

# Predicting the Weather: The Physics of Earth's Atmosphere

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## **Abstract**

It's difficult to imagine a world without some idea of what tomorrow's weather will look like. Since the emergence of modern computing in the 1940s, and the development of weather satellites during the 1960s and 70s, weather prediction has slowly evolved into an integral component of the global economy. But how do scientists go about creating a computer model forecast of the weather, and what physical principles underlie weather phenomena in the first place? In this paper, I explore these questions – using fundamental principles in classical mechanics, thermodynamics, and radiative transfer theory – with a focus on atmospheric dynamics, cloud formation, and satellite observations of the atmosphere.

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# 1 Introduction

Why does the wind blow? The short answer is that the Earth is heated unevenly by the Sun, more at the equator than at the poles, and by different amounts depending on the season, time of day, and reflectiveness of Earth’s surface and cloud cover. These differences in heating cause the gas molecules that make up the atmosphere to either spread out or get closer together. Once this happens, the areas of denser and thinner gas exchange molecules so that everything tends to even out, always moving towards equilibrium, but never reaching it. We feel this flow of gas molecules as wind.

This was the answer that I found as a curious child, and it does get at some important physical principles including heat, pressure-gradient force, and the properties of gases. It leaves out, however, the precision of quantitative description or any notion of how we might try to predict the wind’s future behavior. My goal in this paper is to provide a precise way to answer this question – along with others about the dynamics of temperature, humidity, density, and pressure (Section 2); the formation of clouds and precipitation (Section 3); and how weather satellites observe the state of the atmosphere (Section 4) – based on fundamental physical ideas found in classical mechanics, thermodynamics, and radiative transfer theory. I will conclude by discussing the considerations necessary to implement a numerical weather model (Section 5).

## 1.1 The Atmosphere

Before diving in to the details of geophysical fluid dynamics, it’s important to introduce and summarize basic ideas and jargon relevant to the atmosphere. Essentially all weather phenomena occur in the bottom layer of the atmosphere (see Figure 1), called the troposphere [1–3]. The troposphere mixes very little with the layers above it, separated from the stratosphere above by a border layer called the tropopause. The height of the tropopause varies with latitude: lowest near the poles ( $\sim 9$  km) and highest near the equator ( $\sim 17$  km). Because of this lack of mixing, we need only consider dynamics in this bottom layer to predict and model the weather. Dry air in the troposphere is composed by mass of approximately 78% nitrogen, 21% oxygen, and less than 1% of other gases. When water vapor is included, its concentration reaches as high as 5% [1,2].

While not strictly included in the atmosphere, the hydrosphere and lithosphere<sup>1</sup> provide important boundary conditions and sources/sinks of heat and water vapor, and play an important role in predicting the weather [2,4]. The effects of different weather phenomena including clouds, precipitation, and snow and ice cover also significantly impact atmospheric dynamics [4]. Finally, it is essential to include the external

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<sup>1</sup>These are technical terms for Earth’s ocean and land, respectively.

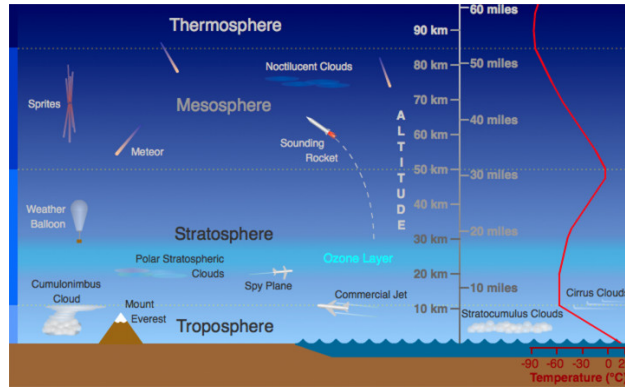


Figure 1: This diagram, courtesy of Randy Russell at the UCAR Center for Science Education, shows the layers of the atmosphere, and common phenomena found within each layer. The focus of this paper is on the bottom layer, called the troposphere, where weather phenomena occur.

inputs and outputs which drive the atmosphere and allow it to exist in the first place: the Sun’s radiation, providing energy to drive weather systems; Earth’s reradiation and reflection of this energy; the Earth’s gravitational field, holding the atmosphere close to the surface; and the rotation and orbit of the Earth within the solar system, creating daily and seasonal cycles [1].

## 1.2 A Brief History of Weather Prediction

Meteorologists have historically approached the task of weather prediction by identifying patterns in the atmosphere, such as the tendency for mid-latitude storms to travel from west to east, following the prevailing winds known as the westerlies. Even with more advanced knowledge about storm tracking and evolution, and with the development of better systems for observation like radar and satellite imaging, forecasts generated by humans alone tend to be crude, and useful for projecting only a day or two into the future.

Fortunately, the development of numerical weather prediction using computers, in conjunction with improvement in observation from satellite imagery, has significantly advanced forecast accuracy and range over the last 70 years. The basic idea of numerical weather prediction is that if you have an understanding of the physical processes occurring then you can model the relationship between different physical quantities which describe the atmosphere – temperature, pressure, humidity, wind velocity, and density – via a set of dynamical equations. Using these equations, and an approximate knowledge of the current conditions of the atmosphere and its boundaries, one can predict the future state of the atmosphere by integrating through time. Automated computing and numerical methods are essential to this process because the complex nonlinear nature of the equations does not allow for general analytical solution.

The most accurate of today's models of the atmosphere are based on just five equations – 4 partial differential equations (PDEs) and one algebraic equation – relating five unknown dynamic quantities which describe the atmosphere. The focus of this paper is on deriving the basic forms of these equations from the fundamental principles familiar to undergraduate physics students.

The equations governing atmospheric dynamics had been mostly developed by the early 20th century, but it would take another 50 years to produce a successful forecast. During the first decade of the 20th century, pioneers of numerical weather prediction, Cleveland Abbe and Vilhelm Bjerknes, were the first to identify that hydrodynamic and thermodynamic principles could be applied to the atmosphere and to develop numerical techniques for doing so [5]. It was not until 1922, when Lewis Fry Richardson published *Weather Prediction by Numerical Process*, that anyone described a serious attempt at a numerical weather forecast [6]. It took Richardson several weeks to do the calculations for a 6-hour forecast, which was not only totally wrong, but physically impossible. It was clear that a human generated numerical forecast was out of reach.

The advent of computers near the end of WWII marked a turning point for numerical weather prediction. Finally, in the year 1950, a team including world famous mathematician John von Neumann, using the ENIAC computer and a simplified version of the governing equations, produced the first successful forecast: a 24-hour forecast completed in just under 24 hours [5]. Within the decade, governments around the world were producing practical numerical forecasts to aid meteorologists [7, 8]. Advances in computing power, remote sensing, numerical techniques, models, and parameterizations have steadily improved forecast range accuracy through the present day.

## 2 Governing Equations of the Atmosphere

The dynamics of the atmosphere can be described by five governing equations which include a vector PDE for conservation of momentum in 3-dimensional space (Section 2.1), a PDE for conservation of total mass (Section 2.2), a PDE for conservation of water vapor mass (Section 2.3), an algebraic equation of state (Section 2.4), and lastly a PDE for conservation of energy (Section 2.5). The rest of this section will be devoted to deriving and understanding these equations from basic physical principles [4, 5, 9–11]. For a comprehensive list of these equations, and the variables and constants they contain, see the summary in Appendix C.

## 2.1 Conservation of Momentum

The motion of a fluid, which is modeled as a continuum rather than a collection of discrete particles, is nonetheless governed by the same fundamental principles as those we apply to particle and solid-body motion – namely, Newton’s laws of motion. What follows is a derivation of conservation of momentum for the atmosphere. Simply stated, the velocity of a fixed-mass parcel of air changes only due to external forces: pressure, gravity, and friction.

In discussing the forces acting on and between different parts of a fluid, it is useful to precisely define the idea of an air parcel. We define the air parcel [1, 12] as an infinitesimal element of fluid mass  $\delta m$ , with infinitesimal dimensions  $\delta x$ ,  $\delta y$ , and  $\delta z$ , designating a parallelepiped in Cartesian coordinates, centered at  $(x, y, z)$ , as indicated in Figure 2. We call the velocity of the parcel  $\mathbf{u} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = (u, v, w)$ . The density  $\rho$  of the parcel is assumed to be uniform. For the atmosphere, which is a compressible fluid, the mass of an air parcel remains constant while its volume may change. We relate the parcel mass to its volume and density by

$$\delta m = \rho \delta x \delta y \delta z. \quad (2.1.1)$$

Now let us consider the forces on the air parcel in an inertial frame. These include the gravitational force  $\mathbf{F}_g$ , the pressure force  $\mathbf{F}_p$ , and the friction force  $\mathbf{F}_f$ .

We begin with gravity. Since we assume the parcel has uniform density, we write the net gravitational force as acting on the center of the parcel, as shown in Figure 2. We write the gravitational force most generally as a product of the mass with the gravitational field of the Earth  $\mathbf{g}$  in an inertial reference frame:

$$\mathbf{F}_g = \mathbf{g} \delta m = \rho \mathbf{g} \delta x \delta y \delta z. \quad (2.1.2)$$

Second, we consider pressure. The pressure force acts on the parcel as a contact force exerted inward on all 6 faces. We find the net pressure force by summing the different forces exerted on each face by the pressure field  $p(x, y, z)$ . Since each face is displaced by half an infinitesimal dimension ( $\delta x$ ,  $\delta y$ , or  $\delta z$ , according to the face) from the center, we consider the average pressure on each face separately, as shown in Figure 3. Using a first order Taylor expansion<sup>2</sup>, we rewrite the pressure on each face using a partial derivative. For example, for the side which faces in the right (positive  $x$ -direction), we find

$$p(x + \frac{1}{2}\delta x, y, z) = p(x, y, z) + \frac{1}{2}\delta x \frac{\partial p}{\partial x}. \quad (2.1.3)$$

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<sup>2</sup>A first order Taylor expansion is exact here because  $\delta x$ ,  $\delta y$ , and  $\delta z$  are infinitesimals.

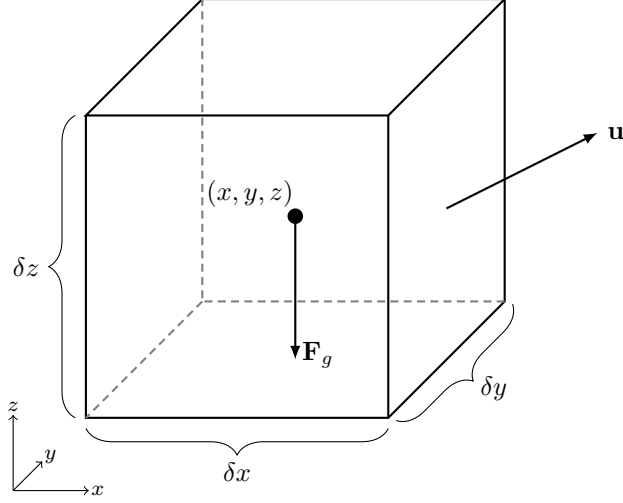


Figure 2: The force of gravity  $\mathbf{F}_g$  acts on an air parcel centered at position  $(x, y, z)$  with dimensions  $\delta x$ ,  $\delta y$ , and  $\delta z$ , moving at velocity  $\mathbf{u}$ .

