

# Weakly-affine monads

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T: I think that “weakly-affine” should be “weakly affine” throughout, yes?

## Abstract

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## 1 Introduction

For context:

► **Proposition 1.1.** *A monoid  $(M, m, e)$  is a group if and only if the associativity square*

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times \text{id}} & M \times M \\ \downarrow \text{id} \times m & & \downarrow m \\ M \times M & \xrightarrow{m} & M \end{array} \quad (1)$$

*is a pullback.*

This statement holds generally for a monoid object in a cartesian monoidal category, where the following elementwise proof still applies by the Yoneda lemma.

**Proof.** The square (1) is a pullback, both of sets and of groups, if and only if given  $a, g, h, c \in M$  such that  $ag = hc$ , there exists a unique  $b \in M$  such that  $g = bc$  and  $h = ab$ . First, suppose that  $G$  is a group. Then the only possible choice of  $b$  is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (1) is a pullback. We can set  $g, h = e$  and  $c = a$  so that  $ae = ea = a$ . Instantiating the pullback property on these elements gives  $b$  such that  $ab = e$  and  $ba = e$ , that is,  $b = a^{-1}$ . ◀

► **Definition 1.1.** *Let  $X$  be a set. Denote by  $MX$  the set of finitely supported measures on  $X$ , i.e. functions  $m : X \rightarrow [0, \infty)$  which are zero for all but a finite number of  $x \in X$ . Given a function  $f : X \rightarrow Y$ , denote by  $Mf : MX \rightarrow MY$  the function sending  $m \in MX$  to the assignment*

$$(Mf)(m) : y \mapsto \sum_{x \in f^{-1}(y)} p(x).$$

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This makes  $M$  into a functor, and even a monad with the following unit and multiplication maps.

$$\begin{array}{ccc} X & \xrightarrow{\delta} & MX \\ x & \longmapsto & \delta_x, \end{array} \quad \begin{array}{ccc} MMX & \xrightarrow{E} & MX \\ \xi & \longmapsto & E\xi, \end{array}$$

where

$$\delta_x(x') = \begin{cases} 1 & x = x', \\ 0 & x \neq x', \end{cases} \quad (E\xi)(x) = \sum_{m \in MX} \xi(m) m(x).$$

Call  $M$  the measure monad on **Set**.

Denote also by  $DX \subseteq MX$  the subset of probability measures, i.e. those finitely supported  $p : X \rightarrow [0, \infty)$  such that

$$\sum_{x \in X} p(x) = 1.$$

$D$  forms a submonad of  $M$  called the distribution monad.

### 1.1 GS-monoidal and Markov categories

The notion of *gs-monoidal category* has been originally introduced in the context of algebraic approaches to term graph rewriting [3], and then developed in a series of papers [4, 6, 5]. We recall here the basic definitions adopting the graphical formalism of string diagrams, referring to [13] for background on various notions of monoidal categories and their associated diagrammatic calculus.

T: The text has contained both terms “gs-monoidal category” and “GS-category”, and it would probably be better to stick with one of these two. I’ve opted for the former now since that’s what it’s in the literature, but “GS category” with or without dash seems fine too (and perhaps the people who prefer to say “CD category” could be convinced to say “GS category” instead)

► **Definition 1.2.** A *gs-monoidal category* is a symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  with a commutative comonoid structure on each object  $X$ , consisting of a comultiplication and a counit,

$$\text{copy}_X = \begin{array}{c} \text{---} \text{---} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array} \quad \text{del}_X = \begin{array}{c} \bullet \\ | \\ X \end{array}$$

which satisfy the commutative comonoid equations:

$$\begin{array}{c} \text{---} \text{---} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} \text{---} \text{---} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} | \\ X \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} \text{---} \text{---} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array}$$

These comonoid structures must be multiplicative with respect to the monoidal structure:

$$\begin{array}{ccc}
\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ X \otimes Y \end{array} & = & \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \bullet \quad \bullet \\ \text{---} \text{---} \\ X \quad Y \end{array} \\
\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ I \end{array} & = & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ I \end{array}
\end{array}$$

► **Definition 1.3.** A morphism  $f : X \rightarrow Y$  in a gs-monoidal category is called *copyable* or *functional* if and only if

$$\begin{array}{ccc}
\begin{array}{c} Y \quad Y \\ \text{---} \text{---} \\ \bullet \\ \text{---} \\ \boxed{f} \\ \text{---} \\ X \end{array} & = & \begin{array}{c} Y \quad Y \\ \text{---} \text{---} \\ \boxed{f} \quad \boxed{f} \\ \text{---} \text{---} \\ \bullet \\ \text{---} \\ X \end{array}
\end{array}$$

