Weakly-affine monads

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Abstract

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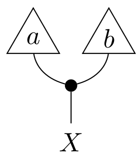
1 Weakly Markov categories and weakly affine monads

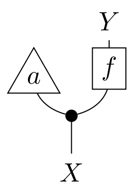
Let $\mathcal{C}$ be a GS-category. For every object X, the set $\mathcal{C}(X, I)$ has a canonical commutative monoid structure as follows: the monoidal unit is the discard map $X \to I$, and given $a, b : X \to I$, their product ab is given by copying, as follows.

How to call them? effects? co-states?



If a morphism $f : X \to Y$ is copyable and discardable, precomposition with f induces a morphism of monoids $\mathcal{C}(Y, I) \to \mathcal{C}(X, I)$.

The monoid $\mathcal{C}(X, I)$ acts on the set $\mathcal{C}(X, Y)$: given $a : X \to I$ and $f : X \to Y$, $a \cdot f$ is given as follows,



and the product $(f, g) \mapsto f \cdot g := (f \otimes g) \circ \mathrm{copy}_X$ is equivariant for this action in both variables.

▶ **Definition 1.1.** A GS-category $\mathcal{C}$ is called *weakly Markov* if for every object X, the monoid $\mathcal{C}(X, I)$ is a group.

Every Markov category is weakly Markov: for each X, the monoid $\mathcal{C}(X, I)$ is the trivial group.

▶ **Definition 1.2.** Given two parallel morphisms $f, g : X \to Y$ in a weakly Markov GS-category $\mathcal{C}$, we say that f and g are *equivalent*, and write $f \sim g$, if they lie in the same orbit for the action of $\mathcal{C}(X, I)$, i.e. if there is $a \in \mathcal{C}(X, I)$ such that $a \cdot f = g$. We say they are *uniquely equivalent* if there is a unique $a \in \mathcal{C}(X, I)$ such that $a \cdot f = g$.

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Let's now consider the case where the GS structure comes from a commutative monad on a cartesian monoidal category $\mathcal{D}$. In this case, the monoid structure of Kleisli morphisms $X \to 1$ comes from the following canonical internal monoid structure of $T1$ in $\mathcal{D}$, given by

$$1 \xrightarrow{\eta} T1, \qquad T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

The monoid structure of Kleisli morphisms $X \to 1$ is now given as follows. The unit is given by

$$X \xrightarrow{!} 1 \xrightarrow{\eta} T1,$$

and the multiplication of the morphisms $f^\sharp, g^\sharp : X \to T1$ is

$$X \xrightarrow{\text{copy}} X \times X \xrightarrow{f^\sharp \times g^\sharp} T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

▶ **Definition 1.3.** A commutative monad T on a cartesian monoidal category is called *weakly affine* if $T1$ with its canonical internal monoid structure is a group.

▶ **Proposition 1.4.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. The Kleisli category of T is weakly Markov if and only if T is weakly affine.*

Proof. First, suppose that $T1$ is an internal group, and denote by $\iota : T1 \to T1$ its inversion map. The inverse of the morphism $f^\sharp : X \to T1$ in $\mathrm{Kl}_T(X, 1)$ is given by $\iota \circ f$: indeed, the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r, "\text{copy}"] \arrow[d, "f^\sharp"'] & X \times X \arrow[d, "f^\sharp \times f^\sharp"'] \arrow[dr, "f \times (\iota \circ f)"] & & \\
T1 \arrow[r, "\text{copy}"] \arrow[d, "!"'] & T1 \times T1 \arrow[r, "\mathrm{id} \times \iota"'] & T1 \times T1 \arrow[r, "c"] & T(1 \times 1) \arrow[d, "\cong"] \\
1 \arrow[rrr, "\eta"] & & & T1
\end{tikzcd}$$

where the bottom rectangle commutes since ι is the inversion map for $T1$. The analogous diagram with $\iota \times \mathrm{id}$ in place of $\mathrm{id} \times \iota$ commutes analogously.