The magnitude of the force exerted on each face is the product of the pressure on the face and the area of the face. The direction of the pressure force is inward, and therefore opposite the normal vector out of the face  $\hat{\mathbf{n}}$ . That is, the force on each face is given by

$$\mathbf{F}_{\text{on face}} = -p_{\text{at face}} A \hat{\mathbf{n}}. \quad (2.1.4)$$

Using the right-side face as an example again, the force  $\mathbf{F}_{x+}$  on the face is

$$\mathbf{F}_{x+} = -[p(x, y, z) + \frac{1}{2} \delta x \frac{\partial p}{\partial x}] \delta y \delta z \hat{\mathbf{x}}. \quad (2.1.5)$$

We name the forces on the other five faces similarly, with the subscript denoting along which axis it faces

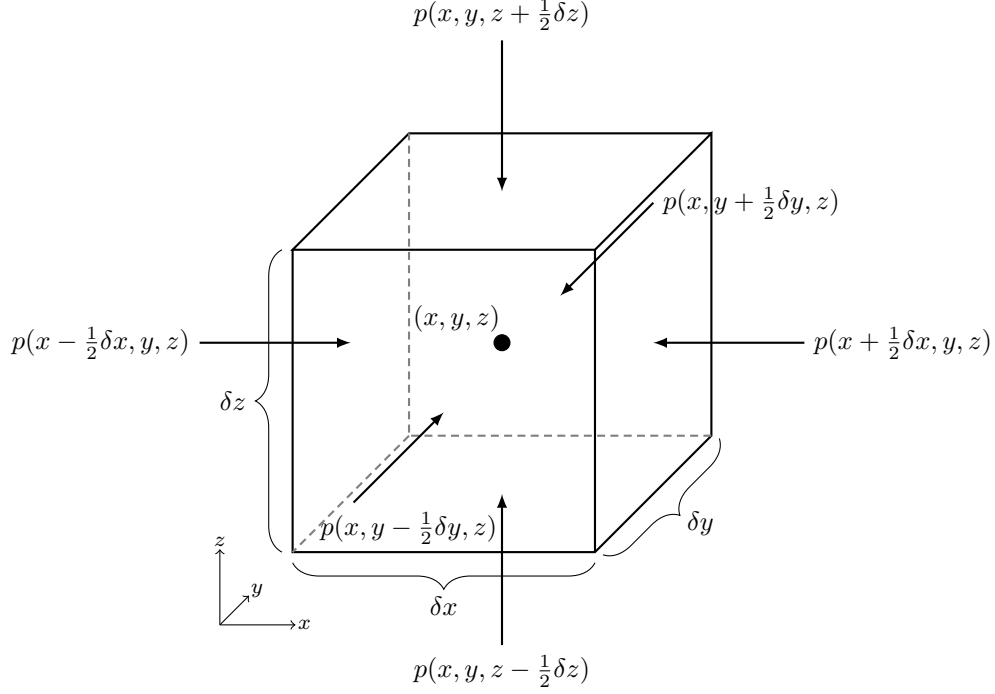


Figure 3: The average pressure on each face of an air parcel is denoted by the pressure function  $p$  at the center of the face. Notice that each pressure value is defined by displacing half an infinitesimal dimension from the parcel's center.

and whether in the positive or negative direction. When we sum all six pressure forces, we find

$$\mathbf{F}_p = \mathbf{F}_{x+} + \mathbf{F}_{x-} + \mathbf{F}_{y+} + \mathbf{F}_{y-} + \mathbf{F}_{z+} + \mathbf{F}_{z-} \quad (2.1.6)$$

$$\begin{aligned} &= -[p(x, y, z) + \frac{1}{2}\delta x \frac{\partial p}{\partial x}] \delta y \delta z \hat{\mathbf{x}} + [p(x, y, z) - \frac{1}{2}\delta x \frac{\partial p}{\partial x}] \delta y \delta z \hat{\mathbf{x}} \\ &\quad - [p(x, y, z) + \frac{1}{2}\delta y \frac{\partial p}{\partial y}] \delta x \delta z \hat{\mathbf{y}} + [p(x, y, z) - \frac{1}{2}\delta y \frac{\partial p}{\partial y}] \delta x \delta z \hat{\mathbf{y}} \\ &\quad - [p(x, y, z) + \frac{1}{2}\delta z \frac{\partial p}{\partial z}] \delta x \delta y \hat{\mathbf{z}} + [p(x, y, z) - \frac{1}{2}\delta z \frac{\partial p}{\partial z}] \delta x \delta y \hat{\mathbf{z}} \end{aligned} \quad (2.1.7)$$

$$= -(\frac{\partial p}{\partial x} \hat{\mathbf{x}} + \frac{\partial p}{\partial y} \hat{\mathbf{y}} + \frac{\partial p}{\partial z} \hat{\mathbf{z}}) \delta x \delta y \delta z \quad (2.1.8)$$

$$= -\nabla p \delta x \delta y \delta z, \quad (2.1.9)$$

which is the opposite of the gradient of the pressure times the volume of the parcel. This makes sense because fluids flow from high pressure to low pressure as they tend toward equilibrium and the magnitude of the force required to do that will increase with a larger volume.

Finally, we consider the friction force. If our goal was to model the entire complexity of a fluid down to the



the smallest scales, we would now proceed to derive the viscous force terms found in the compressible Navier-Stokes momentum equation:  $\mu \nabla^2 \mathbf{u} + \frac{1}{3} \mu \nabla(\nabla \cdot \mathbf{u})$ , where  $\mu$  is the viscosity [12, 13]. In doing this to model the atmosphere, however, we would be attempting to predict all of the atmosphere's turbulence down to the molecular scale, which is much too computationally expensive, even for today's supercomputers. Instead of making it a dynamic variable, we parameterize the friction force using a simplified model of turbulence and surface interaction (see Section 5.1). For now we leave the friction force written as  $\mathbf{F}_f$ , and the friction force per unit mass as  $\mathbf{f} = \mathbf{F}_f / \delta m$ .

The net force on the parcel in an inertial reference frame, is therefore

$$\mathbf{F}_{\text{net}} = \mathbf{F}_g + \mathbf{F}_p + \mathbf{F}_f = \rho \mathbf{g} \delta x \delta y \delta z - \nabla p \delta x \delta y \delta z + \mathbf{f} \delta m, \quad (2.1.10)$$

which is related to the acceleration  $D\mathbf{u}/Dt$  by Newton's second law:

$$\rho \mathbf{g} \delta x \delta y \delta z - \nabla p \delta x \delta y \delta z + \mathbf{f} \delta m = \delta m \frac{D\mathbf{u}}{Dt}. \quad (2.1.11)$$

Note that the acceleration is defined as the material derivative of the velocity  $D\mathbf{u}/Dt$  (see Appendix A), because we have considered the forces on a particular parcel. By canceling out the mass  $\delta m = \rho \delta x \delta y \delta z$  on both sides we find

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{f}. \quad (2.1.12)$$

This is the vector equation of motion for an air parcel in an inertial frame, and is the statement of conservation of momentum for the atmosphere. However, when we observe and model the atmosphere, we find ourselves in the non-inertial reference frame of the rotating oblate spheroidal Earth<sup>3</sup>.

Following the derivations found in Marion and Thornton [14] and Saha [1], we define a relative velocity  $\mathbf{v}$ , the velocity of the air parcel with respect to the Earth's rotating reference frame. The velocity  $\mathbf{u}$ , on the other hand, is defined with respect to the approximately inertial frame of the "fixed stars", a very good approximation for most purposes. In fact, the non-inertial effects found on Earth are so greatly dominated by the Earth's rotation about its own axis with respect to the stars, that we can use the reference frame fixed at the center of the Earth, but not rotating with respect to the stars, as an appropriate approximation for an inertial reference frame [14]. Consider a point P, denoting a moving air parcel near the surface of the Earth, as shown in Figure 4. Its position is given in two ways: 1) by the vector  $(\mathbf{r})_i$  defined in an

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<sup>3</sup>This just means that the Earth is slightly squished from a perfect spherical shape. However, for the purposes of modeling the atmosphere we can assume that the Earth is spherical.

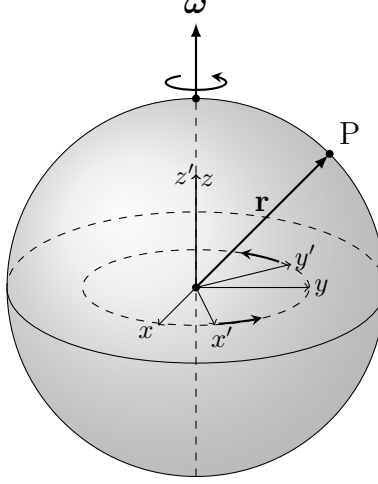


Figure 4: This diagram represents the Earth, rotating with angular velocity  $\omega$ . The vector  $\mathbf{r}$  describes the position of a moving point  $P$  near the Earth's surface. It is defined both with respect to the approximately inertial Earth-centered coordinate system with axes  $x$ ,  $y$ , and  $z$ , and with respect to a non-inertial Earth-centered coordinate system rotating with the Earth designated by the axes  $x'$ ,  $y'$ , and  $z'$ .

approximately inertial coordinate system originating at the center of the Earth and “fixed” with respect to the stars (axes  $x$ ,  $y$ , and  $z$ ), and 2) by the vector  $(\mathbf{r})_r$ , also defined in a coordinate system originating at the center of the Earth, but which rotates with the Earth (axes  $x'$ ,  $y'$ , and  $z'$ ). These are the same vector, but they are defined in different coordinate systems, so they have different component values. Throughout the rest of this section, we will use this notation with parentheses and subscripts  $i$  and  $r$  to denote whether we are referring to a vector in the inertial or rotating reference frames, respectively.

The rotating frame vector  $(\mathbf{r})_r$ , by definition, changes in time at a rate equal to the wind velocity  $\mathbf{v}$  with respect to the surface, that is

$$\left(\frac{D\mathbf{r}}{Dt}\right)_r \equiv \mathbf{v}. \quad (2.1.13)$$

On the other hand, in the inertial frame,  $(\mathbf{r})_i$  changes in time at the absolute velocity  $\mathbf{u}$ , which is equal to the sum of the wind velocity  $\mathbf{v}$  and the tangential velocity of the Earth's surface at the point  $P$  in the inertial frame, which we call  $\mathbf{v}_E$ ; that is

$$\left(\frac{D\mathbf{r}}{Dt}\right)_i \equiv \mathbf{u} = \mathbf{v} + \mathbf{v}_E. \quad (2.1.14)$$

The tangential velocity  $\mathbf{v}_E$  in the inertial frame is given by the cross product of the angular velocity  $\omega$  with the radial position  $\mathbf{r}$ . Therefore,

$$\mathbf{u} = \mathbf{v} + \omega \times \mathbf{r}. \quad (2.1.15)$$

Next we take the derivative, in the inertial frame, of both sides of this equation in order to relate the absolute acceleration  $(D\mathbf{u}/Dt)_i$  to the rotating-frame acceleration  $(D\mathbf{v}/Dt)_r$  [9, 14]. Computing this with the rules of differentiation (see detailed computation in Appendix B) we find

$$\frac{D\mathbf{u}}{Dt} = \frac{D\mathbf{v}}{Dt} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (2.1.16)$$

Note that all velocities and accelerations are defined using the material derivative  $D/Dt$  because they refer to a rates of change in time for a specific air parcel (see Appendix A). It is easiest to interpret this equation as the apparent force  $\mathbf{F}_{\text{apparent}}$  on a body of mass  $m$  in a rotating frame being equal to the sum of the real frame-independent force  $\mathbf{F}$  on the body (from Newton's second law) and a net fictitious force  $\mathbf{F}_{\text{fictitious}}$  (the other terms in the equation). That is,

$$\mathbf{F}_{\text{apparent}} = \mathbf{F} + \mathbf{F}_{\text{fictitious}}, \quad (2.1.17)$$

where  $\mathbf{F}_{\text{apparent}} = m \frac{D\mathbf{v}}{Dt}$ ,  $\mathbf{F} = m \frac{D\mathbf{u}}{Dt}$ , and  $\mathbf{F}_{\text{fictitious}} = m \left[ \frac{D\mathbf{v}}{Dt} - \frac{D\mathbf{u}}{Dt} \right] = m [-2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$ .

