbrace{\int_{-\infty}^{\infty} f(x, s-x)\, dx}_{\geq 0}$$

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f(x, t-x)\, dx.$$

ex: The joint density of X & Y is given by:

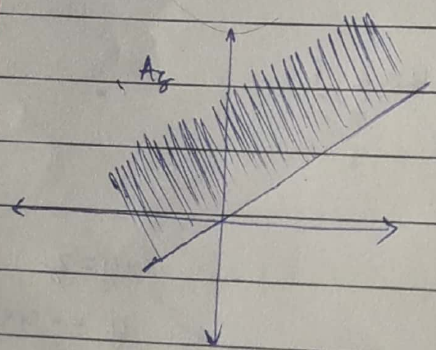
$$f(x,y) = \begin{cases} e^{-(x+y)} ; & 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $\frac{X}{Y}$

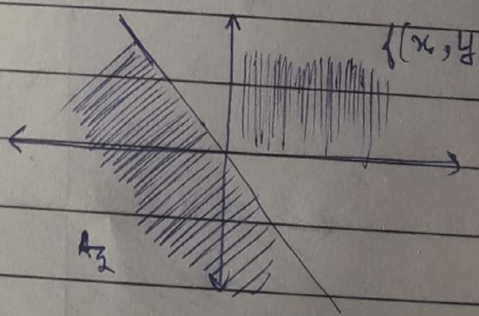
→ $F_Z(z) = P\{Z \leq z\} = P\{(X,Y) \in A_z\}$.

where $A_z = \{(x,y) \in \mathbb{R}^2 \mid \frac{x}{y} \leq z\}$.

$\Rightarrow x \leq zy$



$y = \frac{x}{z}$ when $z \geq 0$.



$y = \frac{x}{z}$ when $z < 0$

~~If $F_Z(z) = P\{Z \leq z\} = P\{(X,Y) \in A_z\}$ where $A_z = \{(x,y) \in \mathbb{R}^2 \mid \frac{x}{y} = z\}$.~~

If $z < 0$ then $A_z \cap \{(x,y)\ f(x,y) > 0\} = \phi \Rightarrow F_Z(z) = 0$

" $z \geq 0$ then $F_Z(z) = \int_0^{\infty}\left(\int_{x/z}^{\infty} e^{-y}\, dy\right) dx$.

$= 1 - \frac{1}{z+1}$

$$F_Z(z) = \begin{cases} 0 & ; \quad z \leq 0 \\ 1 - \frac{1}{z+1} & ; \quad z \geq 0. \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{(z+1)^2} & z \geq 0 \end{cases}$$

$$\int_0^\infty \frac{1}{(1+z)^2}\, dz = -\left[\frac{1}{(1+z)}\right]_0^\infty$$

$$= -[-1]$$

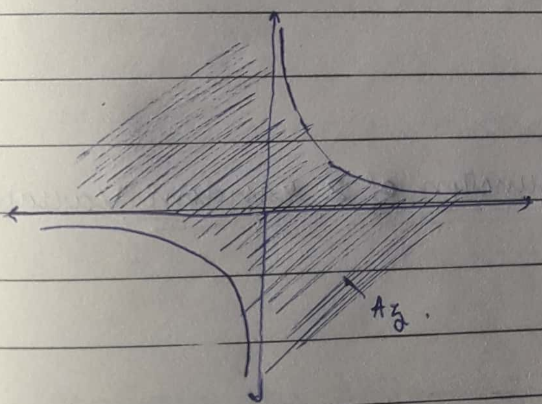
$$= 1.$$

ex: Let X and Y be two independent uniform (0, 1) random variable. Find the pdf of XY (if it exists

→ $Z = XY$

$F_Z(z) = P(Z \leq z) = P\{(X, Y) \in A_z\}$

$A_z = \{(x, y) \in \mathbb{R}^2 \mid xy \leq z\}$

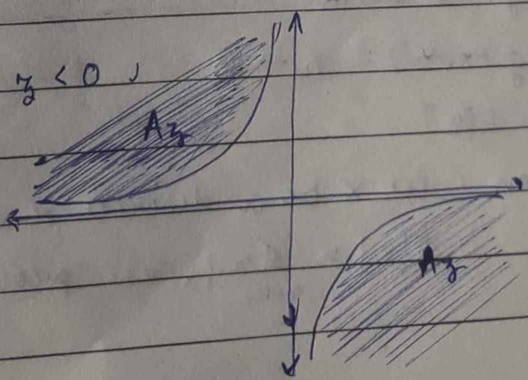


if $z \geq 0$

$xy = z. \Rightarrow y = \frac{z}{x}$

$xy \leq z$

If $z < 0$,



Teacher's Signature...........

$$f_X(x) = \begin{cases} 1 & ; \quad 0 \leq x < 1 \\ 0 & ; \quad \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

$$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} 1 & ; \quad 0 < x, y < 1 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

if $z \geq 1$ then $A_z \supset D$ where

$D = (0,1) \times (0,1)$ Thus

$$F_Z(z) = \iint_D \underbrace{f(x,y)}_{=1} \, dx\, dy$$

$$= 1.$$

if $0 < z < 1$

$$F_Z(z) = (z - 1) \int_z^1 \left(\int_0^{z/y} f(x, y)\, dx \right) dy$$

$$= z - z \ln z$$

$$F_Z(z) = \begin{cases} 0 & ; \quad z \leq 0 \\ z - z \ln z & ; \quad 0 < z < 1 \\ 1 & ; \quad z \geq 1 \end{cases}$$

$$f_Z(z) = \begin{cases} -\ln z & 0 < z < 1. \\ 0 & \text{o.w.} \end{cases}$$

19 March **Expectation of a real-valued function of 2 random variables.**

$X, Y : \Omega \to \mathbb{R}$ r.v.

$g : \mathbb{R}^2 \to \mathbb{R}$

$Z = g(X, Y) : \Omega \to \mathbb{R}$

$E[Z]$

Definition : Let X be a discrete r.v. with the pmf f_X. Then

$$E[X] = \sum_{x \in R_X} x f_X(x) \quad \text{provided} \quad \sum_{x \in R_X} |x| \, \underbrace{f_X(x)}_{\geq 0} < \infty$$

Teacher's Signature..........

R_X = Range of r.v. X.

if R_X is finite then

$$\sum_x x f_X(x) < \infty \iff \sum |x| f_X(x) < \infty$$

$$x \leq |x|.$$

if R_X is finite then

$\sum_x x f_X(x) < \infty$ but $\sum_{x \in R_X} |x| f_X(x)$. may not be ~~fi~~ finite

$\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges but } \sum_{n=1}^{\infty} \frac{1}{n} = \infty \right]$

ex: Let X be a r.v.

the pmf $P\left(X = \frac{(-1)^{n+1} 3^n}{n}\right) = \frac{2}{3^n}$ $(n = 1, 2, \ldots)$

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\left(\frac{1/3}{1 - 1/3}\right) = 1.$$

$$\sum_{n=1}^{\infty} n\, P\left(X = \frac{3^n}{n} \times \frac{2}{3^n}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n} \times \frac{2}{3^n}.$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$< \infty$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \times \frac{2}{3^n} = \infty$$

$\Rightarrow$ E[X] doesn't exist.

ex: $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots$ — (1)

consider the following rearrangement of the series (1)

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots\ldots < \infty$ — (2)

Sum

Another arrangement of 1

$$1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} \quad -(3)$$

(3) converges to $\frac{S}{2}$.

Theorem ⇒ A conditionally convergent series can be made to converge to any arbitrary no. or even ~~value~~ made diverge by suitable rearrangement of its terms.

Theorem ⇒ If the series converges absolutely then every rearrangement of the series converges to the same number

Definition ⇒ Let X be a r.v. with pdf f_X. Then

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \quad \text{provided} \int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$$

Theorem: (i) Let X be a discrete r.v. with pmf f_X and $g: \mathbb{R} \to \mathbb{R}$ be a function then

$$E[g(X)] = \sum_{x \in R_X} g(x) f_X(x) \quad \text{provided} \sum_{x \in R_X} |g(x)| f_X(x) < \infty$$

(ii) Let X be a r.v. with pdf f_X & $g: \mathbb{R} \to \mathbb{R}$ be a "Borel function" then

works even when pdf isn't available ⇐

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \quad \text{provided}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x)\,dx < \infty$$

ex: $X \sim \exp(5)$.