Conversely, suppose that for every X, the monoid structure on $\mathrm{Kl}_T(X, 1)$ has inverses. Then in particular we can take $X = T1$, and the inverse of the Kleisli morphism $\mathrm{id} : T1 \to T1$ is an inversion map for $T1$. ◀

1.1 In terms of the Yoneda embedding

For context:

▶ **Proposition 1.5.** *A monoid (M, m, e) is a group if and only if the associativity square*

$$\begin{tikzcd}
M \times M \times M \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"] & M
\end{tikzcd} \tag{1}$$

is a pullback.

The same can be said more generally for a monoid object in a cartesian monoidal category.

Proof. The square (**??**) is a pullback, both of sets and of groups, if and only if given $a, g, h, c \in M$ such that $ag = hc$, there exists a unique $b \in M$ such that $g = bc$ and $h = ab$. First, suppose that g is a group. The only possible choice of b is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (**??**) is a pullback. We can set $g, h = e$ and $c = a$ so that $ae = ea = a$. Instantiating the pullback property, there is a unique b such that $ab = e$ and $ba = e$, that is, $b = a^{-1}$. ◀

Let $\mathcal{D}$ be a cartesian monoidal category. Consider the presheaf category $[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$, equipped with the Day convolution product,

$$F \boxtimes G \cong \int^{A,B \in \mathcal{D}} \mathcal{D}(-, A \times B) \times F(A) \times G(B).$$

The Yoneda embedding $\mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is strong monoidal: indeed, for each X,

$$1 \cong \mathcal{D}(X, 1),$$

since 1 is terminal, and for each X and Y, by Yoneda reduction,

$$\begin{aligned} \mathcal{D}(-, X) \boxtimes \mathcal{D}(-, Y) &\cong \int^{A,B \in \mathcal{D}} \mathcal{D}(-, A \times B) \times \mathcal{D}(-, X) \times \mathcal{D}(-, Y) \\ &\cong \mathcal{D}(-, X \times Y). \end{aligned}$$

Therefore, and by the universal property of products, at the level of individual hom-sets the Day convolution product of representable presheaves just takes the cartesian products of sets:

$$\big(\mathcal{D}(-, X) \boxtimes \mathcal{D}(-, Y)\big)(A) \cong \mathcal{D}(A, X \times Y) \cong \mathcal{D}(A, X) \times \mathcal{D}(A, Y).$$

Take now an object M of $\mathcal{D}$. Since the Yoneda embedding is fully faithful and strong monoidal, a monoid structure (M, m, e) on M is equivalently a monoid structure on the representable presheaf $\mathcal{D}(-, M)$. This makes the individual hom-sets monoids, with unit and multiplication as follows for each object X:

$$1 \xrightarrow{\quad\cong\quad} \mathcal{D}(X, 1) \xrightarrow{e_*} \mathcal{D}(X, M)$$

$$\mathcal{D}(X, M) \times \mathcal{D}(X, M) \xrightarrow{\cong} \mathcal{D}(X, M \times M) \xrightarrow{m_*} \mathcal{D}(X, M)$$

This is precisely the monoid structure that we have defined in the previous section for $M = T1$.

▶ **Proposition 1.6.** *M is an internal group if and only if all the monoids $\mathcal{D}(X, M)$ are groups.*

Proof. By **??**, M is a group object if and only if its associativity square (**??**) is a pullback. Since the hom-functor preserves and reflects all limits in its second argument, we have that (**??**) is a pullback if and only if for each object X, the following diagram (or equivalently, its bottom right square) is a pullback,

$$\begin{tikzcd}
\mathcal{D}(X,M)\times\mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[rr] \arrow[dd] \arrow[dr, "\cong"'] & & \mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[d, "\cong"] \\
& \mathcal{D}(X,M\times M\times M) \arrow[r, "(m\times\mathrm{id})_*"'] \arrow[d, "(\mathrm{id}\times m)_*"] & \mathcal{D}(X,M\times M) \arrow[d, "m_*"] \\
\mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[r, "\cong"'] & \mathcal{D}(X,M\times M) \arrow[r, "m_*"'] & \mathcal{D}(X,M)
\end{tikzcd}$$

where the unlabelled arrows are the unique ones that make the diagram commute. Again by **??**, the diagram above is a pullback if and only if $\mathcal{D}(X,M)$ is a group. ◀

1.2 Examples of weakly affine monads

▶ **Example 1.7.** We present a family of examples of commutative monads that are weakly affine but not affine. Let A be a commutative monoid (written multiplicatively). Then the functor $T_A \coloneqq A\times -$ on **Set** has a canonical structure of commutative monad, where the lax structure components $c_{X,Y}$ are given by multiplying elements in A while carrying the elements of X and Y along.