Combining Equation 2.1.12 with Equation 2.1.16, we can now write down the equation of motion for an air parcel in the Earth's rotating reference frame:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{f} - 2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (2.1.18)$$

The quantity  $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$  is known as the centrifugal acceleration while  $-2\boldsymbol{\omega} \times \mathbf{v}$  is known as the Coriolis acceleration. The centrifugal acceleration points outwards, perpendicular to the rotation axis<sup>4</sup>, and is constant in time<sup>5</sup>. It is therefore convenient to group this term with the gravitational field  $\mathbf{g}$  using what we will call the effective gravitational field  $\mathbf{g}^*$ , which is the sum of the gravitational field and centrifugal acceleration [1, 2, 4]. We thus express the effective gravitational field

$$\mathbf{g}^* = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad (2.1.19)$$

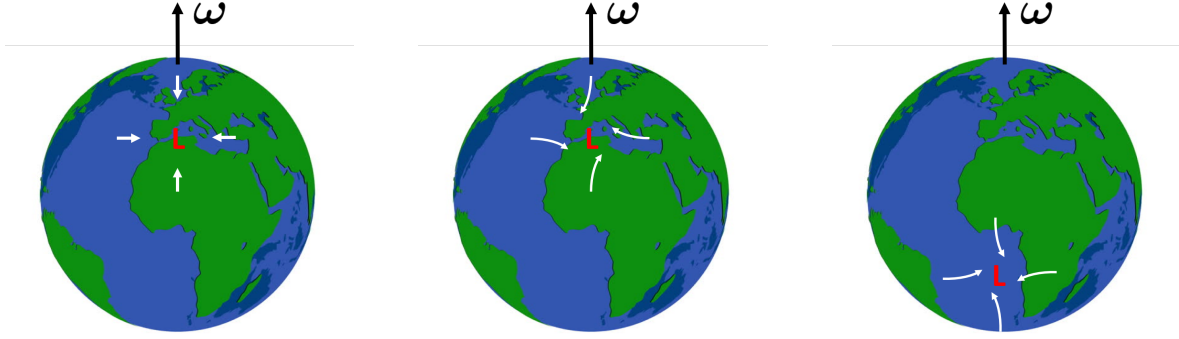
where  $\mathbf{g}$  is given by Newton's law of universal gravitation,

$$\mathbf{g} = -\frac{GM}{r^2} \hat{\mathbf{r}}. \quad (2.1.20)$$

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<sup>4</sup>Centrifugal means center-fleeing.

<sup>5</sup>The angular velocity of the Earth does vary slightly in time, but this variation is insignificant over the timescale of a weather forecast.



(a) Winds flow opposite the pressure gradient, i.e. from higher pressure to lower pressure. This figure shows how winds initially flow from surrounding higher pressure towards a low pressure region.

(b) Winds flowing towards a low pressure area in the northern hemisphere experience a counterclockwise deflection due to the Coriolis effect.

(c) Winds flowing towards a low pressure area in the southern hemisphere experience a clockwise deflection due to the Coriolis effect.

Figure 5: These figures demonstrate the Coriolis effect with a simplified diagram of wind flow towards a low pressure area.

We finally express the equation of motion most concisely as

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^* - 2\boldsymbol{\omega} \times \mathbf{v} + \mathbf{f}. \quad (2.1.21)$$

The Coriolis effect, an implication of the Coriolis acceleration term, can be demonstrated with the simple example of wind flowing towards a low pressure area. Consider a low pressure area centered over Western Europe, with surrounding higher pressure. The pressure gradient will cause wind to begin blowing directly toward the center of low pressure, as shown in Figure 5a. However, these wind velocities will appear to deflect from a straight path because of the Coriolis effect apparent in the rotating frame of the Earth. Computing the cross product, this implies the deflections shown in Figure 5b, which leads to a counterclockwise rotation. A similar effect is observed in the southern hemisphere (Figure 5c), but in the reverse direction because of the curvature of the Earth.

The interplay between the Coriolis effect and the pressure gradient force causes the swirling patterns seen in strong cyclones, most vividly in the near-circular inspiral path of wind in hurricanes. The Coriolis effect is an important consideration in long-range projectile motion as well, but – counter to popular mythology – is not large enough to have any noticeable effect on the direction that a sink or toilet bowl drains.

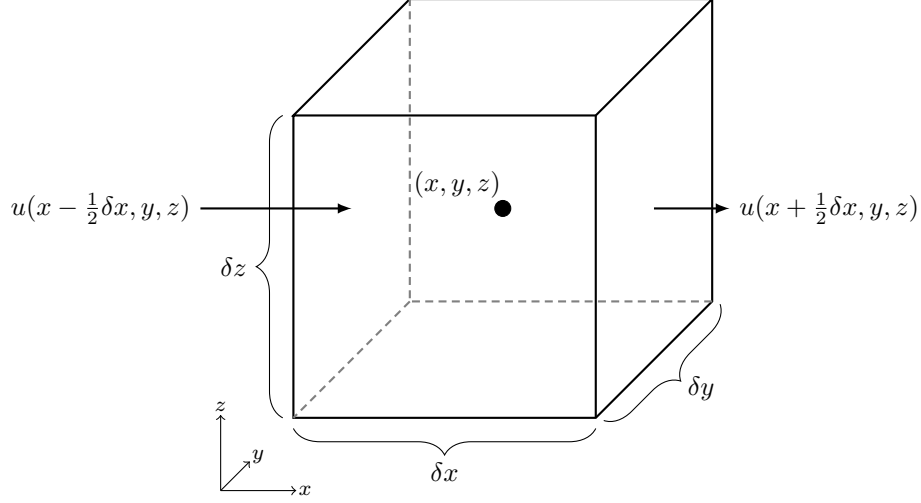


Figure 6: This air parcel diagram illustrates the difference in fluid velocity at opposing faces of the parcel, with the example shown using the left and right faces.

## 2.2 Conservation of Total Mass

Another important aspect of closed classical systems, essential to understanding the behavior of the atmosphere, is the principle of conservation of mass. In fluid mechanics, mass conservation is expressed via the continuity equation, which we derive in this section [1, 9, 12]. We interpret the continuity equation as stating that the rate of change of mass in a system enclosed by a surface (like an air parcel) is equal to the flux of mass across that surface.

Consider again, an infinitesimal air parcel with mass  $\delta m$  and dimensions  $\delta x$ ,  $\delta y$ , and  $\delta z$ , moving a fluid with field velocity  $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = (u, v, w)$  with respect to the rotating frame of the Earth. Conservation of mass implies that

$$\frac{D}{Dt}(\delta m) = 0, \quad (2.2.1)$$

which, written in terms of density and volume, yields

$$\frac{D}{Dt}(\rho \delta x \delta y \delta z) = 0. \quad (2.2.2)$$

Using the product rule, we expand the left-hand side:

$$\frac{D\rho}{Dt}(\delta x \delta y \delta z) + \rho \left[ \frac{D(\delta x)}{Dt}(\delta y \delta z) + \frac{D(\delta y)}{Dt}(\delta x \delta z) + \frac{D(\delta z)}{Dt}(\delta x \delta y) \right] = 0. \quad (2.2.3)$$

Now consider what it means to take the material derivative of an infinitesimal length element,  $\delta x$ . As

$\delta x$  is moving with the fluid at a velocity  $u = \frac{dx}{dt}$ ,  $\delta x$  evolves over an infinitesimal time step  $\delta t$  by an amount equal to the difference in velocities between the two sides of the parcel times the time step (see Figure 6):

$$\delta x(t + \delta t) = \delta x(t) + \left[ u\left(x + \frac{1}{2}\delta x, y, z\right) - u\left(x - \frac{1}{2}\delta x, y, z\right) \right] \delta t. \quad (2.2.4)$$

Using the definition of the derivative again, we find

$$\frac{D(\delta x)}{Dt} = \frac{\delta x(t + \delta t) - \delta x(t)}{\delta t} = u\left(x + \frac{1}{2}\delta x, y, z\right) - u\left(x - \frac{1}{2}\delta x, y, z\right) = \delta x \frac{\partial u}{\partial x}. \quad (2.2.5)$$

Without loss of generality, we therefore also know how the material derivatives of  $\delta y$  and  $\delta z$  relate to their respective velocities  $v$  and  $w$ . Using this fact, we re-express the conservation of mass equation

$$\frac{D\rho}{Dt}(\delta x \delta y \delta z) + \rho \left[ \delta x \frac{\partial u}{\partial x} \delta y \delta z + \delta y \frac{\partial v}{\partial y} \delta x \delta z + \delta x \frac{\partial w}{\partial z} \delta x \delta y \right] = 0, \quad (2.2.6)$$

which we rewrite more compactly using the divergence as

$$\frac{D\rho}{Dt}(\delta x \delta y \delta z) + \rho \nabla \cdot \mathbf{v}(\delta x \delta y \delta z) = 0, \quad (2.2.7)$$

finally yielding the continuity equation:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}. \quad (2.2.8)$$

This equation relates the density of a fluid to its degree of compression/expansion, which in part responsible for constraining temperature and pressure.

## 2.3 Conservation of Water Vapor Mass

The amount of water vapor in the atmosphere plays an important role in atmospheric dynamics because it changes the average density of an air parcel at a given pressure and temperature. The specific humidity  $q$  is the ratio of water vapor mass to total moist air<sup>6</sup> parcel mass. Water vapor is conserved unless there is an exchange with other air parcels, or evaporation/condensation into and from precipitation in proximity to the air parcel. Therefore, the only change in  $q$  over time is driven by the parameterized function  $M$  [4, 5],

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<sup>6</sup>In meteorology, moist air is simply air which contains some water vapor.

the net evaporation/condensation rate per unit mass:

$$\frac{Dq}{Dt} = M. \quad (2.3.1)$$

## 2.4 Equation of State

Most generally, an equation of state relates the pressure of a material to its temperature, density, and composition. In atmospheric dynamics the composition that can vary is the water vapor concentration by mass,  $q$  [4, 5]. Therefore, our aim in this section is to express pressure as a function of these state variables; that is, find an expression for  $p(T, \rho, q)$ .