It is called *discardable* or *full* if

$$\begin{array}{ccc}
\begin{array}{c} \bullet \\ \text{---} \\ \boxed{f} \\ \text{---} \\ X \end{array} & = & \begin{array}{c} \bullet \\ \text{---} \\ X \end{array}
\end{array}$$

► **Example 1.4.** The category **Rel** of sets and relations with the monoidal operation  $\otimes : \mathbf{Rel} \times \mathbf{Rel} \rightarrow \mathbf{Rel}$  the given by the direct product of sets is a gs-monoidal category [6]. In this gs-monoidal category, the copyable arrows are precisely the partial functions, and the discardable arrows are the total relations.

► **Remark 1.5.** It is well-known that if duplicators and dischargers of a given gs-monoidal category  $\mathcal{C}$  satisfy naturality, then the monoidal product is the categorical product, and thus the category is cartesian monoidal [7], i.e. the following conditions are equivalent for a gs-monoidal category  $\mathcal{C}$ :

- $\mathcal{C}$  is cartesian monoidal;
- every morphism is copyable and discardable;
- the copy and discard maps are natural.

In recent works [?] it has been proved that gs-monoidal structure naturally arises in several situations, such as Kleisli categories of commutative monads or span categories. In the following proposition, we recall the result regarding Kleisli categories:

► **Proposition 1.6.** Let  $T$  be a symmetric monoidal (equivalently, commutative) monad on a cartesian monoidal category  $\mathcal{D}$ . Then  $\mathbf{Kl}_T$  is canonically a gs-monoidal category with copy and discard structure induced by that of  $\mathcal{D}$ .

► **Definition 1.7.** A gs-monoidal category is called Markov if any (hence all) of the following equivalent conditions are satisfied:

- The monoidal unit is terminal;
- The discard map is natural;
- Every morphism is discardable.

88 ► **Definition 1.8.** A monad  $T$  on a cartesian monoidal category is called *affine* if  $T1 \cong 1$ .

89 ► **Proposition 1.9.** In the hypotheses of Proposition 1.6, we have that  $\text{Kl}_T$  is Markov if and  
90 only if  $T$  is affine.

91 In this work, we introduce an intermediate level between gs-monoidal and Markov called  
92 *weakly Markov*, and its corresponding notion for monads, which we call *weakly affine*.

## 93 2 Weakly Markov categories and weakly affine monads

### 94 2.1 The monoid of effects

95 In a gs-monoidal category  $\mathcal{C}$  we call a *state* a morphism from the monoidal unit  $p : I \rightarrow X$ ,  
96 and *effect* or *co-state* a morphism to the monoidal unit  $a : X \rightarrow I$ . We represent them as  
97 triangles as follows.



99 Effects, i.e. elements of the set  $\mathcal{C}(X, I)$ , form canonically a commutative monoid as follows:  
100 the monoidal unit is the discard map  $X \rightarrow I$ , and given  $a, b : X \rightarrow I$ , their product  $ab$  is  
101 given by copying:<sup>1</sup>



103 If a morphism  $f : X \rightarrow Y$  is copyable and discardable, precomposition with  $f$  induces a  
104 morphism of monoids  $\mathcal{C}(Y, I) \rightarrow \mathcal{C}(X, I)$ .

105 Let's now consider the case where the gs-monoidal structure comes from a commutative  
106 monad on a cartesian monoidal category  $\mathcal{D}$ . In this case, the monoid structure of Kleisli  
107 morphisms  $X \rightarrow 1$  comes from the following canonical internal monoid structure of  $T1$  in  $\mathcal{D}$ ,  
108 given by [12, Section 10]

109 
$$1 \xrightarrow{\eta} T1, \quad T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

110 For example, for the monad of measures  $M$ , we obtain  $M1 = [0, \infty)$  with its usual multiplic-  
111 ation.

112 The monoid structure of Kleisli morphisms  $X \rightarrow 1$  is now given as follows. The unit is  
113 given by

114 
$$X \xrightarrow{!} 1 \xrightarrow{\eta} T1,$$

115 and the multiplication of the morphisms  $f^\sharp, g^\sharp : X \rightarrow T1$  is

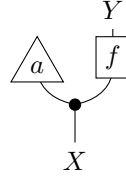
116 
$$X \xrightarrow{\text{copy}} X \times X \xrightarrow{f^\sharp \times g^\sharp} T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

---

<sup>1</sup> See e.g. also the  $\odot$  product in [2, Proposition 3.10].