Since $T_A(1)\cong A$, the monad T_A is weakly affine if and only if A is a group, and affine if and only if $A\cong 1$.

▶ **Example 1.8.** Let $M^* : \mathbf{Set}\to\mathbf{Set}$ be the monad assigning to every set the set of finitely supported discrete *nonzero* measures on M^*, or equivalently let $M^*(X)$ for any set X be the set of nonzero finitely supported functions $X\to[0,\infty)$. The monad structure is defined in terms of the same formulas as for the distribution monad on **Set** and the components $c_{X,Y}$ are also given by the formation of product measures, or equivalently point-wise products of functions $X\to[0,\infty)$.

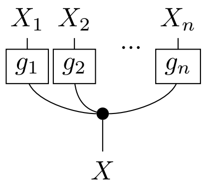
Since $M^*1\cong(0,\infty)\ncong 1$, this monad is not affine. However the monoid structure of $(0,\infty)$ induced by M^* is the usual multiplication of positive real numbers, which form a group. Therefore M^* is weakly affine, and its Kleisli category is weakly Markov.

On the other hand, if the zero measure is included, then we obtain a commutative monad M which can be seen as the monad of semimodules for the semiring of nonnegative reals. Since $M1\cong[0,\infty)$ is not a group under multiplication, M is not weakly affine.

▶ **Example 1.9.** Here is a negative example. Consider the free abelian group monad F on **Set**. Its functor takes a set X and forms the set FX of finite multisets (with repetition, where order does not matter) of elements of X and their formal inverses. We have that $F1\cong\mathbb{Z}$, which is an abelian group under addition. However, the monoid structure on $F1$ induced by the monoidal structure of the monad corresponds to the *multiplication* in $\mathbb{Z}$, which does not have inverses. Therefore F is not weakly affine.

1.3 Conditional independence in weakly Markov categories

▶ **Definition 1.10.** A morphism $f : A\to X_1\otimes\cdots\otimes X_n$ in a A GS-category $\mathcal{C}$ is said to exhibit *conditional independence of the X_i given A* if and only if it can be expressed as a product of the following form.

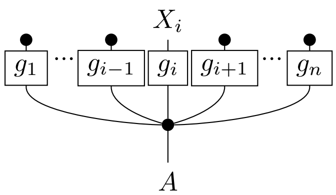


Note that this is slightly different from [**?**, Definition 6.6], although it is equivalent for the case of Markov categories.

▶ **Proposition 1.11.** *Let $f : A \to X_1 \otimes \cdots \otimes X_n$ be a morphism in a GS-category $\mathcal{C}$. Then f exhibits conditional independence of the X_i given A if and only if it is equivalent to the product of all its marginals. Moreover, in that case f is* uniquely *equivalent to the product of its marginals.*

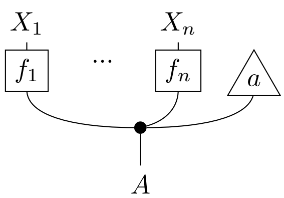
This generalizes the fact that, in Markov categories, a distribution exhibiting conditional independence is the product of its marginals [**?**, Section 12].

Proof. Denote the marginals of f by $f_1, \dots, f_n$. Suppose that f is a product as in Definition 1.7. For each $i = 1, \dots, n$, by marginalizing, we get that f_i is equal to the following.

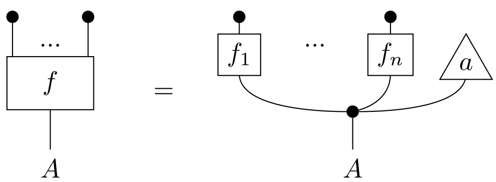


Therefore for each i we have that $f_i \sim g_i$.

Conversely, suppose that f is equivalent to the product of its marginals, i.e. that there exists $a : X \to I$ such that f is equal to the following.