Essentially all models of atmospheric dynamics use the assumption that the air which makes up the atmosphere is well approximated as an ideal gas<sup>7</sup>. But what exactly is an ideal gas, and how do we derive the ideal gas law? Our first task is to describe a microscopic model of an ideal gas, and to use principles from kinetic theory [15] to derive the ideal gas law,

$$p = \rho R_s T. \quad (2.4.1)$$

From here we extend the ideal gas law to an equation of state for a moist atmosphere, where the specific gas constant  $R_s$ , isn't really a constant at all, but varies as a function of the specific humidity  $q$ .

First, let us consider the properties of what we call an ideal gas. We assume that the gas is composed of a very large number of molecules that are small, hard spheres. The molecules are spread out so that the average distances between them is much larger than their size. Collisions with each other and boundaries are assumed to be perfectly elastic and frictionless, acting instantaneously as momentum-conserving point collisions. Finally, the motion of these molecules is assumed to be entirely classical, so Newton's laws apply, but we do not consider other forces or quantum mechanical effects.

Imagine a cube-shaped container, with side length  $L$ , filled with an ideal gas. This is the model we will use to derive the ideal gas law [15]. But to begin, let us consider just one molecule in the container, with mass  $m$  and velocity  $\mathbf{v}$ , as shown in Figure 7. Since we want to know how the pressure exerted by a gas is related to its temperature (and by extension molecule velocity), let us first consider the average "pressure" exerted by one molecule on one face of the box. We choose the right-side face (positive  $x$ -direction outward). When the molecule collides in a perfectly elastic collision with this face, its velocity in the  $x$ -direction is reversed

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<sup>7</sup>Other equations of state are necessary to model dense planetary atmospheres, like Jupiter's.

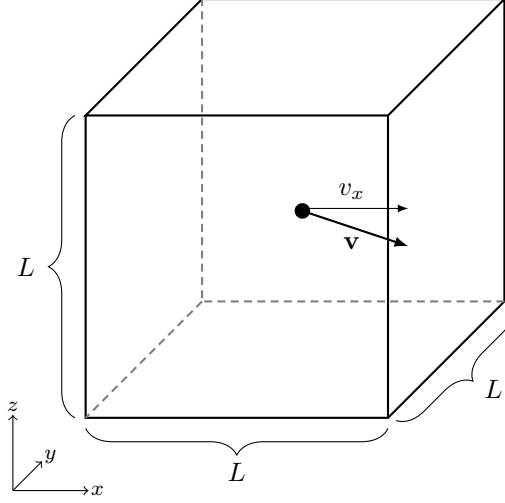


Figure 7: An idealized particle of mass  $m$  moving in a cube-shaped container, – dimensions given by the length  $L$  – with a velocity  $\mathbf{v}$  and x-component of velocity  $v_x$ .

while the other components are unchanged, so its change in velocity is  $\Delta v_x = -2v_x$ . As the molecule bounces back and forth in the container a distance  $L$  each way, traversing the x-direction at a speed  $v_x$ , it makes this type of collision with the right-side face over and over again, at time intervals equal to  $\Delta t = 2L/v_x$ . Using this information, the time-averaged (denoted by overbar) force on the molecule,  $\overline{F}_{\text{on molecule}}$ , by Newton's second law, is

$$\overline{F}_{\text{on molecule}} = m \frac{\Delta v_x}{\Delta t} = -\frac{mv_x^2}{L}. \quad (2.4.2)$$

Since the average pressure  $\overline{p}$  on the face is equal to the force on the face (opposite of the force on the molecule, by Newton's third law) divided by the area of the face  $A = L^2$ , we write the average pressure as

$$\overline{p} = \frac{\overline{F}_{\text{on face}}}{A} = -\frac{\overline{F}_{\text{on molecule}}}{A} = -\left(-\frac{mv_x^2}{L}\right)\left(\frac{1}{L^2}\right) = \frac{mv_x^2}{V}, \quad (2.4.3)$$

where  $V = L^3$  is the volume of the container.

Now imagine the case where the container is filled with a large number of  $N$  identical molecules with randomly distributed positions and velocities. The pressure<sup>8</sup> exerted on the right-side face should now be the sum of many terms like the one in Equation 2.4.3, so

$$p = \frac{m}{V}(v_{1x}^2 + v_{2x}^2 + v_{3x}^2 + \dots), \quad (2.4.4)$$

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<sup>8</sup>We drop the time average with a large number of molecules, since we are now indicating a traditional pressure.



which we express with an average (using overbar notation again) of all the  $v_x^2$  values times the number of molecules by

$$pV = Nm\overline{v_x^2}. \quad (2.4.5)$$

Since we are assuming an ideal gas with no preferred direction, the motion of molecules is distributed randomly, so we expect the pressure exerted on the other five faces to be the same as that exerted on the right-side face. This also implies that the average component velocities in the y- and z-directions should be equal to the average we found for the x-component, so

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}, \quad (2.4.6)$$

and the average square of the magnitude of the total velocity should be equal to the sum of the squares of the average component velocities, by the Pythagorean theorem, implying

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = 3\overline{v_x^2}. \quad (2.4.7)$$

Substituting this into equation 2.4.5, we find

$$pV = \frac{1}{3}Nm\overline{v^2}, \quad (2.4.8)$$

a familiar form that is starting to look like the ideal gas law.

To get from velocity to a temperature relation, we invoke a foundational idea in classical statistical mechanics, the equipartition theorem [15]. The equipartition theorem<sup>9</sup> states that for a system in thermal equilibrium, the energy will be shared equally among all forms, called degrees of freedom. These degrees of freedom might include translational, rotational, or elastic forms of energy, depending on the properties of the system, and each with separate degrees of freedom for each spatial dimension. Quantitatively, the equipartition theorem states that at a temperature  $T$ , the average energy per molecule of any degree of freedom – where the energy is a quadratic function of that degree of freedom – is  $\frac{1}{2}kT$ , where  $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$  is the Boltzmann constant.

In the primarily diatomic gas of the atmosphere, the energy is distributed evenly across three translation degrees of freedom (three spatial dimensions) and two rotational degrees of freedom (two non-zero principal moments of inertia). In this case, however, we consider only the translational kinetic energy, which is

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<sup>9</sup>See Schroeder for a proof of this theorem using Boltzmann factors [15].

responsible for exerting pressure. Therefore, we write the translational kinetic energy in the container

$$\text{KE} = \frac{1}{2}Nmv^2 = \frac{3}{2}NkT \quad (2.4.9)$$

$$\implies Nmv^2 = 3NkT, \quad (2.4.10)$$

and substitute into equation 2.4.8, our state equation, finding

$$pV = NkT. \quad (2.4.11)$$

This is the ideal gas law in the form most familiar to physicists. By definition, the universal gas constant  $R = 8.31 \text{ J K}^{-1} \text{ mol}^{-1}$  is the product of the number of molecules in a gas and the Boltzmann constant, per mole of gas; that is

$$R = \frac{Nk}{n}, \quad (2.4.12)$$

where  $n$  is the number of moles in the gas. This allows us to express the ideal gas law in the form generally used by chemists:

$$pV = nRT. \quad (2.4.13)$$

For the purposes of atmospheric dynamics, we use the form which includes density  $\rho = m/V$  rather than volume, and the specific gas constant  $R_s$  (units are  $\text{J K}^{-1} \text{ kg}^{-1}$ ) rather than the universal one  $R$  (units are  $\text{J K}^{-1} \text{ mol}^{-1}$ ), which are related by  $m$ :  $mR_s = nR$ . This yields the form of ideal gas law as used by meteorologists:

$$p = \rho R_s T. \quad (2.4.14)$$

However, to appropriately model the atmosphere, we need to take into account the variable composition of air. Specifically, we need to use the specific humidity, or mass concentration of water vapor,  $q = M_v/M$ , where  $M_v$  is the mass of water vapor,  $M_d$  is the mass of dry air, and  $M = M_d + M_v$  is the total mass, all in a given volume of moist air. This effectively means we must replace the specific gas constant  $R_s$  by a function of  $q$  [4]. First consider that we may write two versions of the ideal gas law, one for dry air and one for water vapor:

$$p_d = \rho_d R_d T \quad (2.4.15)$$

$$p_v = \rho_v R_v T, \quad (2.4.16)$$

where the specific gas constant of dry air is  $R_d = 287.1 \text{ J kg}^{-1} \text{ K}^{-1}$  and for water vapor is  $R_v = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$  [4]. The subscripts  $d$  and  $v$  denote the state variables for dry air and pure water vapor, respectively. By Dalton's law of partial pressures, the pressure in a moist atmosphere – a mix of dry air and water vapor – is therefore the sum of the two pressures  $p_d$  and  $p_v$ , so

$$p = p_d + p_v, \quad (2.4.17)$$

and likewise for density:

$$\rho = \rho_d + \rho_v. \quad (2.4.18)$$

This allows us to write a combined equation of state:

$$p = (\rho_d R_d + \rho_v R_v) T \quad (2.4.19)$$

$$= \rho R_d T \left( \frac{\rho_d}{\rho} + \frac{\rho_v}{\rho} \frac{R_v}{R_d} \right). \quad (2.4.20)$$

Recall the specific humidity  $q$ , which we rewrite, using the approximation that most of the mass of moist air is in the dry air components ( $M_d > 0.95M$ ):

$$q = \frac{M_v}{M} = \frac{M_v}{M_d + M_v} \approx \frac{M_v}{M_d} = \frac{\rho_v}{\rho_d}. \quad (2.4.21)$$

This implies that  $\rho_d/\rho \approx 1$  and  $\rho_v/\rho \approx q$ . Using these approximations and the calculation

$$\frac{R_v}{R_d} = \frac{461.5 \text{ J kg}^{-1} \text{ K}^{-1}}{287.1 \text{ J kg}^{-1} \text{ K}^{-1}} \approx 1.607, \quad (2.4.22)$$

we express the final form of the equation of state for moist air:

$$p = \rho R_d T (1 + 0.61q). \quad (2.4.23)$$

Since  $q$  is usually quite small in the atmosphere, the equation of state for moist air differs from the ideal gas law up to only a few percent in  $p$ . However, this slight difference has important consequences for the nature of convection and cloud formation in the atmosphere (see Section 3).

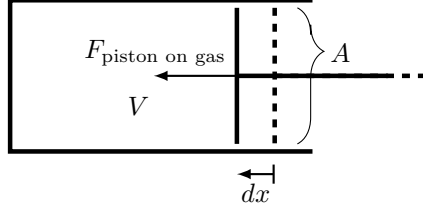


Figure 8: A piston having area  $A$ , is displaced infinitesimally a distance  $dx$  into a gas chamber with volume  $V$  exerting a force  $F_{\text{piston on gas}}$ . This simplified model is used as an example illustrating the first law of thermodynamics. Adapted from Schroeder [15].