117 For the monad of measures  $M$ , Kleisli morphisms  $X \rightarrow 1$  are functions  $X \rightarrow [0, \infty)$ , and  
 118 their monoidal structure is their pointwise product.

119 Note that the commutative monoid  $\mathcal{C}(X, I)$  acts on the set  $\mathcal{C}(X, Y)$ : given  $a : X \rightarrow I$  and  
 120  $f : X \rightarrow Y$ , the resulting  $a \cdot f$  is given as follows,



122 and the product  $(f, g) \mapsto f \cdot g := (f \otimes g) \circ \text{copy}_X$  is equivariant for this action in both  
 123 variables (separately). For the monad of measures  $M$ , this action amounts to a pointwise  
 124 rescaling.

## 125 2.2 Main definitions

126 ► **Definition 2.1.** A gs-monoidal category  $\mathcal{C}$  is called *weakly Markov* if for every object  $X$ ,  
 127 the monoid  $\mathcal{C}(X, I)$  is a group.

128 Every Markov category is weakly Markov: for each  $X$ , the monoid  $\mathcal{C}(X, I)$  is the trivial  
 129 group.

130 T: Instead of saying “weakly Markov GS-category”, perhaps we can just say “weakly Markov category”?

131 ► **Definition 2.2.** Given two parallel morphisms  $f, g : X \rightarrow Y$  in a weakly Markov category  
 132  $\mathcal{C}$ , we say that  $f$  and  $g$  are:

- 133 1. *equivalent*, denoted  $f \sim g$ , if they lie in the same orbit for the action of  $\mathcal{C}(X, I)$ , i.e. if  
 134 there is  $a \in \mathcal{C}(X, I)$  such that  $a \cdot f = g$ .
- 135 2. *uniquely equivalent* if there is a unique  $a \in \mathcal{C}(X, I)$  such that  $a \cdot f = g$ .

136 T: Isn't the equivalence always unique? The action of  $\mathcal{C}(X, I)$  on  $\mathcal{C}(X, Y)$  is free, as one can easily see by discarding  $Y$ . That's basically also the argument used in Proposition 3.5 below

137 T: Is it worth noting that the equivalence classes form a Markov cat, which is isomorphic to the Markov cat of discardable morphisms?

138 Let's now look at the Kleisli case.

139 ► **Definition 2.3.** A commutative monad  $T$  on a cartesian monoidal category is called *weakly affine* if  $T1$  with its canonical internal monoid structure is a group.

141 This choice of terminology is motivated by the following proposition, which can be seen  
 142 as a “weakly” version of Proposition 1.9.

143 ► **Proposition 2.4.** Let  $\mathcal{D}$  be a cartesian monoidal category, and let  $T$  be a commutative  
 144 monad on  $\mathcal{D}$ . Then the Kleisli category of  $T$  is weakly Markov if and only if  $T$  is weakly  
 145 affine.

146 **Proof.** First, suppose that  $T1$  is an internal group, and denote by  $\iota : T1 \rightarrow T1$  its inversion  
 147 map. The inverse of a Kleisli morphism  $a : X \rightarrow 1$  in  $\text{Kl}_T(X, 1)$  represented by  $a^\sharp : X \rightarrow T1$

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is represented by  $\iota \circ a^\sharp$ : indeed, the following diagram in  $\mathcal{D}$  commutes,

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & X & \xrightarrow{\text{copy}} & X \times X & & & \\
 & \downarrow a^\sharp & & \downarrow a^\sharp \times a^\sharp & \searrow a^\sharp \times (\iota \circ a^\sharp) & & \\
 149 & T1 & \xrightarrow{\text{copy}} & T1 \times T1 & \xrightarrow{\text{id} \times \iota} & T1 \times T1 & \xrightarrow{c} T(1 \times 1) \\
 & \downarrow ! & & & & & \downarrow \cong \\
 & 1 & \xrightarrow{\eta} & & & & T1
 \end{array}
 \end{array}$$

where the bottom rectangle commutes since  $\iota$  is the inversion map for  $T1$ . The analogous diagram with  $\iota \times \text{id}$  in place of  $\text{id} \times \iota$  commutes analogously.