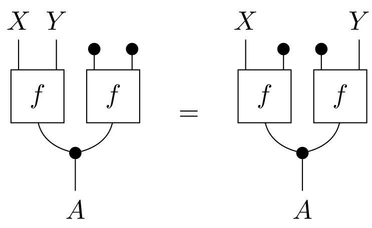


One can then choose $g_i = f_i$ for all $i < n$, and $g_n = a \cdot f_n$, so that f is in the form of Definition 1.7. Moreover, by marginalizing over all the X_i at once, we see that



so that a is uniquely determined. ◀

▶ **Remark 1.12.** For $n = 2$, a morphism $f : A \to X \otimes Y$ in a weakly Markov GS-category $\mathcal{C}$ exhibits conditional independence of X and Y given A if and only if the following equation holds.



▶ **Lemma 1.13.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. A Kleisli morphism $f^\sharp : A \to T(X_1 \times \cdots \times X_n)$ exhibits conditional independence of the X_i given A if and only if it factors as follows*

$$\begin{tikzcd} A \arrow[d, "{(g_1^\sharp,\dots,g_n^\sharp)}"'] \arrow[dr, "f^\sharp"] & \\ TX_1 \times \cdots \times TX_n \arrow[r, "c"'] & T(X_1 \times \cdots \times X_n), \end{tikzcd}$$

for some Kleisli maps $g_i^\sharp : A \to TX_i$, where the map c above is the one obtained by iterating the multiplication of the monoidal structure (such a map is unique by associativity).

Proof. In terms of the base category $\mathcal{D}$, a Kleisli morphism in the form of Definition 1.7 reads as follows.

$$A \xrightarrow{\text{copy}} A \times \cdots \times A \xrightarrow{g_1^\sharp \times \cdots \times g_n^\sharp} TX_1 \times \cdots \times TX_n \xrightarrow{c} T(X_1 \times \cdots \times X_n).$$

Therefore $f^\sharp : A \to T(X_1 \times \cdots \times X_n)$ is exhibiting conditional independence if and only if it is in the form above. ◀

▶ **Definition 1.14.** Let $\mathcal{C}$ be a GS-category. We say that $\mathcal{C}$ satisfies the *localized independence property* if whenever a morphism $f : A \to X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A, as well as conditional independence of X and $Y \otimes Z$ given A, then it exhibits conditional independence of X, Y and Z given A.

▶ **Theorem 1.15.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. The following conditions are equivalent.*

1. *T is weakly affine;*
2. $\mathrm{Kl}(T)$ *is weakly Markov;*
3. $\mathrm{Kl}(T)$ *satisfies the localized independence property;*
4. *For all objects X, Y, and Z, the following associativity diagram is a pullback.*

$$\begin{tikzcd} T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"] \arrow[d, "c_{X,Y} \times \mathrm{id}"'] & T(X) \times T(Y \times Z) \arrow[d, "c_{X, Y \times Z}"] \\ T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"] & T(X \times Y \times Z) \end{tikzcd} \tag{2}$$

Proof. 1 ⇔ 2: see Proposition 1.4.

2 ⇒ 3: Suppose $f : A \to X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A, as well as conditional independence of X and $Y \otimes Z$ given A By marginalizing out X, we have that f_{YZ} exhibits conditional independence of Y and Z given A. Since by hypothesis f exhibits conditional independence of X and $Y \otimes Z$ given A, by Proposition 1.8 we have that f is equivalent to the product of f_X and f_{YZ}. But, again by Proposition 1.8, f_{YZ} is equivalent to the product of f_Y and f_Z, so we have that f is equivalent to the product

of all its marginals. Using Proposition 1.8 in the other direction, thie means that f exhibits conditional independence of X, Y and Z given A.

$3 \Rightarrow 4$: By the universal property of products, a cone over the cospan in (1) consists of maps $g_1^\sharp : A \to TX$, $g_{23}^\sharp : A \to T(Y \times Z)$, $g_{12}^\sharp : A \to T(X \times Y)$ and $g_3^\sharp : A \to TZ$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[drr, bend left, "(g_1^\sharp, g_{23}^\sharp)"] \arrow[ddr, bend right, "(g_{12}^\sharp, g_3^\sharp)"'] & & \\
& T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"'] \arrow[d, "c_{X,Y} \times \mathrm{id}"] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
& T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"'] & T(X \times Y \times Z)
\end{tikzcd}$$