## 2.5 Conservation of Energy

The law of conservation of energy states that the total energy of an isolated system, one which has no interactions with other systems, is constant in time. The first law of thermodynamics extends this idea, stating that the change in total energy of a closed system – one where no *mass* leaves or enters, but energy is able to flow in and out – is equal to the energy that enters the system as heat and work. Using this idea, we derive an equation which relates the rate of heat flow to pressure and temperature changes, and to the properties of air given by heat capacity and density. Stated in differential form, the first law states

$$dU = \delta Q + \delta W, \quad (2.5.1)$$

where  $dU$  is the change in energy of the closed system,  $\delta Q$  is the heat flowing *into* the system, and  $\delta W$  is the work done *on* the system. We use  $\delta$  to denote a small quantity rather than the differential of change  $d$  for  $Q$  and  $W$ , because these are not state variables, but rather, describe processes. In contrast, since  $U$  is a state variable, we can define infinitesimal changes in its value, and therefore use the differential  $d$ .

Work acts on an air parcel via compression. Therefore, in order to apply the first law to the atmosphere, it is helpful to use the example of a piston [15], shown in Figure 8. The piston has a force of magnitude  $F$  pressing with an area  $A$  on a volume of gas with volume  $V$ , and displaces a positive distance  $dx$ . The volume of gas is the system of interest in this example.

To begin, it is useful to express the change in energy  $dU$  in terms of the temperature differential  $dT$  and the material properties of the air parcel, namely mass  $m$  and specific heat at constant volume  $c_v$ . By definition, the heat capacity  $C$  of an object is the heat added to raise an objects temperature, per unit of temperature:

$$C \equiv \frac{Q}{\Delta T}. \quad (2.5.2)$$

Assuming constant volume, and therefore no work being done on the system, then  $Q = \Delta U$ , so we write in differential form:

$$(C)_V = \left( \frac{dU}{dT} \right)_V, \quad (2.5.3)$$

where the subscript  $V$  indicates that the volume is being held constant. Rewriting the heat capacity, given  $(C)_V = m c_v$ , we find

$$dU = m c_v dT. \quad (2.5.4)$$

Now let us consider the work on the system. Since the infinitesimal work on the system  $\delta W$  acts with a force  $F_{\text{piston on gas}}$  parallel to the displacement  $dx$ , then we write the work as the product

$$\delta W = F_{\text{piston on gas}} dx. \quad (2.5.5)$$

The external force exerted on the gas in the chamber by the piston  $F_{\text{piston on gas}}$  is in a Newton's third law pair with the force exerted on the piston by the gas in the chamber  $F_{\text{gas on piston}}$ ; that is

$$F_{\text{piston on gas}} = -F_{\text{gas on piston}}. \quad (2.5.6)$$

Since  $F_{\text{gas on piston}}$  is a gas pressure force, its magnitude is pressure  $p$  times area  $A$ , and its direction is opposite the direction the piston is compressing. Since we define  $dx$  to be a positive displacement, then

$$F_{\text{gas on piston}} = -pA. \quad (2.5.7)$$

Thus we rewrite the work as

$$\delta W = pA dx. \quad (2.5.8)$$

The infinitesimal change in volume is  $dV = -A dx$  (the volume decreases, so  $dV$  is negative), therefore

$$\delta W = -p dV. \quad (2.5.9)$$

We express the volume element  $dV$  in terms of the mass  $m$  and the density  $\rho$ :

$$dV = d \left( \frac{m}{\rho} \right). \quad (2.5.10)$$

Since the density of the gas  $\rho$  is variable while mass in the chamber is fixed, we expand the differential:

$$dV = -\frac{m}{\rho^2}d\rho. \quad (2.5.11)$$

And use the ideal gas law

$$p = \rho RT, \quad (2.5.12)$$

to rewrite  $d\rho$  in terms of  $dp$  and  $dT$ , expanding the differential with the quotient rule:

$$d\rho = d\left(\frac{p}{RT}\right) \quad (2.5.13)$$

$$= \frac{1}{R} \frac{T dp - p dT}{T^2} \quad (2.5.14)$$

$$= \frac{1}{RT} dp - \frac{p}{RT^2} dT \quad (2.5.15)$$

$$= \frac{1}{RT} dp - \frac{\rho RT}{RT^2} dT \quad (2.5.16)$$

$$= \frac{1}{RT} dp - \frac{\rho}{T} dT \quad (2.5.17)$$

So we find

$$\delta W = \frac{m\rho RT}{\rho^2} \left( \frac{1}{RT} dp - \frac{\rho}{T} dT \right) \quad (2.5.18)$$

$$= m \frac{dp}{\rho} - m R dT. \quad (2.5.19)$$

Combining the expressions for  $\delta W$  (Equation 2.5.18) and  $dU$  (Equation 2.5.4), we write an expanded version of the first law

$$m c_v dT = \delta Q + m \frac{dp}{\rho} - m R dT, \quad (2.5.20)$$

which simplifies to

$$(R + c_v)dT = \frac{\delta Q}{m} + \frac{dp}{\rho}. \quad (2.5.21)$$

Changing constants to the specific heat of dry air at constant pressure

$$c_p = R + c_v = 1006 \text{ J kg}^{-1} \text{ K}^{-1}, \quad (2.5.22)$$

further simplifies the equation to

$$c_p dT = \frac{\delta Q}{m} + \frac{dp}{\rho}. \quad (2.5.23)$$

For the sake of evolving these variables over time, we divide both sides by the time step  $\delta t$ , which as  $\delta t \rightarrow 0$  indicates the derivative<sup>10</sup>, so we have the final form of the first law

$$c_p \frac{DT}{Dt} = J + \frac{1}{\rho} \frac{Dp}{Dt}, \quad (2.5.24)$$

where  $J = \frac{1}{m} \frac{DQ}{Dt}$ , the rate of heat flow per unit mass. This equation is responsible for representing the primary flux of energy into the atmosphere, and relating that to changes in state variables temperature, density, and pressure. The parameterization  $J$  takes into account heating from the Sun, which fundamentally drives the motion of the atmosphere.

### 3 Clouds and Precipitation

In Section 2 we described the dynamics of the atmosphere, i.e. how the motion of the atmosphere (wind,  $\mathbf{v}$ ) and state variables ( $T$ ,  $p$ ,  $q$ ,  $\rho$ ) change in time. However, we are missing a conspicuous and important part of the weather: clouds and precipitation. Clouds form by the condensation of water vapor in moist air. This occurs when air becomes supersaturated with water. Saturation of air occurs when the vapor pressure of air is equal to the vapor pressure of liquid water; that is, when the amount of water evaporating from a flat liquid surface is equal to amount of water condensing onto the liquid from the air above, as shown in Figure 9. The point is also called 100% relative humidity. The saturation point depends on temperature – warm air can “hold” more water than cool air – but does not generally exceed a concentration by mass of  $q = 0.05$ . Supersaturation is any point past 100% relative humidity, where the vapor pressure of the air exceeds the vapor pressure of liquid water. This most often occurs when rising warm air parcels cool due to expansion and reduced pressure<sup>11</sup>, increasing the relative humidity. However, clouds do not form on their own from pure water vapor-air mixtures when supersaturation occurs. Instead, every droplet requires an aerosol nucleus (fine dust particle) of large enough size to begin growing. In this section we explore why cloud droplets need an aerosol nucleus to condense.

Let us look at the thermodynamics of phase change as it relates to cloud droplet nucleation. An important quantity in this discussion is the Gibbs free energy  $G$ , defined as the maximum amount of work which can

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<sup>10</sup>This is the material derivative because we are considering the state of a specific air parcel.

<sup>11</sup>Recall that the ideal gas law indicates that pressure and temperature are inversely proportional.

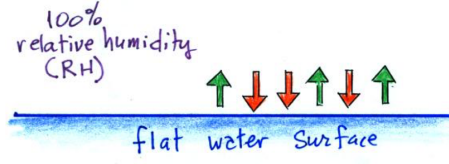


Figure 9: This diagram, courtesy of Chuck Weidman at the University of Arizona Department of Hydrology and Atmospheric Sciences, shows how saturation of air with water vapor (100% humidity) occurs when there is equilibrium between evaporation from a flat water surface and condensation onto the surface.

be extracted from a substance in a closed system<sup>12</sup>, without any change in volume or temperature [15].

Imagine a spherical droplet of water with  $N_l$  molecules surrounded by  $N - N_l$  molecules of water in vapor form, so that the total number of molecules is  $N$ . We express the Gibbs free energy per molecule, also called the chemical potential, as  $\mu_l = G_l/N_l$  and  $\mu_g = G_v/(N - N_l)$ , for the liquid and vapor phases, respectively. To write down the Gibbs free energy of the system, we simply sum the components:

$$G = G_l + G_v = N_l\mu_l + (N - N_l)\mu_g. \quad (3.0.1)$$

We rewrite  $N_l$  in terms of volume per molecule of liquid  $v_l$  and droplet radius  $r$ , using

$$v_l N_l = \frac{4}{3}\pi r^3, \quad (3.0.2)$$

to find

$$G = -\frac{4\pi r^3}{3v_l}(\mu_g - \mu_l) + N\mu_g. \quad (3.0.3)$$

However, this quantity is missing an important source of Gibbs free energy in a cloud system, called the interfacial free energy. This takes into account the fact that stretching a volume of water into a shape with larger surface area requires energy, because it takes energy to break the hydrogen bonds in water. This is why droplets form spheres, minimizing surface area because of the cohesion, or mutual attraction, of water molecules. According to the definition of surface tension  $\sigma$ , a surface has given amount of interfacial Gibbs free energy  $G_{\text{surf}}$  which is related to the area of the surface  $A = 4\pi r^2$  (area of a sphere) by

$$\sigma = \frac{G_{\text{surf}}}{A} = \frac{G_{\text{surf}}}{4\pi r^2}. \quad (3.0.4)$$

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<sup>12</sup>One in which no mass enters or leaves the system, but transfers of heat and energy are allowed.