Conversely, suppose that for every  $X$ , the monoid structure on  $\text{Kl}_T(X, 1)$  has inverses. Then in particular we can take  $X = T1$ , and the inverse of the Kleisli morphism  $\text{id} : T1 \rightarrow T1$  is an inversion map for  $T1$ . ◀

This result can also be thought of in terms of the Yoneda embedding, see the details in Appendix A.

### 2.3 Examples of weakly affine monads

Every affine monad is a weakly affine monad. Here are less trivial examples.

► **Example 2.5.** Let  $M^* : \mathbf{Set} \rightarrow \mathbf{Set}$  be the monad assigning to every set the set of finitely supported discrete *nonzero* measures on  $M^*$ , or equivalently let  $M^*(X)$  for any set  $X$  be the set of nonzero finitely supported functions  $X \rightarrow [0, \infty)$ . The monad structure is defined in terms of the same formulas as for the monad of measures  $M$  (Definition 1.1) and the components  $c_{X,Y}$  are also given by the formation of product measures, or equivalently pointwise products of functions  $X \rightarrow [0, \infty)$ .

Since  $M^*1 \cong (0, \infty) \not\cong 1$ , this monad is not affine. However the monoid structure of  $(0, \infty)$  induced by  $M^*$  is the usual multiplication of positive real numbers, which form a group. Therefore  $M^*$  is weakly affine, and its Kleisli category is weakly Markov.

T: More generally, we could consider nonzero measures with values in any positive semi-field, see the corresponding monads considered in arXiv:2108.10718. Not sure though if it's interesting enough to mention?

On the other hand, if the zero measure is included, we have  $M1 \cong [0, \infty)$  which is not a group under multiplication, so  $M$  is not weakly affine.

► **Example 2.6.** Let  $A$  be a commutative monoid. Then the functor  $T_A := A \times -$  on  $\mathbf{Set}$  has a canonical structure of commutative monad, where the lax structure components  $c_{X,Y}$  are given by multiplying elements in  $A$  while carrying the elements of  $X$  and  $Y$  along.

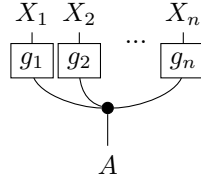
Since  $T_A(1) \cong A$ , the monad  $T_A$  is weakly affine if and only if  $A$  is a group, and affine if and only if  $A \cong 1$ .

► **Example 2.7.** Here is a negative example. Consider the free abelian group monad  $F$  on  $\mathbf{Set}$ . Its functor takes a set  $X$  and forms the set  $FX$  of finite multisets (with repetition, where order does not matter) of elements of  $X$  and their formal inverses. We have that  $F1 \cong \mathbb{Z}$ , which is an abelian group under addition. However, the monoid structure on  $F1$  induced by the monoidal structure of the monad corresponds to the *multiplication* in  $\mathbb{Z}$ , which does not have inverses. Therefore  $F$  is not weakly affine.

### 3 Conditional independence in weakly Markov categories

Markov categories have a rich theory of conditional dependence and independence [10]. Some of those ideas can be translated and generalized to the setting of weakly Markov categories.

► **Definition 3.1.** A morphism  $f : A \rightarrow X_1 \otimes \cdots \otimes X_n$  in a gs-monoidal category  $\mathcal{C}$  is said to exhibit *conditional independence of the  $X_i$  given  $A$*  if and only if it can be expressed as a product of the following form.



Note that this is slightly different from [1, Definition 6.6], although it is equivalent for the case of Markov categories.

How is it different? It looks to me like they consider it only for conditionals of states and require that equation to hold only with almost sure equality, but that means that it's slightly different already for Markov cats

Here is what conditional independence looks like in the Kleisli case.

► **Proposition 3.2.** Let  $\mathcal{D}$  be a cartesian monoidal category, and let  $T$  be a commutative monad on  $\mathcal{D}$ . A Kleisli morphism represented by  $f^\sharp : A \rightarrow T(X_1 \times \cdots \times X_n)$  exhibits conditional independence of the  $X_i$  given  $A$  if and only if it factors as follows

$$\begin{array}{ccc} A & \xrightarrow{f^\sharp} & T(X_1 \times \cdots \times X_n) \\ (g_1^\sharp, \dots, g_n^\sharp) \downarrow & & \uparrow c \\ TX_1 \times \cdots \times TX_n & \xrightarrow{c} & T(X_1 \times \cdots \times X_n) \end{array}$$

for some Kleisli maps  $g_i^\sharp : A \rightarrow TX_i$ , where the map  $c$  above is the one obtained by iterating the multiplication of the monoidal structure (such a map is unique by associativity).