By Lemma 1.10, this amounts to a Kleisli map $f^\sharp : A \to T(X \times Y \times Z)$ exhibiting conditional independence of X and $Y \otimes Z$ given A, as well as of $X \otimes Y$ and Z given A. By the localized independence property, we then have that f exhibits conditional independence of all X, Y and Z given A, and so, again by Lemma 1.10, $f^\sharp$ factors through the product $TX \times TY \times TZ$. More specifically, by marginalizing over Z, we have that $g_{12}^\sharp$ factors through $TX \times TY$, i.e. the following diagram on the left commutes for some $h_1^\sharp : A \to TX$ and $h_2^\sharp : A \to TY$, and similarly, by marginalizing over X, the diagram on the right commutes for some $\ell_2^\sharp : A \to TY$ and $\ell_3^\sharp : A \to TZ$.

$$\begin{tikzcd}
A \arrow[d, "(h_1^\sharp, h_2^\sharp)"'] \arrow[dr, "g_{12}^\sharp"] & \\
TX \times TY \arrow[r, "c"'] & T(X \times Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "(\ell_2^\sharp, \ell_3^\sharp)"'] \arrow[dr, "g_{23}^\sharp"] & \\
TY \times TZ \arrow[r, "c"'] & T(Y \times Z)
\end{tikzcd}$$

In other words, the upper and the left curved triangles in the following diagram commute.

$$\begin{tikzcd}
A \arrow[drr, bend left, "(g_1^\sharp, g_{23}^\sharp)"] \arrow[dr, shift left, "(g_1^\sharp, \ell_2^\sharp, \ell_3^\sharp)"] \arrow[dr, shift right, "(h_1^\sharp, h_2^\sharp, g_3^\sharp)"'] \arrow[ddr, bend right, "(g_{12}^\sharp, g_3^\sharp)"'] & & \\
& T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"'] \arrow[d, "c_{X,Y} \times \mathrm{id}"] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
& T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"'] & T(X \times Y \times Z)
\end{tikzcd}$$

By marginalizing over Y and Z, there exists a unique $a^\sharp : A \to T1$ such that $h_1 = a \cdot g_1$. Therefore

$$g_{12} = h_1 \cdot h_2 = (a \cdot g_1) \cdot h_2 = g_1 \cdot (a \cdot h_2),$$

and so in the diagram above we can equivalently replace h_1 and h_2 with g_1 and $a \cdot h_2$. Similarly by marginalizing over X and Y, there exists a unique $c^\sharp : A \to T1$ such that $\ell_3 = c \cdot g_3$, so that

$$g_{23} = \ell_2 \cdot \ell_3 = \ell_2 \cdot (c \cdot g_3) = (c \cdot \ell_2) \cdot g_3$$

and in the diagram above we can replace ℓ_2 and ℓ_3 with $c \cdot \ell_2$ and g_3, as follows.

$$
\begin{tikzcd}
A \arrow[dr, shift left, "{(g_1^\sharp,(c\cdot\ell_2)^\sharp,g_3^\sharp)}"] \arrow[dr, shift right, "{(g_1^\sharp,(a\cdot h_2)^\sharp,g_3^\sharp)}"'] \arrow[drr, bend left, "{(g_1^\sharp,g_{23}^\sharp)}"] \arrow[ddr, bend right, "{(g_{12}^\sharp,g_3^\sharp)}"'] & & \\
& T(X)\times T(Y)\times T(Z) \arrow[r, "\mathrm{id}\times c_{Y,Z}"'] \arrow[d, "c_{X,Y}\times\mathrm{id}"] & T(X)\times T(Y\times Z) \arrow[d, "c_{X,Y\times Z}"] \\
& T(X\times Y)\times T(Z) \arrow[r, "c_{X\times Y,Z}"'] & T(X\times Y\times Z)
\end{tikzcd}
$$

Now, marginalizing over Z and Z, we see that necessarily $a \cdot h_2 = c \cdot \ell_2$. Therefore there is a unique map $A \to TX \times TY \times TZ$ making the whole diagram commute, which means that (1) is a pullback.