Thus we write the total Gibbs free energy of the cloud system:

$$G_{\text{total}} = 4\pi r^2 \sigma - \frac{4\pi r^3}{3v_l}(\mu_g - \mu_l) + N\mu_g. \quad (3.0.5)$$

Let us consider the shape of the curve of  $G_{\text{total}}$  as  $r$  varies, and its implications for cloud droplet nucleation. We are interested in how the shape of the curve changes as the difference in chemical potentials  $\mu_g - \mu_l$  varies. For simplicity we drop the constant  $N\mu_g$  term so we have a change in Gibbs free energy,

$$\Delta E = 4\pi r^2 \sigma - \frac{4\pi r^3}{3v_l}(\mu_g - \mu_l). \quad (3.0.6)$$

By the definition of the Gibbs free energy and of chemical potential, at constant temperature and number, we have the Gibbs thermodynamic identity,

$$dG = Vdp = Nd\mu, \quad (3.0.7)$$

implying that

$$d\mu_g = \frac{V_l}{N_l} dp d\mu_g = v_g dp \quad (3.0.8)$$

and

$$d\mu_l = v_l dp. \quad (3.0.9)$$

The differential representing an infinitesimal change in the quantity  $\mu_g - \mu_l$ , is therefore given by

$$d(\mu_g - \mu_l) = (v_g - v_l)dp \approx v_g dp. \quad (3.0.10)$$

The approximation  $(v_g - v_l) \approx v_g$  is appropriate because the volume occupied by a liquid water molecule is insignificant in comparison to that of a vapor molecule. We use the ideal gas law for the volume of a single molecule  $pv_g = kT$ , to find

$$d(\mu_g - \mu_l) = v_g dp \quad (3.0.11)$$

$$= \frac{kT}{p} dp. \quad (3.0.12)$$

Integrating this differential from equilibrium, i.e.  $\mu_g = \mu_l$  or  $p = p_s$ , where  $p_s$  is the pressure at saturation,

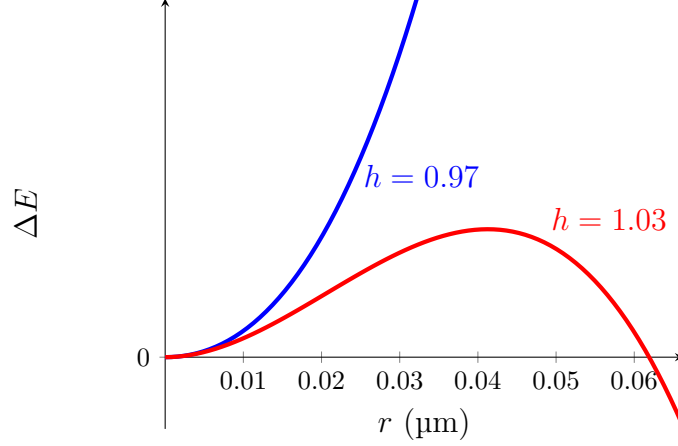


Figure 10: A plot of the energy required  $\Delta E$ , given by Equation 3.0.17, to create a water droplet of radius  $r$  from water vapor, at different levels of saturation. At under-saturation  $h = 0.97$  (blue), the expansion of a water droplet always requires additional energy, which leads to the droplet evaporating. At supersaturation  $h = 1.03$  (red), however, there is a critical value  $r_c$ , located where the curve peaks, after which point water droplets expand spontaneously because condensation begins to release energy.

we find

$$\mu_g - \mu_l = \int_0^{\mu_g - \mu_l} d(\mu_g - \mu_l) \quad (3.0.13)$$

$$= kT \int_{p_s}^p \frac{1}{p} dp \quad (3.0.14)$$

$$= kT \ln \frac{p}{p_s} \quad (3.0.15)$$

$$= kT \ln h, \quad (3.0.16)$$

where  $h = p/p_s$  is the relative humidity.

We rewrite the change in Gibbs free energy

$$\Delta E = 4\pi r^2 \sigma - \frac{4\pi r^3}{3} n k T \ln h, \quad (3.0.17)$$

where  $n = 1/v_l$ , the number of molecules per unit volume. Given we know the following approximately constant values:  $\sigma \approx 7.5 \cdot 10^{-2} \text{ N m}^{-1}$ ,  $n \approx 3.3 \cdot 10^{28} \text{ m}^{-3}$ ,  $k = 1.38 \cdot 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1}$ , and  $T \approx 0^\circ \text{C} \approx 270 \text{ K}$ , we plot  $\Delta E$  in Figure 10 with values  $h = 0.97$  and  $h = 1.03$ . We can see that for  $h < 1$ , the  $\Delta E$  curve increases monotonically. This means that for below-saturation water droplets  $\Delta E$  decreases when the droplet shrinks (goes to smaller  $r$ ), so the tendency of droplets will be to evaporate. This principle is also true for supersaturation ( $h > 1$ ) up until a critical value (the peak of the curve), at which point runaway growth of

water droplets may occur. The measured level of supersaturation rarely exceeds  $h = 1.03$  in clouds, so the peak of the curve for this value, around  $r_c = 0.04 \mu\text{m}$ , is a generous lower bound on critical droplet radius.

So how likely is it that a droplet of liquid water would reach this size by the chance of collision alone? To estimate the probability of this occurring we use a Boltzmann type relation [16], which allows us to find a ratio between an all-vapor state and all-liquid state, for a given number of molecules. It is appropriate to use the Boltzmann distribution if we assume the system of interest in thermodynamics equilibrium with its surroundings. Since clouds form in large regions of supersaturated air, we can assume that this condition is met for a small interior part of such a region. We express the Boltzmann distribution, a probability distribution that gives the probability of a state given its energy and temperature, by

$$p_i \propto e^{-\epsilon_i/kT}, \quad (3.0.18)$$

where  $p_i$  is the probability of a state  $i$  occurring, and  $-\epsilon_i$  is the energy of that state. The ratio of the probabilities of two states  $i$  and  $j$ , called the Boltzmann factor, is therefore given by

$$\frac{p_i}{p_j} = e^{(\epsilon_j - \epsilon_i)/kT}, \quad (3.0.19)$$

which is a crude estimate for the the probability that a collection of particles is in state  $i$  relative to state  $j$ . Applying this to water droplet condensation, we are finding the probability of a collection of water molecules being in a liquid state  $l$ , having a droplet radius  $r_c$ , relative to the probability that the same number of molecules is in a dispersed gaseous state  $g$ . Given that the energy difference is  $-\Delta E$  (Equation 3.0.6), and assuming the approximate conditions described previously, we find that

$$\frac{p_l}{p_g} = e^{-(4\pi r^2 \sigma - \frac{4\pi r^3}{3} n k T \ln h)/kT} \approx e^{-10^{10}} \quad (3.0.20)$$

Therefore, the chance of droplets forming to a critical size for droplet formation, by chance collisions alone, is so incredibly low so as to be essentially impossible [2, 16]. The implication of this is that homogeneous (water only) droplet nucleation is essentially impossible in our atmosphere, which is supported by experiments done in cloud chambers [16]. The only way a droplet can form is using an already large nucleus on the order of tenths of microns to start. Atmospheric aerosols, usually in the form of dust, smoke, or pollen are generally  $0.1 \mu\text{m}$  or larger, which provide ample size to begin cloud droplet formation at marginal supersaturation. From here, cloud droplets grow into rain droplets, at first by condensation, and then by collision as they

become heavy enough to overcome buoyancy and rising air currents, beginning to fall through the cloud under the force of gravity.

## 4 Observing the Atmosphere: Satellite Remote Sensing

An accurate estimate of the current conditions of the atmosphere is essential to producing an effective numerical weather forecast. Before the development of weather satellites during the 1960s and 70s, weather models were severely limited in ability because there was little to no data on the conditions of the atmosphere over the oceans and less populated land areas. However, by the 1980s satellite data started to be used to make estimates of initial conditions for numerical weather models. Satellite data has been the dominant source of atmospheric data input for weather models since the 1990s [17]. In this section, I explore the physics involved in weather satellites remotely sensing atmospheric quantities including temperature, humidity, density, pressure, and wind velocity, along with estimates of cloud- and snow-cover, cloud heights and types, and precipitation rates.

Weather satellites cannot measure any part of the atmosphere directly [17]. Fundamentally, weather satellites directly measure one quantity: electromagnetic radiation. However, an understanding of how different types and phases of matter interact with electromagnetic radiation allows us to make useful inferences about almost all aspects of the state of the atmosphere from this single type of measurement.

More precisely, weather satellites measure the spectral radiance  $I_\nu$ , which is the rate of energy flux per unit solid angle per unit area per unit frequency [2, 18]. Qualitatively speaking, the spectral radiance is the intensity of radiation measured for a given frequency. We will focus on the example of remote sensing temperature to illustrate how spectral radiance data can be used infer atmospheric quantities. This means we ignore the effects due to clouds and precipitation, as well as scattering and spectral line emission. These effects play an important role in determining other atmospheric quantities, which we will discuss later. For the purpose of temperature estimation, we assume that our satellite measures attenuated emission from the Earth and atmosphere as blackbody radiation only [18]. The spectral radiance of a blackbody with temperature  $T$  at frequency  $\nu$  is given by Planck's law

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}, \quad (4.0.1)$$

where  $k$  is the Boltzmann constant,  $h$  is the Planck constant, and  $c$  is the speed of light.

We consider the attenuated blackbody emissions from Earth's surface and from the vertical profile of the

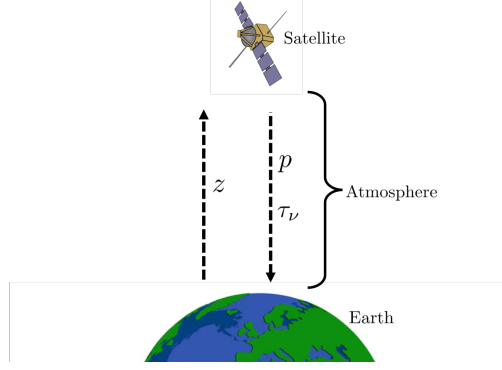


Figure 11: This diagram indicates how the height  $z$ , pressure  $p$ , and spectral optical depth  $\tau_\nu$  vary in the atmosphere. The arrows indicate the direction of increasing value for these quantities.

atmosphere separately. That is, we define the total spectral radiance to be equal to the sum of the spectral radiances from the surface  $I_{\nu, \text{surf}}$  and from the atmosphere  $I_{\nu, \text{atm}}$ :

$$I_\nu = I_{\nu, \text{surf}} + I_{\nu, \text{atm}}. \quad (4.0.2)$$

Both of these spectral radiances are attenuated from the original blackbody emissions (Equation 4.0.1) due to absorption in the atmosphere. We quantify this attenuation with the spectral optical depth  $\tau_\nu(z)$  which is the natural logarithm of the ratio of incident to transmitted spectral radiation through the atmosphere from some height above the surface  $z$  up to the height of the satellite  $s$ . The spectral optical depth increases from  $\tau_\nu(s) = 0$  as the height  $z$  decreases from  $s$  to 0 because we are moving below additional layers of the atmosphere with different densities and compositions which cumulatively absorb radiation, as shown in Figure 11. We call the total spectral optical depth of the atmosphere  $\tau_\nu^* = \tau_\nu(0)$ .

We find the spectral radiance from the surface by simply multiplying the blackbody emission from Planck's law by a transmittance factor which decreases exponentially with the optical depth:

$$I_{\text{surf}} = B_\nu(T_s) e^{-\tau_\nu^*}. \quad (4.0.3)$$

Finding the spectral radiance from the atmosphere itself is more complicated because radiation is being emitted throughout the vertical profile of the atmosphere, and is absorbed by different amounts depending on what height it is emitted from. Therefore, to calculate this spectral radiance we integrate Planck's law times the transmittance factor over the optical depth from the surface ( $\tau_\nu(0) = \tau_\nu^*$ ) to the top of the atmosphere

( $\tau_\nu(s) = 0$ ):

$$I_{\text{atm}} = \int_{\tau_\nu^*}^0 B_\nu(T(z)) e^{-\tau_\nu(z)} d\tau_\nu. \quad (4.0.4)$$

In order to rewrite this with the differential  $dz$  instead of  $d\tau_\nu$ , consider that

$$\alpha_\nu(z) \equiv \frac{d\tau_\nu}{dz}, \quad (4.0.5)$$

by definition of the spectral absorption coefficient  $\alpha_\nu(z)$ , which is the spectral optical depth per unit thickness.