**Proof.** In terms of the base category  $\mathcal{D}$ , a Kleisli morphism in the form of Definition 3.1 reads as follows.

$$A \xrightarrow{\text{copy}} A \times \cdots \times A \xrightarrow{g_1^\sharp \times \cdots \times g_n^\sharp} TX_1 \times \cdots \times TX_n \xrightarrow{c} T(X_1 \times \cdots \times X_n).$$

Therefore  $f^\sharp : A \rightarrow T(X_1 \times \cdots \times X_n)$  exhibits the conditional independence if and only if it is of the form above. ◀

► **Example 3.3.** In the Kleisli category of the distribution monad  $D$ , which is Markov, a morphism  $f : A \rightarrow X \otimes Y$  exhibits conditional independence if and only if it is the product of its marginals [8, Section 12].

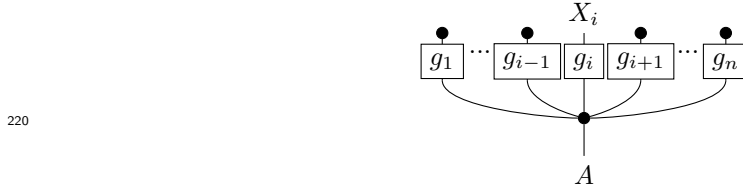
► **Example 3.4.** In the Kleisli category of the measure monad  $M$ , the zero measure always displays conditional independence of its outputs given its inputs: for example, for  $A = 1$ , the zero measure on  $X \times Y$  is the product of the zero measure on  $X$  and the zero (or any other) measure on  $Y$ . Notice that both marginals of the zero measure are zero measures—therefore, the factors appearing in the product are not necessarily related to the marginals.

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212 In a weakly Markov category, the situation is similar to the Markov case, but up to  
213 equivalence.

214 ► **Proposition 3.5.** *Let  $f : A \rightarrow X_1 \otimes \cdots \otimes X_n$  be a morphism in a weakly Markov category*  
215  *$\mathcal{C}$ . Then  $f$  exhibits conditional independence of the  $X_i$  given  $A$  if and only if it is equivalent*  
216 *to the product of all its marginals. Moreover, in that case  $f$  is uniquely equivalent to the*  
217 *product of its marginals.*

218 **Proof.** Denote the marginals of  $f$  by  $f_1, \dots, f_n$ . Suppose that  $f$  is a product as in Defini-  
219 tion 3.1. For each  $i = 1, \dots, n$ , by marginalizing, we get that  $f_i$  is equal to the following.

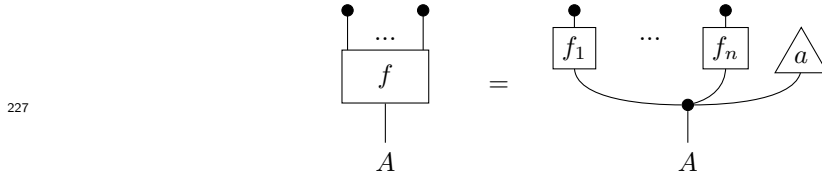


221 Therefore for each  $i$  we have that  $f_i \sim g_i$ .

222 Conversely, suppose that  $f$  is equivalent to the product of its marginals, i.e. that there  
223 exists  $a : X \rightarrow I$  such that  $f$  is equal to the following.

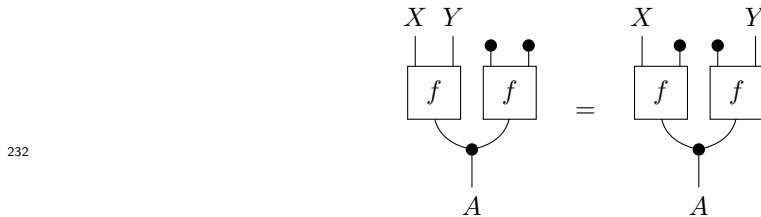


225 One can then choose  $g_i = f_i$  for all  $i < n$ , and  $g_n = a \cdot f_n$ , so that  $f$  is in the form of  
226 Definition 3.1. Moreover, by marginalizing over all the  $X_i$  at once, we see that



228 so that  $a$  is uniquely determined. ◀

229 ► **Remark 3.6.** For  $n = 2$ , a morphism  $f : A \rightarrow X \otimes Y$  in a weakly Markov category  $\mathcal{C}$   
230 exhibits conditional independence of  $X$  and  $Y$  given  $A$  if and only if the following equation  
231 holds.