$g(x) = \min\{x, 10\}$. → continuous

$g(X)$ doesn't have pdf

$$E[g(X)] = \int_{0}^{\infty} \min\{x, 10\}\, 5e^{-5x}\,dx \qquad x \in [0, \infty)$$

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$$= \int_0^{10} x \cdot 5e^{-5x}\,dx + \int_{10}^{\infty} 10e^{-5x}\,dx$$

$$= 10 \underbrace{\int_{10}^{\infty} e^{-5x}\,dx}_{\text{exist \& is finite}}$$

Theorem : (i) Let X and Y be discrete r.v. with pdf $f(x,y)$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a function then

$$E[g(X,Y)] = \sum_{x \in R_X,\, y \in R_Y} \sum g(x,y) f(x,y) \text{ provided } \sum_{x \in R_X,\, y \in R_Y} \sum |g(x,y)| f(x,y) < \infty$$

(ii) Let X & Y be r.v. with a pdf f_X & $g : \mathbb{R}^2 \to \mathbb{R}$ be a "Borel" function then

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy \text{ provided } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)| f(x,y)\,dx\,dy < \infty$$

ex:

X \ Y	-1	0	2	6
-2	1/9	1/27	1/27	1/9
1	2/3	0	1/9	1/9
3	0	0	-1/9	4/27

$g(x,y) = |y-x|$

$Z = |Y-X|$

$$\cancel{E[|Y-X|] = \sum_{x \in R_X,\, y \in R_Y} |y-x| f(x,y)}$$

$E[|Y-X|] = ?$

→ $$E[|Y-X|] = \sum_{x \in R_X} \sum_{y \in R_Y} |y-x| f(x,y).$$

$$= \sum_{y \in R_Y} |y+2| f(-2,y) + \sum_{y \in R_Y} |y-1| f(1,y) + \sum_{y \in R_Y} |y-3| f(3,y)$$

$$\sum |y+2| f(-2,y) = 1 \times \frac{1}{9} + 2 \times \frac{1}{27} + 4 \times \frac{1}{27} + 8 \times \frac{1}{9} = \frac{11}{9}.$$

$$\sum_{y \in R_Y} |y-1| f(1,y) = 2 \times \frac{2}{9} + 1 \times 0 \quad 1 \times \frac{1}{9} \times 5 \times \frac{1}{9} = \frac{10}{9}.$$

$$\sum |y-3| f(3,y) = 2 \times \frac{2}{9} + 1 \times 0 + 1 \times \frac{1}{9} + 5 \quad = \frac{5}{9}$$

Teacher's Signature..........

PMF that we have calculated earlier ⇒

$P(Z=1) = \frac{1}{3}$, $P(Z=2) = \frac{7}{27}$, $P(Z=3) = \frac{4}{27}$

$P(Z=4) = \frac{1}{27}$, $P(Z=5) = \frac{1}{9}$, $P(Z=8) = \frac{1}{9}$

$$E[|Y-X|] = \sum_{z \in R_Z} z\, P(Z=z)$$

$$= \frac{1}{3} + \frac{14}{27} + \frac{12}{27} + \frac{4}{27} + \frac{5}{9} + \frac{8}{9}$$

$$= \frac{9+14+12+4+15+24}{27}$$

$$= \frac{78}{27} = \frac{26}{9}$$

20 Mar

ex: Let X and Y be independent exponential (λ) random variable. Find the mean of max $\{X, Y\}$.

→ X and Y are independent

$X \sim exp(\lambda)$, $Y \sim exp(\lambda)$

$$f_X(x) = \begin{cases} 0 & ; x < 0 \\ \lambda e^{-\lambda x} & ; x \geq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & ; y < 0 \\ \lambda e^{-\lambda y} & ; y \geq 0 \end{cases}$$

$$E[\max\{X, Y\}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max\{x, y\}\, f(x,y)\, dx\, dy$$

Joint density of X & Y.

$f(x,y) = f_X(x) f_Y(y)$

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & ; 0 \leq x, y < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

Teacher's Signature........

$$I = \int_0^\infty \int_0^\infty \max\{x, y\} f(x, y)\, dx\, dy$$

$$I = \int_0^\infty \int_0^\infty \max\{x, y\} \lambda^2 e^{-\lambda(x+y)}\, dx\, dy$$

$$I = I_1 + I_2$$

$$I_1 = \int_0^\infty \left(\int_0^\infty x \lambda^2 e^{-\lambda(x+y)}\, dy \right) dx.$$

$$= \frac{3}{4\lambda}$$

$$I_2 = \int_0^\infty \left(\int_x^\infty y \lambda^2 e^{-\lambda(x+y)}\, dy \right) dx$$

$$= \frac{3}{4\lambda}$$

20 March

Conditional Distributions.

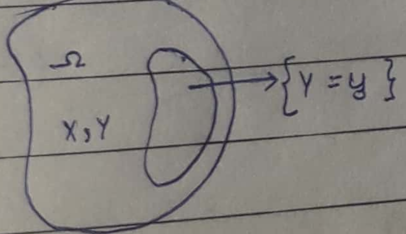
Suppose X and Y are 2 r.v. on sample space Ω & X, Y are dependent. "Quantify the independence"

Definition ⇒ Let X and Y be discrete r.v. with joint pmf $f(x,y)$. Then the conditional pmf of r.v. X given $Y = y$ is defined:

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_X(y)} & : f(y) > 0 \\ 0 & : f_Y(y) = 0 \end{cases}$$

$$f_Y(y) = P(Y=y)$$

$$= \begin{cases} P(X = x \mid Y=y) ; & \text{if } P(Y=y) > 0 \\ 0 & \text{if } P(Y=y) = 0. \end{cases}$$



$f_X(x) \to (\Omega, P)$.

$\{Y=y\} \to f_{X|Y}(x|y)$.

Let $P\{Y=y\}>0$.

$$\sum_{x \in R_X} f_{X|Y}(x|y) = \sum_{x \in R_Y} \frac{f(x,y)}{f_Y(y)}.$$

$$= \frac{1}{f_Y(y)} \sum_{x \in R_X} f(x,y) = \frac{f_Y(y)}{f_Y(y)} = 1.$$

Conditional Distribution Function

$F_X(x) = P\{X \leq x\} \quad \forall x \in R.$

$F_X : R \to [0,1]$

If X is discrete with pmf f_X then $F_X(x) = \sum_{t \in R_X : t \leq x} f_X(t)$

$$\underset{P(X \leq x \mid Y=y)}{\overset{}{F_X(x|y)}} = \sum_{t \in R_X : t \leq x} f_{X|Y}(t|y).$$

ex: Let the joint pmf of X and Y be given as

X \ Y	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

$\to f_X(-1) = 1/4$

$\to f_X(0) = 1/2.$

$\to f_X(1) = 1/4.$

Find the conditional pmf of X given $Y=0$. Also find the conditional D.F. of X given $Y=0$.

Sol: $f_Y(0) = P(Y=0)$.

$$= \sum_{x \in R_X} f(x,0) = \frac{1}{2}.$$

Teacher's Signature..........

Distribution f^n is right continuous.

$$\frac{f(x,0)}{f_Y(0)}$$

$$f_{X|Y}(x|0) = \begin{cases} 1/2 & x=-1 \\ 0 & x=0 \\ 1/2 & x=1 \end{cases}$$

$$F_X(x|0) = \begin{cases} 0 & \text{if } x<-1 \\ 1/2 & \text{if } -1 \le x < 0 \\ 1/2+0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$f_X(x) = \sum_{y \in R_Y} f(x,y) = \sum_{y \in R_Y} f_{X|Y}(x,y) f_Y(y)$$

Ex: ~~Suppose~~ $f_Y(y) = \begin{cases} 5/6 & ; y=10^2 \\ 1/6 & ; y=10^4 \end{cases}$

$$f_{X|Y}(x|10^2) = \begin{cases} 1/2 & x=10^{-2} \\ 1/3 & x=10^{-1} \\ 1/6 & x=1. \end{cases}$$

$$f_{X|Y}(x|10^4) = \begin{cases} 1/2 & \text{if } x=1. \\ 1/3 & x=10 \\ 1/6 & x=100. \end{cases}$$

Find $f_X(x)$

→ $R_X = \{10^{-2}, 10^{-1}, 1, 10, 10^2\}$

$$f_X(10^{-2}) = \sum_{y \in R_Y} f_{X|Y}(10^{-2}|y) f_Y(y).$$
$$= \frac{1}{2} \times \frac{5}{6}$$

$$f_X(10^{-1}) = \frac{1}{3} \times \frac{5}{6} + 0 \times \frac{1}{3} = \frac{5}{18}.$$

$$f_X(1) = \frac{1}{6} \times \frac{5}{6} + \frac{1}{2} \times \frac{5}{6} = \frac{2}{9}.$$

Teacher's Signature..........