$4 \Rightarrow 1$: If T is weakly affine, then taking $X = Y = Z = 1$ in (1) shows that this monoid must be an abelian group: we obtain a unique arrow $\iota\colon T(1) \to T(1)$ making the following diagram commute,

$$
\begin{tikzcd}
T1 \arrow[dr, "{(\mathrm{id},\iota,\mathrm{id})}"] \arrow[drr, bend left, "{(\mathrm{id},\eta_1 !)}"] \arrow[ddr, bend right, "{(\eta_1 !,\mathrm{id})}"'] & & & \\
& T1\times T1\times T1 \arrow[r, "\mathrm{id}\times c_{1,1}"] \arrow[d, "c_{1,1}\times\mathrm{id}"'] & T1\times T(1\times 1) \arrow[r, "\cong"] \arrow[d, "c_{1,1\times 1}"] & T1\times T1 \arrow[d, "c_{1,1}"] \\
& T(1\times 1)\times T1 \arrow[r, "c_{1\times 1,1}"] \arrow[d, "\cong"'] & T(1\times 1\times 1) \arrow[r, "\cong"] \arrow[d, "\cong"] & T(1\times 1) \arrow[d, "\cong"] \\
& T1\times T1 \arrow[r, "c_{1,1}"'] & T(1\times 1) \arrow[r, "\cong"'] & T1
\end{tikzcd}
$$

and the commutativity shows that ι satisfies the equations making it the inversion map for a group structure. ◀

2 Additional material (to be added to section)

▶ **Proposition 2.1.** *Let $(G, \cdot, 1)$ be a group and let X be a set. A function $\alpha : M \times X \to X$ determines a left action if and only if the square*

$$
\begin{tikzcd}
G\times G\times X \arrow[r, "\cdot\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\alpha"] & M\times X \arrow[d, "\alpha"] \\
G\times X \arrow[r, "\alpha"] & X
\end{tikzcd}
\tag{3}
$$

commutes and it is a pullback.

Proof. By definition, the square (2) commutes if and only if α and $\cdot$ are compatible. Now we show that the commutative square (2) is a pullback if and only if α satisfies the identity axiom, i.e. $\alpha(e, x) = x$ for every x in X. Now, if (2) is a pullback, then there exists a function $\beta : X \to G$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "{\langle e!,\beta,\mathrm{id}\rangle}"] \arrow[drr, bend left, "{\langle e!,\mathrm{id}\rangle}"] \arrow[ddr, bend right, "{\langle e!,\mathrm{id}\rangle}"'] & & \\
& G \times G \times X \arrow[r, "id\times\alpha"] \arrow[d, "\cdot\times id"'] & G \times X \arrow[d, "\alpha"] \\
& G \times X \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

commutes, where $e! : X \to G$ is the function assigning the identity element e to every element x of X. Now, since the left triangle commutes, then we have that $e = e \cdot \beta(x)$ for every x of X, i.e. $\beta(x) = e$ for every x of X. Now, since the right triangle commutes, we can conclude that $\alpha(\beta(x), x) = \alpha(e, x) = x$ for every x in X.

Now we show that $\alpha(e, x) = x$ implies that the commutative square (2) is a pullback. Let us consider a set Y and the functions $\langle f_1, f_2\rangle : Y \to G \times X$ and $\langle g_1, g_2\rangle : Y \to G \times X$ such that $\alpha(f_1(y), f_2(y)) = \alpha(g_1(y), g_2(y))$. By applying $\alpha(f_1(y)^{-1}, -)$ to both sides, and then combining the compatibility of α with the assumption that $\alpha(e, x) = x$, we can conclude that $f_2(y) = \alpha(f_1(y)^{-1} \cdot g_1(y), g_2(y))$. Therefore, we can conclude that the diagram

$$\begin{tikzcd}
Y \arrow[dr, "{\langle f_1,\gamma,g_2\rangle}"] \arrow[drr, bend left, "{\langle f_1,f_2\rangle}"] \arrow[ddr, bend right, "{\langle g_1,g_2\rangle}"'] & & \\
& M \times M \times X \arrow[r, "id\times\alpha"] \arrow[d, "\cdot\times id"'] & M \times X \arrow[d, "\alpha"] \\
& M \times X \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

commutes, where the function $\gamma : Y \to M$ is defined by $\gamma(y) := f_1^{-1}(y) \cdot g_1(y)$. By the unicity of the inverse in a group, this function is also unique, and hence we can conclude that the commutative square (2) is a pullback. ◀