### 3.1 Main result

The concept of conditional independence for general weakly Markov categories allows us to give an equivalent characterization of weakly affine monads. The condition is in terms of a pullback condition on the associativity diagram, and can be seen as a generalization of Proposition 1.1.

► **Theorem 3.7.** *Let  $\mathcal{D}$  be a cartesian monoidal category, and let  $T$  be a commutative monad on  $\mathcal{D}$ . Then the following conditions are equivalent.*

1.  *$T$  is weakly affine;*
2. *The Kleisli category  $\text{Kl}_T$  is weakly Markov;*
3. *For all objects  $X, Y$ , and  $Z$ , the following associativity diagram is a pullback.*

$$\begin{array}{ccc}
 T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\
 c_{X,Y} \times \text{id} \downarrow & & \downarrow c_{X,Y \times Z} \\
 T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z)
 \end{array} \quad (2)$$

We prove the theorem by means of the following property of weakly Markov categories.

► **Lemma 3.8** (localized independence property). *Let  $\mathcal{C}$  be a weakly Markov category. Whenever a morphism  $f : A \rightarrow X \otimes Y \otimes Z$  exhibits conditional independence of  $X \otimes Y$  (jointly) and  $Z$  given  $A$ , as well as conditional independence of  $X$  and  $Y \otimes Z$  given  $A$ , then it exhibits conditional independence of  $X, Y$  and  $Z$  given  $A$ .*

**Proof of Lemma 3.8.** Suppose  $f : A \rightarrow X \otimes Y \otimes Z$  exhibits conditional independence of  $X \otimes Y$  (jointly) and  $Z$  given  $A$ , as well as conditional independence of  $X$  and  $Y \otimes Z$  given  $A$ . By marginalizing out  $X$ , we have that  $f_{YZ}$  exhibits conditional independence of  $Y$  and  $Z$  given  $A$ . Since by hypothesis  $f$  exhibits conditional independence of  $X$  and  $Y \otimes Z$  given  $A$ , by Proposition 3.5 we have that  $f$  is equivalent to the product of  $f_X$  and  $f_{YZ}$ . But, again by Proposition 3.5,  $f_{YZ}$  is equivalent to the product of  $f_Y$  and  $f_Z$ , so we have that  $f$  is equivalent to the product of all its marginals. Using Proposition 3.5 in the other direction, this means that  $f$  exhibits conditional independence of  $X, Y$  and  $Z$  given  $A$ . ◀

We are now ready to prove the theorem.

**Proof of Theorem 3.7.**  $1 \Leftrightarrow 2$ : see Proposition 2.4.

$1 \Rightarrow 3$ : By the universal property of products, a cone over the cospan in (2) consists of maps  $g_1^\sharp : A \rightarrow TX$ ,  $g_{23}^\sharp : A \rightarrow T(Y \times Z)$ ,  $g_{12}^\sharp : A \rightarrow T(X \times Y)$  and  $g_3^\sharp : A \rightarrow TZ$  such that the following diagram commutes.

$$\begin{array}{ccc}
 A & \xrightarrow{(g_1^\sharp, g_{23}^\sharp)} & T(X) \times T(Y \times Z) \\
 (g_{12}^\sharp, g_3^\sharp) \searrow & & \downarrow c_{X,Y} \times \text{id} \quad \downarrow c_{X,Y \times Z} \\
 T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z)
 \end{array}$$

By Proposition 3.2, this amounts to a Kleisli map  $f^\sharp : A \rightarrow T(X \times Y \times Z)$  exhibiting conditional independence of  $X$  and  $Y \otimes Z$  given  $A$ , as well as of  $X \otimes Y$  and  $Z$  given  $A$ . By the localized independence property (Lemma 3.8), we then have that  $f$  exhibits conditional