$f_X(10) = \frac{1}{18}$

$f_X(100) = \frac{1}{56}$.

Conditional pdf

Let X and Y be continuous r.v. with joint pdf (x,y) the conditional pdf of X given $Y=y$ is defined as :

$$f_{X|Y}(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & f_Y(y)>0 \\ 0 & f_Y(y)=0 \end{cases}$$

Let $f_Y(y)>0$.

$$\int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1.$$

** $P(A|B) = \frac{P(A\cap B)}{P(B)}$ where $P(B)>0$.

Let X has pdf f_X then $P(X \in A) = \int_A f_X(x)dx$ for any 'Borel' $A \subseteq \mathbb{R}$.

$$P(X\in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx.$$

If Y is a continuous r.v. with pdf then $P(Y=y)=0 \cdot \forall y \in \mathbb{R}$

$$f_Y(y) \neq P\{Y=y\}$$

ex: Let X and Y be 2 r.v. with joint pdf

$$f(x,y) = \begin{cases} 2 & \text{if } 0<x<y<1. \\ 0 & \text{otherwise} \end{cases}$$

Find the $P\left(X \le \frac{2}{3} \mid Y=\frac{3}{4}\right)$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

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$$= \begin{cases} 0 & y < 0 \\ y\cdot\int_0 2\,dx & 0 \le y \le 1 \\ 0 & y \ge 1 \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} 2/2y & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P\left(X \le \frac{2}{3} \,\middle|\, Y = \frac{3}{4}\right) = \int_{-\infty}^{2/3} f_{X|Y}\left(x \,\middle|\, \frac{3}{4}\right) dx$$

$$= \int_0^{2/3} \frac{1}{(3/4)}\, dx$$

$$= \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

21 Mar

ex: (Another method of previous question)

joint pdf of X and Y

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

To find $P\left(X \le \frac{2}{3} \,\middle|\, Y = \frac{3}{4}\right)$

$$= P\left(X \in \left(-\infty, \frac{2}{3}\right] \,\middle|\, Y = \frac{3}{4}\right)$$

$$= \int_{-\infty}^{2/3} f_{X|Y}\left(x \,\middle|\, \frac{3}{4}\right) dx = ?$$

$$f_{X|Y}\left(x \,\middle|\, \frac{3}{4}\right) = \begin{cases} \dfrac{f(x, 3/4)}{f_Y(3/4)} & f_Y\left(\frac{3}{4}\right) > 0 \\ 0 & ;\ f_Y\left(\frac{3}{4}\right) = 0. \end{cases}$$

$$f_Y\left(\frac{3}{4}\right) = \int_{-\infty}^{\infty} f\left(x, \frac{3}{4}\right) dx = \int_0^{3/4} 2\,dx = \frac{3}{2} > 0$$

$$f_{X|Y}\left(x \,\middle|\, \frac{3}{4}\right) = \begin{cases} \dfrac{2}{(3/2)} = \dfrac{4}{3}; & 0 < x < \dfrac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

Teacher's Signature..........

$$\therefore \int_0^{2/3} f_{X|Y}\left(x \mid \tfrac{3}{4}\right) dx = \int_0^{2/3} \frac{4}{3}\, dx = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

ex: Let X and Y be independent r.v. with pdfs f_X and f_Y, respectively. Compute the conditional density of X + Y given X.

→ $Z = X + Y$

$$\begin{cases} \dfrac{g(z, x)}{f_X(x)}, & f_X(x) > 0. \end{cases}$$

where g is the joint pdf of Z & X. If we can express conditional distribution function of Z given X as

$\int_{-\infty}^{z} (\ \ )\, dt$ then integrand in the conditional pdf of Z given X.

$$F_{Z|X}(z|x) = P(Z \leq z \mid X = x)$$
$$= P(X + Y \leq z \mid X = x)$$
$$= P(x + Y \leq z \mid X = x)$$
$$= P(Y \leq z - x \mid X = x)$$
$$= P(Y \leq z - x) = \int_{-\infty}^{z-x} f_Y(t)\, dt$$

$t = s - x.$

$$= \int_{-\infty}^{z} f_Y(s - x)\, ds$$

$$f_{Z|X}(s|x) = f_Y(s - x)$$

ex: Let X and Y be iid geometric r.v. with parameter p. Find the conditional pmf of Y given X+Y = n where n > 2

→ X takes 1, 2, 3, ...

$$P(X = k) = p(1-p)^{k-1}$$

$$\frac{g(z, y)}{f_Z(z)}$$

Conditional pmf of Y given Z = X + Y is $f_{Y|Z}(y|z) =$

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specively

df of

$$\begin{cases} P(Y=y \mid Z=z) & \text{if } P(Z=z)>0 \\ 0 & \text{if } P(Z=z)=0 \end{cases}$$

if $Z=n$ $\quad n \geq z$

$P(Y=y \mid Z=n)=0$ if $y \geq n$

$X+Y=n$

if $y \in \{1,2,\dots\dots n-1\}$ Then $P(Y=y \mid X+Y=n)=$

$$\frac{P(Y=y,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(Y=y,\ X+Y=n)}{P\left(\bigcup_{k=1}^{n-1}\{X=k,\ Y=n-k\}\right)}$$

$$= \frac{P(Y=y,\ X=n-y)}{\sum_{k=1}^{n-1} P(X=k,\ Y=n-k)}$$

$$= \frac{P(Y=y)\,P(X=n-y)}{\sum_{k=1}^{n-1} P(X=k)\,P(Y=n-k)}$$

$$= \frac{p(1-p)^{y-1}\cdot p\cdot(1-p)^{n-y-1}}{\sum_{k=1}^{n-1} p(1-p)^{k-1}\ p(1-p)^{n-k-1}}$$

$$= \frac{p^2(1-p)^{n-2}}{\sum_{k=1}^{n-1} p^2(1-p)^{n-2}}$$

$$= \frac{1}{\sum_{k=1}^{n-1} 1}$$

$$= \left(\frac{1}{n-1}\right).$$

Teacher's Signature..........

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Discrete uniform R.V. $\{1, 2, \ldots n-1\}$

Conditional pmf of Y given $Z = X+Y$ is $f_{X|Y}(y|n) = \begin{cases} \frac{1}{(n-1)} & y = 1, \\ 0 & \end{cases}$

* <u>Total Probability Theorem.</u>

Let Ω be sample space & $\{A_1, A_2, \ldots A_N\}$ be the atmost countable position Ω $\bigcup_{i=1}^{N} A_i = \Omega$ & $A_i \cap A_j = \phi$ (if $i = j$)

Then for any event B

$$P(B) = \sum_{i=1}^{N} P(B/A_i) P(A_i).$$

Suppose X is a discrete r.v. with pmf $f_X \Rightarrow \{\{X = x\}\}$

$\bigcup_{x \in R_X} \{X = x\} = \Omega$

For any event R.

$$P(B) = \sum_{x \in R_X} P(B | X = x) P(X = x) = f_X(x)$$

ex: ~~Let X and Y be~~ If X is continuous r.v. with pdf f_X. Then

$$P(B) = \int_{-\infty}^{\infty} P(B | X = x) f_X(x) dx \qquad P\{X = x\} = 0.$$

ex: Let X and Y be iid uniform $(0, 1)$ random variables. Find $P(X^3 + Y > 1)$.

(i) joint pdf of X and Y, $f(x,y) = f_X(x) f_Y(y)$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = \begin{cases} 1 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Teacher's Signature........

Discrete uniform R.V. $\{1, 2, \ldots, n-1\}$

Conditional pmf of Y given $Z = X+Y$ is $f_{X|Y}(y|n) = \begin{cases} \frac{1}{(n-1)} & y = 1, 2, \ldots \\ 0 & y \geq n \end{cases}$

* Total Probability Theorem.

Let Ω be sample space & $\{A_1, A_2, \ldots, A_N\}$ be the atmost countable position Ω $\bigcup_{i=1}^{N} A_i = \Omega$ & $A_i \cap A_j = \phi$ $(i \neq j)$

Then for any event B

$$P(B) = \sum_{i=1}^{N} P(B/A_i) P(A_i).$$

Suppose x is a discrete r.v. with pmf $f_X \Rightarrow \{\{X = x\}\}_{x \in R_X}$

$$\bigcup_{x \in R_X} \{X = x\} = \Omega$$

For any event B.

$$P(B) = \sum_{x \in R_X} P(B | X = x) P(X = x) = f_X(x)$$

~~ex: Let X and Y be~~ If x is continuous r.v. with pdf f_X. Then,

$$P(B) = \int_{-\infty}^{\infty} P(B | X = x) f_X(x) dx \qquad P\{X = x\} = 0.$$

ex: Let X and Y be iid uniform $(0, 1)$ random variables. Find $P(X^3 + Y > 1)$.