This allows us to rewrite the integral as

$$I_{\text{atm}} = \int_0^s B_\nu(T(z)) e^{-\tau_\nu(z)} \alpha_\nu(z) dz. \quad (4.0.6)$$

Combining the surface and atmospheric radiation terms, we find an expression for the total spectral radiance observed by a satellite in terms of  $T_s$ ,  $T(z)$ , the spectral optical depth  $\tau_\nu(z)$ , and the spectral absorption coefficient  $\alpha_\nu(z)$ :

$$I_\nu = B_\nu(T_s) e^{-\tau_\nu^*} + \int_0^s B_\nu(T(z)) e^{-\tau_\nu(z)} \alpha_\nu(z) dz. \quad (4.0.7)$$

The factors  $e^{-\tau_\nu(z)}$  and  $\alpha_\nu(z)$  are usually grouped together into one factor  $w_\nu(z)$  called the weighting function:

$$w_\nu(z) = e^{-\tau_\nu(z)} \alpha_\nu(z). \quad (4.0.8)$$

This allows us to write a concise form of Equation 4.0.7 for the spectral radiance observed by a satellite:

$$I_\nu = B_\nu(T_s) e^{-\tau_\nu^*} + \int_0^s B_\nu(T(z)) w_\nu(z) dz. \quad (4.0.9)$$

The weighting function  $w_\nu(z)$  is applied to satellite data according to the frequency band that a particular channel on the satellite measures. Each radiation frequency band is emitted and absorbed differently throughout the atmosphere, so weighting functions allow us to estimate from which height range certain emissions came. Knowing this, we can solve for the mean temperature over that height range using the Stefan-Boltzmann law. As an example of what different weighting functions look like for a typical weather satellite, Figure 12 shows the weighting functions for each channel of the NOAA-2 Vertical Temperature Profile Radiometer (active 1972-1980), one of the first satellites to make temperature inferences [18]. Note

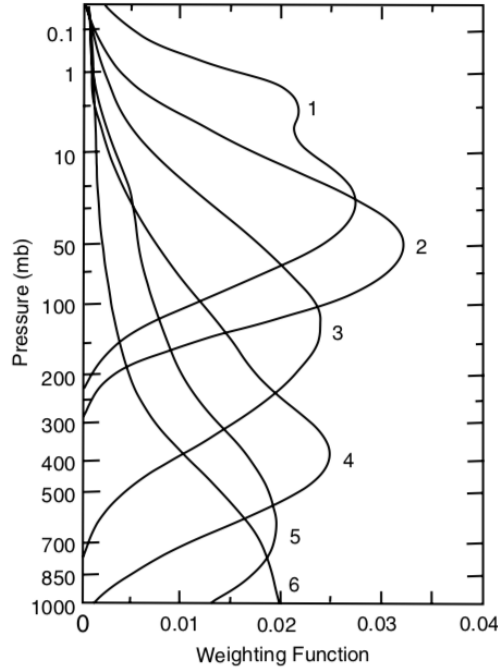


Figure 12: This diagram, adapted from K. N. Liou, shows weighting functions for each channel of the NOAA-2 Vertical Temperature Profile Radiometer, plotted with pressure as the vertical coordinate versus the weighting [18].

that the figure uses pressure as a vertical coordinate, a common practice in atmospheric sciences, because it varies inversely with height  $z$  in a predictable manner. The curves denote different weighting function for each of the six observing channels within the CO<sub>2</sub> absorption band. The peak of each curve gives an indication of where the bulk of a certain frequency band of radiation is being emitted.

There are many additional analysis techniques used to infer other atmospheric quantities. Infrared wavelengths near 6  $\mu\text{m}$  are well absorbed by water vapor, allowing satellites to infer humidity levels, and use weighting functions to estimate the height in the vertical profile of the atmosphere [18]. The temperature of cloud tops (and by extension, cloud heights) can be straightforwardly determined by using the characteristic infrared blackbody curve in areas with visible cloud cover, because clouds are mostly opaque in the infrared. Visible light imaging of clouds, smoke, vegetation, and snow cover utilizes the reflection of sunlight to observe these phenomena during the day-time. Visible and infrared imaging of clouds allows for estimates of wind speed and direction by tracking the evolution of clouds.

## 5 Implementing a Weather Model

### 5.1 Parameterization

As alluded to throughout Section 2, the governing equations of the atmosphere include three parameterized functions: friction  $\mathbf{f}$  (Section 2.1), water vapor flux  $M$  (Section 2.3), and heat flux  $J$  (Section 2.5) [5]. These processes are parameterized, rather than modeled explicitly, because they occur on a scale smaller in space and/or time than the resolution of any current weather models. The effects of the processes are nonetheless essential in a model of the atmosphere<sup>13</sup>. Parameterizations use a statistical approach to account for how these small scale-processes have large-scale effects.

Each layer of the atmosphere has a different kind of parameterization, depending on what mechanisms are primarily responsible for friction, heating, and exchanging water vapor. Clouds play an important role in regulating the reflection and absorption of radiation. The Earth's surface provides powerful sources and sinks of momentum, heat, and humidity. And precipitation does the same through its interaction with the surface convective currents.

### 5.2 Numerical Methods

The advent of computers marked the beginning of useful numerical weather prediction because they allowed the analytically intractable non-linear partial differential equations of the atmosphere to be approximated by many simple arithmetic calculations. The process of turning the calculus problem of integration into an arithmetic problem is called the finite difference method [4, 5]. Rather than using differential equations, these equations are approximated as difference equations with small finite differences, rather than true infinitesimal differentials.

To implement the governing equations of the atmosphere into finite difference processes, the equations must first be transformed into the Eulerian description (see Appendix A) and rewritten in an Earth-centered spherical coordinate system. These transformations allow for appropriate use of boundary conditions, and express the differential equations using partial derivatives only, which are the differentials we will approximate with finite differences. These finite differences in space designate a 3D grid extending from the Earth's surface upward through the troposphere. This grid breaks up the atmosphere into finite parcels, whose states we evolve through finite difference arithmetic.

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<sup>13</sup>One cannot get very far without taking the Sun into account.



### 5.3 Climate Modeling

So far we have focused our attention on numerical modeling applications in weather prediction. However, the same five governing equations are the basis of global climate models, as well [4]. This might seem counter-intuitive, given that chaos theory (see Appendix D) implies that numerical weather models at best can produce usefully accurate forecasts out only a few weeks from initialization. However, the weather can be thought of as a random perturbation on the seasonal and annual averages which define the climate. Therefore, the problem of predicting climate is quite different from predicting weather.

Imagine a global model of the atmosphere, much like those used to predict the weather, but projecting decades into the future. Integrate this model a number of times with slightly different initial conditions, constituting an ensemble [4,9]. We expect these simulations to make very different predictions of the future weather, none of which will accurately reflect what the weather will be like on a given day years from now. However, when we take the time average of these projections on the scale of months to years, we can get an idea of what the average weather. i.e. climate, will look like. Together with projections of future greenhouse gas concentrations – with and without significant reductions in fossil fuel use – models like these can offer insight into what the future climate of Earth might look like.

## 6 Conclusion

The development of numerical weather modeling is one of the greatest scientific revolutions of the twentieth century, albeit a slow one. Numerical modeling of the weather has become steadily more useful in nearly every aspect of life. Today, computer generated forecasts are an essential tool throughout our global economy, from logistics and agriculture to resource use and extraction. Most importantly, loss of life in weather-related disasters has dropped dramatically over the last century, in large part due to advanced warning of powerful storms. Finally, the future of humanity and all life on Earth will rely on our ability to reduce greenhouse gas emissions, as informed by models of our world’s future climate.

## 7 Acknowledgements

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feedback on my presentation in the days before my public talk. Lastly, a special thanks to my supportive physics peers, friends, and family.

## Appendix A The Lagrangian and Eulerian Specifications of Flow

An understanding of the Lagrangian and Eulerian specifications of flow is essential to a mathematical discussion of the dynamics of the atmosphere. The Lagrangian specification refers to the state of a parcel of fluid, following that parcel as it moves through space [3, 12]. We use the material derivative  $\frac{D}{Dt}$ , also called the Lagrangian or total derivative (along with many other names), in this specification, which is the time rate of change of a physical quantity describing a parcel of fluid (a state variable, e.g. temperature), moving along with that parcel as it moves and deforms spatially. This contrasts with the Eulerian specification, in which one describes the changing state of a fluid as it flows past a particular point in space. We use the partial derivative  $\frac{\partial}{\partial t}$  in this description, which is the time rate of change of a physical quantity at a particular position in space (in the case of the atmosphere, the position is usually fixed with respect to the Earth's surface). The two types of derivatives, demonstrated here with scalar field  $f$  describing a fluid moving with flow velocity  $\mathbf{u}$ , are related by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f, \quad (\text{A.0.1})$$

which follows from applying the multivariate chain rule (see Appendix A.1). For a vector field, we simply treat each component separately like a scalar field to get the relationship between derivatives.

We use the example of a flowing river to develop an intuition for these two specifications of flow. Imagine you are floating down a river in a boat, without any propulsion or steering. You stick a thermometer in the water to measure the temperature  $T$  along your journey. Since the boat is flowing along with the water below, you are measuring the temperature in the Lagrangian specification, and would therefore determine a time rate of change of temperature which is the material derivative  $\frac{DT}{Dt}$ . Now imagine you are sitting on the bank of the river for some time, measuring the temperature of the water with your thermometer fixed at a particular position in the river. In this case you determine a time rate of change which is the partial derivative  $\frac{\partial T}{\partial t}$ .

It is important to note that the Lagrangian and Eulerian specifications of flow do not correspond to specific reference frames or coordinate systems. Instead, they refer to what we are measuring in the fluid (state of particle/parcel vs. state at position), and both can be defined with respect to either inertial or non-inertial reference frames, as we will do throughout this paper.

So why do we need both of these specifications and types of derivatives to talk about fluids? It turns out that we use either the Lagrangian and Eulerian specification to describe how quantities evolve through time in the atmosphere, depending on the context. When we apply physical laws to air parcels, as we will do in deriving the equations of the atmosphere, we must use the Lagrangian specification. This is because forces act on objects (like molecules or parcels of fluid), not on positions in a fluid. Likewise, velocities and accelerations of fluid parcels are always defined using the Lagrangian specification. We use the Eulerian specification to rework the equations of the atmosphere so that they make sense within the framework of observing and predicting the atmosphere at arbitrary locations. After all, weather stations are fixed on the Earth's surface, and observing aircraft and weather balloons follow paths mostly independent of the flow of the atmosphere. The Eulerian specification is also used during the numerical computing process because a position-defined perspective is the framework within which we specify initial and boundary conditions, parameterize effects of the Sun, land, and ocean interacting with the atmosphere, and make predictions about what the weather will be like at a given time and location in the future.