▶ **Proposition 2.2.** *If T is weakly affine, then for every object X, the morphism $c_{1,X}$: $T(1) \times T(X) \to T(X)$ determines a (left) group action.*

Proof. The compatibility axiom follows from the fact that the diagram

$$\begin{tikzcd}
T(1) \times T(1) \times T(X) \arrow[r, "\mathrm{id}\times c_{1,X}"] \arrow[d, "c_{1,1}\times\mathrm{id}"'] & T(1) \times T(X) \arrow[d, "c_{1,X}"] \\
T(1) \times T(X) \arrow[r, "c_{1,X}"'] & T(X)
\end{tikzcd}$$

commutes for every strong and commutative monad. Moreover, following the same proof used for Proposition 2.1, we can conclude that the identity axiom is satisfied since T is weakly affine. In particular, because $T(1)$ is a group by Lemma 3.3, and the previous square is a pullbackc (by definition of weakly affine monad). ◀

▶ **Proposition 2.3.** *Let T be a weakly affine monad. If the diagram*

$$\begin{tikzcd} T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\ T(1) \arrow[r, "T(\eta_1)"'] & T^2(1) \end{tikzcd}$$

commutes, then $T^2(1) \cong T(1)$ in $\mathcal{A}$.

Proof. To prove the result it is enough to show that $T(1) \cong 1$ in the Kleisli category $\mathcal{A}_T$. We know from Lemma that $T(1)$ is a group in $\mathcal{A}$, where the arrow $\eta_1 \colon 1 \to T(1)$ is the unit of the group, and $\iota \colon T(1) \to T(1)$ is the inversion map. Therefore, we have that the composition $\iota\eta_1 \colon 1 \to T(1)$ has to be equal to η_1. Therefore, we can consider the arrows $1 \to T(1)$ and $T(1) \to 1$ in the Kleisli category $\mathcal{A}_T$ given by $T(\eta_1)\eta_1$ and ι respectively. The composition $T(\eta_1)\eta_1$ with ι in $\mathcal{A}_T$ is given by $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$. Employing the naturality of η_1 and the fact that $\iota\eta_1 = \eta_1$, it is direct to check that $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$, that is the identity $1 \to 1$ in $\mathcal{A}_T$. Now to show that the other composition gives the identity on $T(1)$ in $\mathcal{A}_T$, it is enough to show that $T(\eta_1)\iota = \eta_{T(1)}$, but this follows by hypothesis. ◀

(Paolo) Credo che $T(\eta_1)\iota \neq \eta_{T(1)}$ nell'esempio delle misure nonzero. Per ogni x in $(0, \infty) = T1$ abbiamo che $\eta_{T(1)}(x) = \delta_x$ (delta di Dirac), mentre $T\eta_1(\iota(x)) = T\eta_1(1/x) = 1/x\, \delta_1$.

▶ **Corollary 2.4.** *Let T be a weakly affine monad. If the diagram*

$$\begin{tikzcd} T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\ T(1) \arrow[r, "T(\eta_1)"'] & T^2(1) \end{tikzcd}$$

commutes, then $T(1)$ is an idempotent group, namely $\iota = \mathrm{id}_{T1}$.

Proof. By Lemma 3.3 we have that $T(1)$ is a group. If $\eta_{T1} = T(\eta_1)\iota$, then we can apply the multiplication of the monad to both sides, obtaining $\iota = \mathrm{id}_{T1}$. ◀

The following result shows that weak affinity occurs frequently. Recall that a strong monad $T \colon \mathcal{A} \to \mathcal{A}$ on a category $\mathcal{A}$ with finite products is **affine** if $T(1) \cong 1$ (see also Remark **??**). Three relevant examples of affine monads are the distribution monad on **Set** (for discrete probability), the Giry monad on the category of measurable spaces (for measure-theoretic probability, see Examples **??** and **??**), and the expectation monad, see [**?**].

▶ **Remark 2.5.** We are not aware of any relation between weakly affine monads in our sense and Jacobs' *strongly affine* monads [**?**], other than the fact that strongly affine implies affine implies weakly affine.