1st Method → (i) joint pdf of X and Y, $f(x,y) = f_X(x) f_Y(y)$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = \begin{cases} 1 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Teacher's Signature..........

$P(X^3 + Y > 1) = P((X, Y) \in A)$

where $A = \{ (x, y) \in \mathbb{R}^2 \mid x^3 + y > 1 \}$

$= \iint_A f(x, y)\, dx\, dy$

Draw: $x^3 + y = 1$.

$y = 1 - x^3$.

$f(x) = 1 - x^3$.

1st ⇒ Domain $= \mathbb{R}$, $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$ f is continuous

By IVP, so range of $f\ (-\infty, \infty)$

2nd ⇒ $f'(x) = -3x^2 \leq 0 \ \forall x \in \mathbb{R}$ f is decreasing on $\mathbb{R}$

3rd ⇒ $f'(x) = 0 \Rightarrow x = 0$ no local extrema

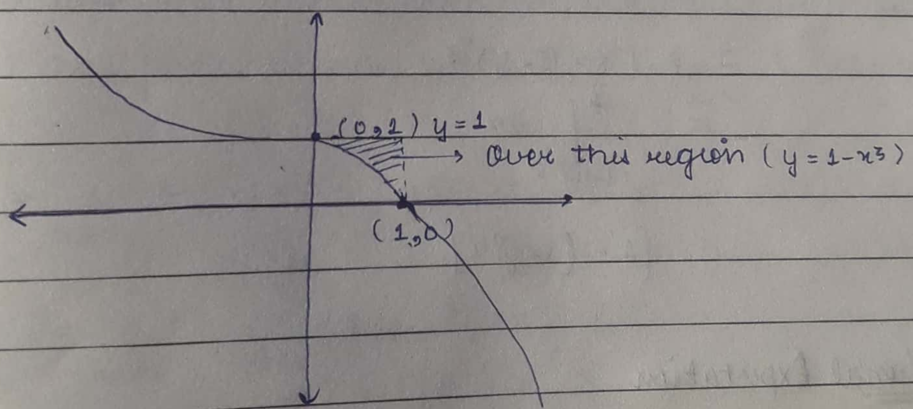
4th ⇒ $f''(x) = -6x$.

on $(-\infty, 0)$ $f''(x) > 0$ convex

on $(0, \infty)$ $f''(x) < 0$ concave.

$x = 0$ is the pt. of inflexion.

5th ⇒ ~~Graph~~ Curve intersects with graph at $(0,1)$ and $(1,0)$



$$\therefore \iint_A f(x, y)\, dx\, dy = \int_0^1 \left(\int_{1-x^3}^{1} dy \right) dx$$

$$= \int_0^1 \left(1 - (1 - x^3) \right) dx$$

$$= \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

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2nd Method

$$P(X^3+Y>1) = \int_{-\infty}^{\infty} P(X^3+Y>1 \mid X=x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(x^3+Y>1 \mid X=x)\, f_X(x)\, dx$$

$$= \int_{0}^{1} P(Y>1-x^3 \mid X=x)\, dx.$$

$$= \int_{0}^{1} P(Y>1-x^3)\, dx.$$

$$P(Y>1-x^3) = \int_{1-x^3}^{1} dy.$$

$$\left(\begin{array}{c} 0<x<1 \\ x^3<1 \\ 1-x^3>0 \end{array}\right) = x^3$$

3rd Method

$$P(X^3+Y>1) = \int_{-\infty}^{\infty} P(X^3+Y>1 \mid Y=y)\, f_Y(y)\, dy.$$

$$= \int_{0}^{1} P(X^3>1-y)\, dy.$$

$$= P(X>(1-y)^{1/3})$$

$$= \int_{(1-y)^{1/3}}^{1} dx$$

$$= \left[1-(1-y)^{1/3}\right]$$

Conditional Expectation

Let X and Y be discrete random variables with conditional ~~pent~~ pmf of X given $Y=y$. Then the conditional expectation of X given $Y=y$ is defined as:

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2nd Method

$$P(x^3+Y>1) = \int_{-\infty}^{\infty} P(x^3+Y>1 \mid X=x)\, f_X(x)\,dx$$

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$$= \int_0^1 P(Y>1-x^3 \mid X=x)\,dx.$$

$$= \int_0^1 P(Y>1-x^3)\,dx.$$

$$P(Y>1-x^3) = \int_{1-x^3}^{1} dy.$$

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3rd method

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Conditional Expectation

Let X and Y be discrete random variables with conditional ~~pmf~~ pmf of X given $Y=y$. Then the conditional expectation of X given $Y=y$ is defined as:

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$$E[X \mid Y=y] = \sum_{x \in R_X} x f_{X|Y}(x|y) \quad \text{provided } \sum_{x} |x| f_{X|Y}(x|y) < \infty$$

Ex: Let X and Y be iid geometric (p) random variables. Find $E[Y \mid X+Y = n]$ where $n \geq 2$.

→ $Z = X+Y$

$$f_{Y|Z}(y|n) = \begin{cases} \frac{1}{n-1}, & y = 1, 2, \dots n-1. \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y \mid Z=n] = \sum_{y=1}^{n-1} y f_{Y|Z}(y|n)$$

$$= \sum_{y=1}^{n-1} \frac{y}{n-1} = \frac{1}{n-1} \sum_{y=1}^{n-1}$$

$$= \frac{1}{(n-1)} \left(\frac{(n-1)(n-1+1)}{2} \right)$$

$$= \frac{n}{2}.$$

Total Expectation Theorem.

Let X and Y be discrete random variables with joint pmf $f(x,y)$. If Y has finite means then $E[Y] = \sum_{x \in R_X} E[Y \mid X=x] f_X(x)$.

Proof ⇒ $\sum_{x \in R_Y} E[Y \mid X = x] f_X(x)$.

$$= \sum_{x \in R_X} \left(\sum_{y \in R_Y} y f_{Y|X}(y|x) \right) f_X(x)$$

$$= \sum_{y \in R_Y} y \left(\sum_{x \in R_X} \underbrace{f_{Y|X}(y|x) f_X(x)}_{f(x,y)} \right)$$

$$= \sum_{y \in R_Y} y \left(\sum_{x \in R_X} f(x,y) \right)$$

$$= \sum_{y \in R_Y} y f_Y(y) \quad = Y$$

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Celebrations
Dt. / /

24 Mar 18 Q: Let X and Y be continuous r.v. with joint pdf

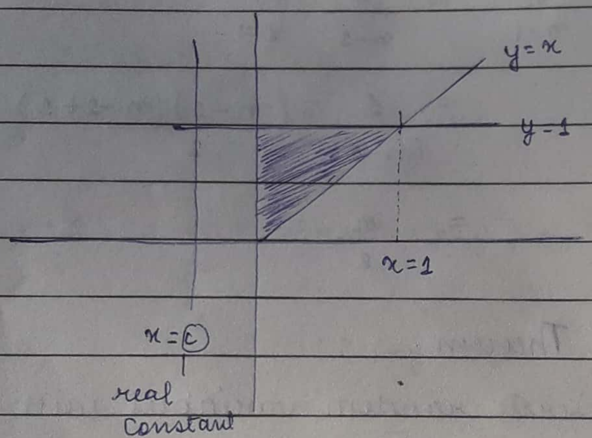
$$f(x,y) = \begin{cases} 6(y-x) & 0 \le x \le y \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

Find the $E[Y \mid X = x]$ and hence calculate $E[Y]$

→ $E[Y \mid X = x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$

$$f_{Y|X}(y|x) = \begin{cases} \frac{f(x,y)}{f_X(x)} & , f_X(x) > 0 \\ 0 & f_X(x) = 0. \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$



if $x < 0$ then $f_X(x) = 0$

if $x > 1$ then $f_X(x) = 0$.

if $0 \le x \le 1$ then $f_X(x) = \int_{x}^{1} 6(y-x)\, dy$.

$$= \left[\frac{6y^2}{2} - 6xy \right]_x^1 .$$

$$= 6 \left[\left(\frac{1}{2} - x \right) - \left(\frac{x^2}{2} - x^2 \right) \right]$$

$$= 6 \left[\frac{x^2}{2} - x + \frac{1}{2} \right]$$

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$= 3(x-1)^2$

for $0 < x < 1$.