## A.1 The Material Derivative

The capital “D” notation for the material derivative  $\frac{D}{Dt}$  is simply a shorthand for writing the total derivative of a field quantity which depends on time and position in three-dimensional space. That is, if we have a scalar field  $f(t, x, y, z)$ , then

$$\frac{Df}{Dt} \equiv \frac{d}{dt}f(t, x, y, z). \quad (\text{A.1.1})$$

Expanding the derivative using the multivariate chain rule, we find

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} \frac{\partial t}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}. \quad (\text{A.1.2})$$

And recognizing that the partial derivatives of the position coordinates with respect to time are simply the components of the velocity  $\mathbf{u} = (u, v, w)$  as we move through space, we rewrite this as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w. \quad (\text{A.1.3})$$

Which we express more compactly using the dot product and gradient:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f. \quad (\text{A.1.4})$$

## Appendix B Derivatives of Velocities in Inertial and Rotating Frames

We take the inertial-frame derivative of both sides of equation 2.1.15:

$$\left(\frac{D\mathbf{u}}{Dt}\right)_i = \left(\frac{D}{Dt}[\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}]\right)_i; \quad (\text{B.0.1})$$

and simplify/expand the derivative on the right-hand side using the product rule:

$$\left(\frac{D\mathbf{u}}{Dt}\right)_i = \left(\frac{D\mathbf{v}}{Dt}\right)_i + \left(\frac{D}{Dt}[\boldsymbol{\omega} \times \mathbf{r}]\right)_i \quad (\text{B.0.2})$$

$$= \left(\frac{D\mathbf{v}}{Dt}\right)_i + \left(\frac{D\boldsymbol{\omega}}{Dt}\right)_i \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{D\mathbf{r}}{Dt}\right)_i \quad (\text{B.0.3})$$

Using a similar logic to that of Equation 2.1.15, we can rewrite the derivative of the rotating frame velocity in the inertial frame as the derivative of the rotating frame velocity in the rotating frame plus the centripetal acceleration:

$$\left(\frac{D\mathbf{v}}{Dt}\right)_i = \left(\frac{D\mathbf{v}}{Dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v}, \quad (\text{B.0.4})$$

so we have:

$$\left(\frac{D\mathbf{u}}{Dt}\right)_i = \left(\frac{D\mathbf{v}}{Dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v} + \left(\frac{D\boldsymbol{\omega}}{Dt}\right)_i \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{D\mathbf{r}}{Dt}\right)_i. \quad (\text{B.0.5})$$

Subbing in  $\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}$  for  $\left(\frac{D\mathbf{r}}{Dt}\right)_i$ , and  $\dot{\boldsymbol{\omega}}$  for  $\left(\frac{D\boldsymbol{\omega}}{Dt}\right)_i$ , we find:

$$\left(\frac{D\mathbf{u}}{Dt}\right)_i = \left(\frac{D\mathbf{v}}{Dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \quad (\text{B.0.6})$$

$$= \left(\frac{D\mathbf{v}}{Dt}\right)_r + 2\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (\text{B.0.7})$$

Assuming that  $\boldsymbol{\omega}$  is constant for the Earth (which is a reasonable assumption over the timescales of a weather forecast), then  $\dot{\boldsymbol{\omega}} = 0$ , yielding

$$\left(\frac{D\mathbf{u}}{Dt}\right)_i = \left(\frac{D\mathbf{v}}{Dt}\right)_r + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (\text{B.0.8})$$

Which we write, finally, assuming the velocities derivatives are taken in the frames in which they are defined:

$$\frac{D\mathbf{u}}{Dt} = \frac{D\mathbf{v}}{Dt} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (\text{B.0.9})$$

## Appendix C Summary of Governing Equations

This appendix section provides a concise summary, in Lagrangian form, of the equations of the atmosphere, including the definitions of all the variables and constants used.

1. Conservation of momentum:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^* - 2\boldsymbol{\omega} \times \mathbf{v} + \mathbf{f} \quad (\text{C.0.1})$$

2. Conservation of total mass:

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \mathbf{v} \quad (\text{C.0.2})$$

3. Conservation of water vapor mass:

$$\frac{Dq}{Dt} = M \quad (\text{C.0.3})$$

4. Equation of state:

$$p = \rho R_d T (1 + 0.61q) \quad (\text{C.0.4})$$

5. Conservation of energy:

$$c_p \frac{DT}{Dt} = J + \frac{1}{\rho} \frac{Dp}{Dt} \quad (\text{C.0.5})$$

Related to each other by these equations, there are five unknown physical atmospheric quantities at each point in the troposphere  $\mathbf{v} = (u, v, w)$ ,  $\rho$ ,  $p$ ,  $T$ , and  $q$  which evolve dependent on time  $t$ , where  $u$  is the zonal velocity (positive eastward),  $v$  is the meridional velocity (positive northward),  $w$  is the vertical velocity (positive upward),  $\rho$  is the density,  $p$  is the pressure,  $T$  is the temperature, and  $q$  is the specific humidity. All are defined with respect to a rotating Earth-centered reference frame. These equations also contain constants including the specific heat  $c_p$  and specific gas constant of dry air  $R_d$ , as well as well-measured geophysical quantities, which do not vary appreciably with time, including the angular velocity vector of the Earth  $\boldsymbol{\omega}$ , and the effective gravitational field of the Earth  $\mathbf{g}^*$ . Finally, there are three parameterized functions, which aim to take into account phenomena occurring on shorter and smaller time scales than resolved by the dynamic model. These include the friction per unit mass  $\mathbf{f}$ , accounting for effects due to the viscosity of the fluid (especially along the boundary with the Earth's surface), the rate of heat flow per unit mass  $J$ , driven by radiation from the Sun, Earth's surface, and exchanges between air parcels, and the net evaporation/condensation rate per unit mass  $M$ , caused by exchanges with the Earth's surface and between air parcels. These equations, combined with estimates of the parameterized functions, boundary conditions

from the Earth's surface and stratosphere, and the current conditions of the atmosphere from observation, form the basis of a numerical model for the atmosphere.

## Appendix D Chaos

A discussion of numerical weather prediction naturally leads to the question: Why is forecasting still so inaccurate beyond about a week's time? And why, on occasion, does the forecast even for tomorrow go horribly wrong? With orders of magnitudes increase in computing power and observational data, why has the range of accurate forecasting only improved by a few days in the last half century?

An important part of the answer to these questions is rooted in a phenomenon which is pervasive in the natural world, but was not well realized or understood until the late 1970s: chaos [1, 3]. An early pioneer of chaos theory, Edward Lorenz, stumbled upon the phenomenon in 1961 while he was running simple weather simulations. He found that small changes in the initial conditions of the forecast, in his case introduced accidentally by rounding, produced totally different outcomes within a short timescale. Further development of the theory showed that chaos is pervasive in the natural world. In the case of weather modeling, it puts fundamental physical limits on the predictability of the weather.

Even with massive advances in measuring the initial conditions of our atmosphere, and models which get closer to describing physical reality at smaller and smaller grid scales, we will never be able to make completely accurate predictions more than a few weeks into the future. Still, there is a lot of progress to be made before forecast accuracy and range nears the physical limits on predictability implied by chaos theory.

## References

- [1] K. Saha, *The Earth's Atmosphere: Its Physics and Dynamics* (Springer-Verlag, 2008)

ANNOTATION: This text provides helpful derivations for atmospheric dynamics and radiative transfer.

- [2] J. M. Wallace and P. V. Hobbs, *Atmospheric Science: An Introductory Survey* (Elsevier Academic Press, 2006)

ANNOTATION: This text provides a broad overview of atmospheric science. Helpful for putting my topics into the larger context of atmospheric science. Also used for its detailed treatment of radiative transfer in the atmosphere.

- [3] G. Visconti, *Fundamentals of Physics and Chemistry of the Atmosphere* (Springer-Verlag, 2001)

ANNOTATION: This book contains a broad overview of physics applied to the atmosphere, and a helpful section on the implications of chaos theory for the atmosphere.

- [4] J. P. Peixoto and A. H. Oort, *Physics of Climate* (American Institute of Physics, 1992)

ANNOTATION: An excellent source for concise summary of the basic equations of the atmosphere, and a description of how they can be used in climate modeling.

- [5] J. Coiffier, *Fundamentals of Numerical Weather Prediction* (Cambridge University Press, 2011)

ANNOTATION: Helpful in understanding how to implement a numerical weather model from the basic equations using numerical analysis techniques, and how model parameterizations work.

- [6] L. F. Richardson, *Weather Prediction by Numerical Process* (Cambridge University Press, 1922)

ANNOTATION: The first paper to describe an attempt at numerically predicting the weather.

- [7] A. Kasahara and W. M. Washington, *NCAR Global General Circulation Model of the Atmosphere*, Mon. Weather Rev. **95** (1967) (7), pp. 389–402

ANNOTATION: This article describes an early weather prediction model developed at NCAR using height as a vertical coordinate, rather than pressure.

- [8] E. N. Lorenz, *The Nature and Theory of the General Circulation of the Atmosphere* (World Meteorological Organization, 1967)

ANNOTATION: An early summary of the foundations of numerical weather prediction.

- [9] J. Marshall and R. A. Plumb, *Atmosphere, Ocean and Climate Dynamics : An Introductory Text* (Elsevier Academic Press, 2007)

ANNOTATION: A good overview of the dynamics topics covered in this paper. Especially helpful with derivations of conservation of mass and momentum equations.

- [10] C. Eckart and H. G. Ferris, *Equations of Motion of the Ocean and Atmosphere*, Rev. Mod. Phys. **28** (1956), pp. 48–52

ANNOTATION: An early text summarizing the basic equations of the atmosphere, and their physical origins.

- [11] C. Eckart, *Hydrodynamics of Oceans and Atmospheres* (Pergamon Press, 1960)

ANNOTATION: A pioneering text in geophysical fluid dynamics.

- [12] P. K. Kundu and I. M. Cohen, *Fluid Mechanics*, 4th ed. (Elsevier Academic Press, 2008)

ANNOTATION: Used as a resource for fluid dynamics fundamentals. Especially helpful with understanding Lagrangian and Eulerian specifications of flow.

- [13] H. Lamb, *Hydrodynamics*, 6th ed. (Dover Publications, 1945)

ANNOTATION: One of the first major texts on fluid dynamics.

- [14] J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems*, 4th ed. (Saunders College Publishing, 1995)

ANNOTATION: This source mainly provided the background necessary for discussion of non-inertial frames on a rotating Earth.

- [15] D. V. Schroeder, *Thermal Physics* (Addison Wesley Longman, 2000)

ANNOTATION: Used frequently for references to fundamental ideas in thermodynamics and statistical mechanics. Particularly helpful for derivations of the ideal gas law and conservation of energy.

- [16] R. R. Rogers, *A Short Course in Cloud Physics* (Pergamon Press, 1976)

ANNOTATION: This text focuses on the thermodynamics and chemistry of clouds.

- [17] J. N. Thépaut, *Satellite data assimilation in numerical weather prediction: an overview*, Meteorological Training Course Lecture Series, ECMWF (2003)

ANNOTATION: This source describes how satellite observations of radiation from Earth are interpreted in order to estimate the state of the atmosphere.

- [18] K. N. Liou, *An introduction to atmospheric radiation* (Elsevier Academic Press, 2002)



ANNOTATION: A detailed overview of radiative transfer in the atmosphere, and how satellites estimate the state of the atmosphere from radiation measurements.

- [19] A. A. White, B. J. Hoskins, I. Roulstone and A. Staniforth, *Consistent approximate models of the global atmosphere: shallow, deep, hydrostatic, quasi-hydrostatic and non-hydrostatic*, Quart. J. Roy. Meteorol. Soc. **131** (2005) (609), pp. 2081–2107

ANNOTATION: An overview of different approximations used in numerical weather models.

- [20] P. V. Hobbs, *Basic Physical Chemistry for the Atmospheric Sciences*, 2nd ed. (Cambridge University Press, 2000)

ANNOTATION: Chemical and thermodynamic principles of cloud growth are outlined.