$$E[Y \mid X = x] = \int_x^1 \frac{y\, 6(y-x)}{3(x-1)^2}\, dy$$

$$= \frac{2}{(x-1)^2} \int_x^1 (y^2 - xy)\, dy$$

$$= \frac{2}{(x-1)^2} \left[\frac{y^3}{3} - \frac{xy^2}{2} \right]_x^1$$

$$= \frac{2}{(x-1)^2} \left[\left(\frac{1}{3} - \frac{x}{2} \right) - \left(\frac{x^3}{3} - \frac{x^3}{2} \right) \right]$$

$$= \frac{2}{(x-1)^2} \left[\frac{(2-3x) - 2x^3 + 3x^3}{6} \right]$$

$$= \frac{\cancel{2}(x^3 - 3x + 2)}{3(x-1)^2}$$

$$= \frac{(x-1)(x^2 + x - 2)}{3(x-1)^2}$$

$$= \frac{x^2 + x - 2}{3(x-1)}.$$

By Total Expectation Theorem

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X = x] . f_X(x)\, dx$$

$$= \int_0^1 \frac{(x^2 + x - 2)}{3(x-1)} \times 3(x-1)^2\, dx$$

$$= \int_0^1 (x^3 - 3x + 2)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_0^1$$

$$\therefore = \frac{1}{4} - \frac{3}{2} + 2 = \frac{3}{4}$$

Theorem : Let X and Y be discrete r.v. with conditional ~~pmf of X~~ given $Y=y$. If $g : \mathbb{R} \to \mathbb{R}$ is any function then $E[g(x)|Y=y]$

$$= \sum_{x \in R_X} g(x) f_{X|Y}(x|y)$$

Similar version for continuous case.

Covariance

The covariance of two random variables X and Y is defined as.

$$\text{cov}(X,Y) = E[(X-EX)(Y-EY)]$$ ~~when cov(X,Y~~

When $\text{cov}(X,Y)=0$, we say that X and Y are uncorrelated

$X - EX > 0$
$Y - EY > 0$] When X and Y both tends to take large values together.

if $\text{cov}(X,Y) > 0$ then we say that X and Y are +vely correlated.
" " < 0 " " " " X and Y are -vely correlated.

$$\text{cov}(X,Y) = E[XY - XEY - YEX + EXEY]$$
$$\cancel{= E[XY] - EYEX - E}$$
$$= E[XY] - E[XEY] - E[YEX] + E[EXEY]$$
$$= E[XY] - EYEX - EXEY + EXEY$$
$$= E[XY] - EYEX$$

ex: Suppose the joint pmf of X and Y is given as :

X \ Y	-1	0	1	
-1	0	1/4	0	$\Rightarrow 1/4 = P(X=-1)$
0	1/4	0	1/4	$\Rightarrow 1/2 = P(X=0)$
1	0	1/4	0	$\Rightarrow 1/4 = P(X=1)$

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ex: Find cov (X, Y)

→ $R_{XY} = \{-1, 0, 1\}$

$P(XY = -1) = P(X = 1, Y = -1) + P(X = -1, Y = 1)$

$= 0 + 0$

$= 0$

$P(XY = 0) = 1$

$P(XY = 1) = 0$

$E[XY] = 0 \times P(XY = 0)$

$= 0.$

$EX = -1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} = 0.$

If X and Y are independent. $E[XY] = EX\,EY$. Hence $corr(X, Y) = 0$.
Converse isn't true.

$P(X = 0, Y = 0) = 0 \neq \frac{1}{4}$

$P(X = 0) = \frac{1}{2}$, $P(Y = 0) = \frac{1}{2}$

X and Y are dependent.

<u>Preposition</u> ⇒ Let X, Y and Z be random variables & $a, b \in \mathbb{R}$. Then,

(i) $\text{cov}(X, X) = \text{Var}(X)$

(ii) $\text{cov}(X, Y) = \text{cov}(Y, X)$

(iii) $\text{cov}(X, aY + b) = a\,\text{cov}(X, Y)$

(iv) $\text{cov}(X, Y + Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$

Proof (i) ⇒ $\text{cov}(X, X) = E[X^2] - EX\,EX$

$= \text{Var}\,X.$

(iii) ⇒ $\text{cov}(X, aY + b) = E[X(aY + b)] - E[X]\,E[aY + b]$

$= E[aXY + bX] - EX\,[aEY + b]$

$= aE[XY] + bEX - aEX\,EY - bEX$

$= a\,[E[XY] - EX\,EY]$

$= a\,\text{cov}(X, Y).$

Teacher's Signature..........

(iv) $\text{cov}(X, Y+Z) = E[X(Y+Z)] - EX\, E[Y+Z]$

$= E[XY + XZ] - EX[EY + EZ]$

$= E(XY) + E(XZ) - EXEY - EXEZ$

$= \text{cov}(X,Y) + \text{cov}(X,Z)$

ex: Let X and Y be 2 independent $N(0,1)$ random variables &

$Z = 1 + X + XY^2$

$W = 1 + X$

Find $\text{cov}(Z, W)$

→ $\text{cov}(Z,W) = \text{cov}(1 + X + XY^2, 1 + X)$

$= \text{cov}(X + XY^2, X)$

$= \text{cov}(X, X) + \text{cov}(XY^2, X)$

$= \text{Var}(X) + E[X^2Y^2] - \underbrace{E[XY^2]EX}_{=0}$

$= 1 + E[X^2Y^2]$

$= 1 + \underbrace{E(X^2)}\, E(Y^2)$.

$= 1 + 1$ ↳ At $\text{var}(X) = E[X^2] - (E[X])^2$

$= 2$. $= 1 + 0$

$= 1$.

Let X and Y be r.v. then

$\text{Var}(X+Y) = \text{cov}(X+Y, X+Y)$

$= \text{cov}(X, X+Y) + \text{cov}(Y, X+Y)$.

$= \text{cov}(X, X) + \text{cov}(X,Y) + \text{cov}(Y,X), \text{cov}(Y,Y)$

$\Rightarrow \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X,Y)$.

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Covariance:

$\text{cov}(X, Y) = E[(X - EX)(Y - EY)]$

↳ measure of linear relationship b/w X and Y.

If $(X - EX)$ and $(Y - EY)$ have same sign then $\text{cov}(X,Y) > 0$

" " " " " opposite " " " — < 0

Correlation Coefficient.

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

provided $var(X) > 0$, $var(Y) > 0$

Theorem: The correlation coefficient $\rho(x,y)$ satisfies the following properties:

(i) $|\rho(X,Y)| \leq 1$

(ii) $|\rho(X,Y)| = 1 \Rightarrow \exists\, a, b \in \mathbb{R},\ a \neq 0$ s.t. $Y = aX + b$

Moreover, if $\rho(X,Y) = 1$ then $a > 0$ and if $\rho(X,Y) = -1$ then $a < 0$.

ex: Let $X \sim N(0,1)$, Then $EX = 0$, $EX^2 = 1$, $EX^3 = 0$, $EX^4 = 3$. Let $Y = a + bX + cX^2$. Then find $\rho(X,Y)$

$\rightarrow$ $cov(X, a + bX + cX^2) = cov(X, bX + cX^2)$

$= cov(X, bX) + cov(X, cX^2)$.

$= b\,cov(X,X) + c\,cov(X, X^2)$

$= b\,Var(X) + c\,\{E[X \cdot X^2] - EX\,EX^2]$

$= b + 0$

$var(X) = 1$.

$Var(Y) = cov(Y,Y)$.

$= cov(a + bX + cX^2, a + bX + cX^2)$.

$= cov(bX + cX^2, bX + cX^2)$

$= cov(bX, bX + cX^2) + cov(cX^2, bX + cX^2)$

$= cov(bX, bX) + cov(bX, cX^2) + cov(cX^2, bX) + cov(cX^2, cX^2)$

$= b^2 cov(X,X) + bc\,\underset{=0}{cov(X,X^2)} + bc\,\underset{=0}{cov(X,X^2)} + c^2 cov(X^2, X^2)$.

$= b^2 + c^2 [E(X^2Y^2) - EY^2 EX^2]$

$= b^2 + 2c^2$.

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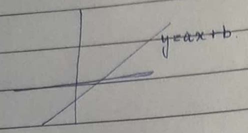
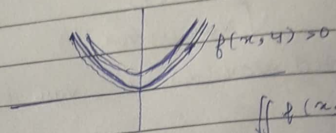
$\rho(X,Y) = \dfrac{b}{\sqrt{b^2+2c^2}} < 1$ if $c \neq 0$.





$\iint f(x,y)\,dx\,dy = 1$

$\text{cov}(X,Y) = 0$

$\rho(X,Y) = 0$.

ex. Let X and Y be 2 r.v. such that $\text{var}(X) = 4$ and $\text{var}(Y) = 9$. If random variables $2X-Y$ and $X+Y$ are independent then find $\rho(X,Y)$

→ $Z = 2X - Y$

$W = X + Y$

Z & W are independent.

$\Rightarrow \text{cov}(Z, W) = 0$.

$\Rightarrow \text{cov}(2X-Y, X+Y) = 0$.

$\Rightarrow \text{cov}(2X, X) + \text{cov}(2X, Y) - \text{cov}(Y, X) - \text{cov}(Y, Y) = 0$

$\Rightarrow 2[\text{cov}(X,X) + \text{cov}(X,Y)] - [\text{cov}(Y,X) + \text{cov}(Y,Y)] = 0$

$\Rightarrow 2\,\text{var}(X) + \text{cov}(X,Y) - \text{var}(Y) = 0$

$\Rightarrow \text{cov}(X,Y) = 1$.

$\rho(X,Y) = \dfrac{1}{\sqrt{4\times 9}} = \dfrac{1}{6}$

$X: \Omega \to \mathbb{R}$ Real Valued r.v.

$Z: \Omega \to \mathbb{C}$ Complex ~~Rando~~ -valued r.v.

$Z(\omega) = X(\omega) + iY(\omega)$ $\boxed{Z = X + iY}$

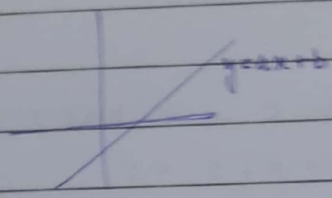
$X: \Omega \to \mathbb{R}$ IR.

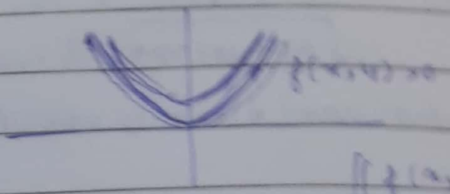
$E[Z] = EX + iEY$

provided EX & EY exists

Teacher's Signature..........

$\rho(x,y) = \dfrac{b}{\sqrt{b^2+2c^2}} < 1$ if $c \neq 0$.





$\iint f(x,y)\,dx\,dy = 1$

$\text{cov}(x,y) = 0$

$\rho(x,y) = 0$

Ex. Let X and Y be 2 r.v. such that $\text{var}(X) = 4$ and $\text{var}(Y) = 9$. If random variables $2X-Y$ and $X+Y$ are independent then find $\rho(X,Y)$

→ $Z = 2X - Y$

$W = X + Y$

Z & W are independent.

$\Rightarrow \text{cov}(Z, W) = 0$.

$\Rightarrow \text{cov}(2X - Y, X + Y) = 0$.

$\Rightarrow \text{cov}(2X, X) + \text{cov}(2X, Y) - \text{cov}(Y, X) - \text{cov}(Y, Y) = 0$

$\Rightarrow 2[\text{cov}(X, X) + \text{cov}(X, Y)] - [\text{cov}(Y, X) + \text{cov}(Y, Y)] = 0$

$\Rightarrow 2\,\text{var}(X) + \text{cov}(X, Y) - \text{var}(Y) = 0$

$\Rightarrow \text{cov}(X, Y) = 1$.

$\rho(X, Y) = \dfrac{1}{\sqrt{4 \times 9}} = \dfrac{1}{6}$

$Y : \Omega \to R$ Real Valued r.v.

$Z : \Omega \to \mathbb{C}$ Complex ~~Rando~~ -valued r.v. $\boxed{Z = X + iY}$

$Z(\omega) = X(\omega) + iY(\omega)$

$X : \Omega \to \mathbb{R}$ IR.

$E[Z] = EX + iEY$

provided EX & EY exists

Characteristic Equation

For any r.v. X its characteristic function $\phi_X : \mathbb{R} \to \mathbb{C}$ is defined as:

$$\phi_X(t) = E[e^{itX}]$$

$$e^{itx} = \cos tx + i\sin tx$$

$|\cos(tx)| \leq 1 \quad \forall \in \mathbb{R}, \forall x$ r.v.

$|(\cos tx)| \leq 1$

$$\left|\sum_{x \in R_X} \cos tx f_X(x)\right| \leq \sum_{x \in R} |\cos tx| + f_X(x)$$
$$= \sum_{x \in R_X} f_X(x) = 1.$$

ex: Let $X \sim$ Bernoulli (p). Find the characteristic f^n of X

$P(X=0) = 1-p$

$P(X=1) = p$

$\phi_X(t) = E[e^{itX}]$.

$E[g(X)] = e^{it0} P(X=0) + e^{it1} P(X=1)$. $\qquad g(x) = e^{itx}$

$= (1-p) + e^{it}p$.

ex: Let $X \sim N(0,1)$. Then find characteristic f^n of X.

→ $\phi_X(t) = E[e^{itX}]$.

$= E[\cos tX + i\sin tX]$

$= E[\cos tX] + i\, E[\sin tX]$

$$E[\cos tX] = \int_{-\infty}^{\infty} \cos(tx) f_X(x)\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{\cos(tx)}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$E[\sin tX] = \int_{-\infty}^{\infty} \frac{\sin(tx)}{\sqrt{2\pi}} e^{-x^2/2} dx$$

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If $\int_{-\infty}^{0} f(x)\,dx < \infty$ and $\int_{0}^{\infty} f(x)\,dx < \infty$ then $\int_{-\infty}^{\infty} f(x)\,dx < \infty$

Ex: $\int_{-\infty}^{\infty} \sin x\,dx$

$\int_{0}^{\infty} \sin x\,dx$ diverges

If $\int_{-\infty}^{\infty} f(x)\,dx$ converges then $\left[\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx\right]$ (1)

Thm: If $\int_{-\infty}^{\infty} f(x)\,dx < \infty$ then (1) holds.

$$\int_{-\infty}^{\infty} \sin tx\, e^{-x^2/2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \sin tx\, e^{-x^2/2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{\sqrt{2\pi}} e^{-x^2/2}\,dx = I(t) \quad \text{Improper Integral.}$$

$$I'(t) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x \sin(tx)\, e^{-x^2/2}\,dx$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} (\sin tx)\left(x e^{-x^2/2}\right) dx\right]$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\underbrace{\left(-\sin tx\, e^{-x^2/2}\right)_{-\infty}^{\infty}}_{=0} + \underbrace{\int_{-\infty}^{\infty} t(\cos tx)\, e^{-x^2/2}\,dx}_{I}\right]$$

$$= \frac{-t}{\sqrt{2\pi}} \left[\cdot\right] = -tI$$

$\Rightarrow \quad I'(t) = -tI.$

$\Rightarrow \quad \frac{I'(t)}{I} = -t$

$\Rightarrow \quad \frac{dI}{I} = -t\,dt.$

$\Rightarrow \ln I(t) = \frac{-t^2}{2} + c$

$\Rightarrow \quad I(t) = k e^{-t^2/2} \qquad \Rightarrow k = 1.$

At $t = 0$ $\quad I(0) = 1$

$\therefore \phi_X(t) = e^{-t^2/2}.$

ex: Let X be a r.v. and $a, b \in \mathbb{R}$ Then $Y = aX + b$ $\phi_Y(t) = E$

$\phi_Y(t) = E[e^{it(aX+b)}]$

$= E[e^{itaX} \cdot e^{itb}] = e^{itb} E[e^{itaX}]$

$= e^{itb} \phi_X(at).$

$= \cancel{e^{itb}}$

ex: Let $X \sim N(\mu, \sigma^2)$ it is implicit that $\sigma > 0$. ~~Define $Y = \frac{X-\mu}{\sigma}$~~

Define $Y = \frac{X-\mu}{\sigma}$, Claim : $Y \sim N(0, 1)$.

$F_Y(y) = P(Y \leq y)$

$= P\left(\frac{X-\mu}{\sigma} \leq y\right) = P(X-\mu \leq \sigma y) = P(X \leq \sigma y + \mu)$

$$= \int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

but $x = \sigma u + \mu$

~~$dx = \sigma du$~~ $dx = \sigma du$

$$\therefore = \int_{-\infty}^{y} \frac{1}{(\sqrt{2\pi})\cancel{\sigma}} e^{-\left(\frac{\sigma^2 u^2}{2\sigma^2}\right)} \cancel{\sigma}\, du$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

$X = \sigma Y + \mu$ where $Y \sim N(0, 1)$

$\phi_X(t) = e^{it\mu} e^{-\sigma^2 \frac{t^2}{2}}$

ex: Let X and Y be 2 independent random variables. Then characteristic function of $X+Y$ is : $\phi_{X+Y}(t) = E[e^{it(X+Y)}]$

$\phi_{X+Y}(t) = E[e^{it(X+Y)}]$

$= E[e^{itX} \cdot e^{itY}]$

$= E[e^{itX}] E[e^{itY}]$

$= \phi_X(t) \phi_Y(t)$

if X and Y are independent & $g: \mathbb{R} \to \mathbb{C}$ then $g(X)$ and $g(Y)$ are independent. $g(x) = e^{itx}$

Teacher's Signature

More generally if $X_1, X_2, \ldots X_n$ are independent random variable

Then $\phi_{X_1+X_2+\cdots+X_n}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) \ldots \phi_{X_n}(t)$.

ex: Let $X \sim B(n,p)$

$$\phi_X(t) = E[e^{itX}]$$

$$= \sum_{k=0}^{n} e^{itk} P(X=k)$$

$X = \sum_{i=1}^{n} X_i$ where $X_i \sim$ Bernoulli (p).

$$\phi_X(t) = \phi_{X_1}(t) \ldots \phi_{X_n}(t)$$

$$= [e^{it}p + (1-p)]^n$$

** <u>Uniqueness Theorem</u> ⇒ Let X and Y be two random variables s.t. $\phi_X(t) = \phi_Y(t)$ $\forall$ $t \in \mathbb{R}$. Then X and Y have the same probability distribution.

ex: Let $X \sim B(n_1,p)$ & $Y \sim B(n_2,p)$ be 2 independent r.v. Then show that $X+Y \sim B(n_1+n_2, p)$.

→ $\phi_X(t) = [e^{it}p + (1-p)]^{n_1}$

$\phi_Y(t) = [e^{it}p + (1-p)]^{n_2}$

$$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t)$$

$$= [e^{it}p + (1-p)]^{n_1} \cdot [e^{it}p + (1-p)]^{n_2}$$

$$= [e^{it}p + (1-p)]^{n_1+n_2}.$$

By uniqueness theorem, we are done.

ex: Let $X \sim N(\mu_1, \sigma^2)$ & $Y \sim N(\mu_2, \sigma^2)$ be independent r.v. Then show that

$$X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

→ $\phi_X(t) = e^{it\mu_1} e^{-\frac{\sigma_1^2 t^2}{2}}$

$\phi_Y(t) = e^{it\mu_2} e^{-\frac{\sigma_2^2 t^2}{2}}$.

Teacher's Signature..........

$\phi_{X+Y}(t) = e^{i(\mu_1+\mu_2)t} \cdot e^{-(\sigma_1^2+\sigma_2^2)t^2/2}$

By uniqueness theorem.

4 Mar **Definition 4** Let $I \subseteq \mathbb{R}$ be an interval & $f: I \to \mathbb{R}$ be a fn. We say that f is convex on I if for any two pts. $x_1, x_2 \in I$ and $\forall t \in (0,1)$ we have

$$f(tx_1 + (1-t)x_2) \le t f(x_1) + (1-t) f(x_2)$$

f is concave if

$$f(tx_1 + (1-t)x_2) \ge t f(x_1) + (1-t) f(x_2)$$

Jensen's Inequality: Let $f: I \to \mathbb{R}$ be a convex function (where I is an interval in $\mathbb{R}$) and let X be a real-valued random variable s.t. (i) $X \in I$ (ii) $EX < \infty$ (iii) $E(f(X)) < \infty$. Then $f(EX) \le E(f(X))$

Observation: f is convex $\Leftrightarrow$ $-f$ is concave.

If $f: I \to \mathbb{R}$ is concave then $f(EX) \ge E(f(X))$

$-f$ is convex

$$-f(EX) \le E(-f(X)) = -E(f(X))$$

Eg. $f(x) = |x|$

$|EX| \le E|X|$ — (1)

Definition 5 Let $r > 0$, $r \in \mathbb{R}$ and X be a r.v. Then $E[X^r]$ is called r^{th}-order central moment of r.v. X.

$E[(X-a)^r]$ → r^{th} moment of X around a.

$E[|X|^r]$ is called r^{th} absolute central moment of X.

Replace X with X^r in (1)

$E[X^r] \le |E(X^r)| \le E(|X|^r)$.

Recall $\quad EX = \sum_{x \in R_X} x f_X(x)$ with

$= \sum_{x \in R_X} |x| f_X(x) < \infty \quad = E|X|$

ex: If the central moment of order $q > 0$ exist then show that moments of order r, where $0 < p < q$ exist.

→ Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = x^r$ where $r > 1$

$f'(x) = r x^{r-1}$

$f''(x) = r(r-1) x^{r-2} > 0$

f is convex on $(0, \infty)$

$\Rightarrow f(E|X|) \le E(f|X|)$

$= (E|X|)^r \le E(|X|^r)$

$\Rightarrow E|X| \le (E|X|^r)^{1/r}$ — ②

$E(X^q) < \infty$ this means $E(|X|^q) < \infty$.

$E(|X|^p) < \infty \quad \forall \quad p \in (0, q)$

Let p be given $0 < p < q \quad \Rightarrow \frac{q}{p} > 1$ in ②

Take $r = \frac{q}{p}$

$E(|X|) \le (E|X|^{q/p})^{p/q}$ — ③

Replace $|X|$ with $|X|^p$ in ③

$E(|X|^p) \le \underbrace{(E|X|^q)}_{<\infty}^{p/q}$

$\Rightarrow E(|X|^p) < \infty$.

ex: Let X be a r.v. with $EX = 10$. Then show that $E[\ln\sqrt{X}] \le \frac{1}{2}\ln 10$

→ $E[\ln\sqrt{X}] = \begin{cases} \sum_{x \in R_X} \ln\sqrt{x}\, f_X(x) \\ \int_{-\infty}^{\infty} \ln\sqrt{x}\, f_X(x)\,dx \end{cases}$

$f(x) = \ln\sqrt{x} = \frac{1}{2}\ln x, \quad x \in (0, \infty)$

$f'(x) = \frac{1}{2x}, \quad f''(x) = \frac{-1}{2x^2} < 0 \quad \forall (0, \infty)$

Teacher's Signature.

f is concave.

$E[\ln\sqrt{x}] \le f(E(x)) = \frac{1}{2}\ln 10$

$\overset{\shortparallel}{E}[f(x)]$

Markov Inequality: Let X be a non-ve r.v. with finite mean n^{th} moment. Then $\forall\ t>0$

$$P(X \geq \varepsilon) \le \frac{E(x^n)}{\varepsilon^n}$$

Chebyshev's Inequality: Let X be a r.v. with finite mean μ and variance σ^2. Then $\forall\ \varepsilon>0$.

$$P(|x-\mu| \geq \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

Proof $\Rightarrow$ $|x-\mu| \geq 0$.

$\Uparrow$

Apply Markov

$\sigma^2 = Var(x) = E[(x-\mu)]^2$.

$$P(|x-\mu| \geq \varepsilon) \le \frac{E((x-\mu)^2)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}$$

$\shortparallel$

$x \geq \varepsilon+\mu$

$x \le -\varepsilon+\mu$

Ex: Let $x \sim B(n,p)$. Estimate $P(x \geq \alpha n)$ where $p<\alpha<1$ using Markov in eq. (with first moment) & Chebyshev's Inequality. Compare both the estimates for $p=\frac{1}{2}$ and $\alpha=\frac{3}{4}$

$\rightarrow$ Markov (with first Moment).

$$P(x \geq \alpha n) \le \frac{E(x)}{\alpha n} = \frac{np}{\alpha n} = \frac{p}{\alpha}.$$

Chebyshev's Inequality

$P(|X-np| \geq \alpha n)$

$P(x \geq \alpha n) = P(x - np \geq \alpha n - np)$

$\le P(|x-np| \geq (\alpha n - np))$

Teacher's Signature

Celebrations Dt. / / Page

$\{|Y| \geq a\} = \{Y \leq -a\} \cup \{Y \geq a\}$.

$\leq \dfrac{Var(X)}{(\alpha n - np)^2} = \dfrac{np(1-p)}{n^2(\alpha - p)^2} = \dfrac{p(1-p)}{n(\alpha - p)^2}$

$p = 1/2, \alpha = 3/4$

$P(X \geq \frac{3}{4}n) \leq \frac{1}{2} \times \frac{4}{1} = \frac{2}{3}$

$P(X \geq \frac{3}{4}n) \leq \dfrac{\frac{1}{2} \times \frac{1}{2}}{n \times (1/4)^2} = \dfrac{4}{n}$ if $n \geq 6$ chebyshev will estimate more accurate results.

2nd April

Law of Large Nos.

Weak law | Strong law

Definition →

Let $A_1, A_2, A_3, \ldots A_n$ be events we say that $A_1, A_2, \ldots A_n$ are mutually (totally) independent if

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \ldots P(A_{i_m})$$

where $\{i_1, i_2, \ldots i_m\} \subset \{1, 2, 3, \ldots n\}$ & $n \geq m \geq 2$

Definition →

We say that r.v.'s $X_1, X_2, \ldots X_n$ are independent if $\{X_1 \in A_1\}$, $\{X_2 \in A_2\}$, $\{X_3 \in A_3\}, \ldots \{X_n \in A_n\}$ are independent if ~~event~~ every Borel $A_1, A_2, \ldots A_n \subseteq \mathbb{R}$.

Definition →

We say that a countable collection of random variable $\{X_1, X_2, \ldots\}$ is mutually independent if every finite subcollection $\{X_{i_1}, X_{i_2}, \ldots\}$ where $i_k \in \mathbb{N}$ are independent.

- **Weak law of Large Nos.**

Let (X_n) be a seq of iid r.v.'s with $EX_1 = \mu < \infty$. Then $\forall\ \epsilon > 0$

Teacher's Signature

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| \geq \delta \right) = 0$$

or equivalently

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| < \delta \right) = 1 \quad \text{where } S_n = \sum_{i=1}^{n} X_i$$

* $\Omega = \{w_1, w_2\}$.

$X(w_1) = 1 \qquad Y(w_1) = 2$

$X(w_2) = 2 \qquad Y(w_2) = 1$.

$P(w_1) = P(w_2) = \frac{1}{2}$

$P(X = 1) = 1/2 \qquad P(X = 2) = 1/2$

$P(Y = 1) = 1/2 \qquad P(Y = 2) = 1/2$.

* 1st observation.

$\forall \delta > 0, \epsilon > 0 \; \exists \; n_0 \in \mathbb{N}$ s.t. $\left| P\left(\left| \frac{S_n}{n} - \mu \right| < \delta \right) - 1 \right| < \epsilon \quad \forall n \geq n_0$

$$1 - P\left(\left| \frac{S_n}{n} - \mu \right| < \delta \right) < \epsilon$$

$$\Rightarrow P\left(\left| \frac{S_n}{n} - \mu \right| < \delta \right) > 1 - \epsilon \quad \forall n \geq n_0$$

2nd observation

$X_1, X_2, \ldots, X_n$ a r.v.

$\frac{S_n}{n} \to$ r.v.

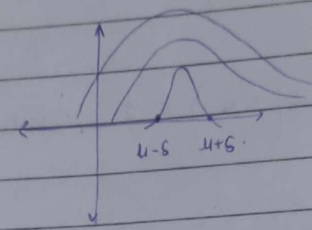
$$\left| \frac{S_n}{n} - \mu \right| < \delta$$

$$\frac{S_n}{n} \in (\mu - \delta, \mu + \delta) \quad \forall n \geq n_0$$

3rd observation

PMF of $\frac{S_n}{n}$ is concentrated in nbd. of μ.

PDF of $\frac{S_n}{n}$



ex: Let $X_1, X_2, X_3, \ldots X_n$ be a seq. of iid r.v. with $EX_i = 0$ and var $= 1$. Let $S_n = \sum_{i=1}^{n} X_i$. Then $\forall\ x > 0$, find $\lim_{n\to\infty} P(-nx < S_n < nx)$

→ $\mu = 0$, $P\{-nx < S_n < nx\}$

$= P\{|S_n - 0| < nx\}$

$= P\left\{\left|\frac{S_n}{n} - 0\right| < x\right\}$

$\to 1$ as $n \to \infty$.

δ = accuracy level ϵ = confidence level.

★ **Random Sample.**

μ = Average ~~(true~~
(true mean)

r.v. → $\frac{S_n}{n} = \frac{X_1 + X_2 \ldots X_n}{n}$ → Sample Mean

3 April

What is the meaning of following student : Probability of head in a 'simple' coin toss is 1/2.

ex: Let A be an event in some random experiment & $P(A) = p$ where $p \in [0,1]$. We consider 'n' ~~repitions~~ repetitions of the random experiment.

M_n = fraction of time event A has occured in these 'n' repitions of experiment.

Define $X_i(w) = \begin{cases} 1 & w \in A \\ 0 & w \in A^c \end{cases}$

$P(X_1 = 1) = P(w : w \in A) = P(A) = p$

$P(X_i = 0) = P(A^c) = 1 - p$.

Teacher's Signature.........

$x_1, x_2, x_3, \ldots\ldots x_n$ are indipendent Bernoulli (p)

$$M_n = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n}$$

$E x_i = p$. By ~~WWL~~ WLLN, $\forall\ \delta > 0$.

$$\lim_{n \to \infty} P\left(|M_n - p| < \delta \right) = 1.$$

δ small enough

ex: Toss a coin infinitely many times, with probability of head $= p$ in each ~~coin~~ toss.

→ $\Omega = \{ w : w = (w_1, w_2, \ldots\ldots) \text{ where each } w_i \text{ is either H or T} \}$

HTHTHT......

HHHH.......

TTT......

HHTHHTHHT.......

Ω is un-countable

$$X_1(w) = \begin{cases} 1 & , w_1 = H \\ 0 & , w_1 = T \end{cases}$$

$$X_2(w) = \begin{cases} 1 & , w_2 = H \\ 0 & , w_2 = T \end{cases}$$

$x_1, x_2, \ldots\ldots x_n$ are independent

$E x_i = p \quad \forall\ i = 1, 2, 3, \ldots\ldots$

$$\frac{S_n(w)}{n} = \frac{X_1(w) + X_2(w) + \ldots\ldots X_n(w)}{n}$$

$=$ relative frequency of head in 'n' coin toss

By WLLN $\forall\ \delta > 0$

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - p \right| \geq \delta \right) = 0$$

$\forall\ \varepsilon > 0 \quad \delta > 0$ → accuracy.

Teacher's Signature..........

$$P\left(\left|\frac{S_n}{n}-p\right| < \delta\right) > 1-\epsilon$$ ↳ confidence level for large enough 'n'

$$P\left(\left|\frac{S_n}{n}-p\right| \geq \delta\right) = P\left(|S_n - np| \geq n\delta\right)$$

$$\leq \frac{Var(S_n)}{n^2\delta^2} = \frac{np(1-p)}{n^2\delta^2} = \frac{p(1-p)}{n\delta^2}$$

$f(p) = p(1-p)$ where $p \in [0,1]$

$f'(p) = 1-2p$, $p = 1/2$

$f''(p) = -2$

If $\frac{1}{4n\delta^2} < \epsilon$

$\Rightarrow n > \frac{1}{4\delta^2 \epsilon}$

$\epsilon = 0.05$, $\delta = 0.02$

$\Rightarrow n > \frac{1}{4(0.02)^2(0.05)}$

$> \frac{1}{4(0.0004)(0.05)}$

$> \frac{10000 \times 100}{4\times 4\times 5}$

$\boxed{n > 12500}$

$\boxed{n > 2500}$

$P\left(\left|\frac{S_n}{n} - p\right| < 0.02\right) > 0.95$ if $n > 12500$

relative frequency

unknown

Monte Carlo Method.

② ~~Strong Law of Week~~

Strong Law of Large Nos :

Let $X_1, X_2, X_3, \ldots\ldots X_n$ be a sequence of iid r.v. with $EX_i = \mu < \infty$. Then

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1 \quad \text{or equivalently}$$

$\forall \delta > 0$:

$$P\left(\lim_{n\to\infty} \left|\frac{S_n - \mu}{n}\right| < \delta\right) = 1.$$

$\frac{S_n(w)}{n}$ is a set in real number.

$\frac{S_n(w)}{n} \to \mu$ for almost all $w \in \Omega$, that is, $\exists N \subset \Omega$.

$\frac{S_n(w)}{n} \not\to \mu \quad P(N) = 0$

$\Omega = \{w : w = (w_1, w_2 \ldots\ldots)\}$