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independence of all  $X, Y$  and  $Z$  given  $A$ , and so, again by Proposition 3.2,  $f^\#$  factors through the product  $TX \times TY \times TZ$ . More specifically, by marginalizing over  $Z$ , we have that  $g_{12}^\#$  factors through  $TX \times TY$ , i.e. the following diagram on the left commutes for some  $h_1^\# : A \rightarrow TX$  and  $h_2^\# : A \rightarrow TY$ , and similarly, by marginalizing over  $X$ , the diagram on the right commutes for some  $\ell_2^\# : A \rightarrow TY$  and  $\ell_3^\# : A \rightarrow TZ$ .

$$\begin{array}{ccc} A & \xrightarrow{g_{12}^\#} & TX \times TY \\ (h_1^\#, h_2^\#) \downarrow & \searrow & \downarrow c \\ TX \times TY & \xrightarrow{c} & T(X \times Y) \end{array} \quad \begin{array}{ccc} A & \xrightarrow{g_{23}^\#} & TY \times TZ \\ (\ell_2^\#, \ell_3^\#) \downarrow & \searrow & \downarrow c \\ TY \times TZ & \xrightarrow{c} & T(Y \times Z) \end{array}$$

In other words, the upper and the left curved triangles in the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{(g_1^\#, g_{23}^\#)} & T(X) \times T(Y \times Z) \\ \downarrow (g_1^\#, \ell_2^\#, \ell_3^\#) & \searrow & \downarrow c_{X, Y \times Z} \\ A & \xrightarrow{(h_1^\#, h_2^\#, g_3^\#)} & T(X) \times T(Y) \times T(Z) \xrightarrow{\text{id} \times c_{Y, Z}} T(X) \times T(Y \times Z) \\ \downarrow (h_1^\#, h_2^\#, g_3^\#) & \searrow & \downarrow c_{X, Y \times Z} \\ A & \xrightarrow{(g_{12}^\#, g_3^\#)} & T(X \times Y) \times T(Z) \xrightarrow{c_{X \times Y, Z}} T(X \times Y \times Z) \end{array}$$

By marginalizing over  $Y$  and  $Z$ , and by weak affinity of  $T$ , there exists a unique  $a^\# : A \rightarrow T1$  such that  $h_1 = a \cdot g_1$ . Therefore

$$g_{12} = h_1 \cdot h_2 = (a \cdot g_1) \cdot h_2 = g_1 \cdot (a \cdot h_2),$$

and so in the diagram above we can equivalently replace  $h_1$  and  $h_2$  with  $g_1$  and  $a \cdot h_2$ . Similarly by marginalizing over  $X$  and  $Y$ , there exists a unique  $c^\# : A \rightarrow T1$  such that  $\ell_3 = c \cdot g_3$ , so that

$$g_{23} = \ell_2 \cdot \ell_3 = \ell_2 \cdot (c \cdot g_3) = (c \cdot \ell_2) \cdot g_3$$

and in the diagram above we can replace  $\ell_2$  and  $\ell_3$  with  $c \cdot \ell_2$  and  $g_3$ , as follows.

$$\begin{array}{ccc} A & \xrightarrow{(g_1^\#, g_{23}^\#)} & T(X) \times T(Y \times Z) \\ \downarrow (g_1^\#, (c \cdot \ell_2)^\#, g_3^\#) & \searrow & \downarrow c_{X, Y \times Z} \\ A & \xrightarrow{(g_1^\#, (a \cdot h_2)^\#, g_3^\#)} & T(X) \times T(Y) \times T(Z) \xrightarrow{\text{id} \times c_{Y, Z}} T(X) \times T(Y \times Z) \\ \downarrow (g_1^\#, (a \cdot h_2)^\#, g_3^\#) & \searrow & \downarrow c_{X, Y \times Z} \\ A & \xrightarrow{(g_{12}^\#, g_3^\#)} & T(X \times Y) \times T(Z) \xrightarrow{c_{X \times Y, Z}} T(X \times Y \times Z) \end{array}$$

Now, marginalizing over  $X$  and  $Z$ , we see that necessarily  $a \cdot h_2 = c \cdot \ell_2$ . Therefore there is a unique map  $A \rightarrow TX \times TY \times TZ$  making the whole diagram commute, which means that (2) is a pullback.

3  $\Rightarrow$  1: If  $T$  is weakly affine, then taking  $X = Y = Z = 1$  in (2) shows that this monoid must be an abelian group: we obtain a unique arrow  $\iota : T(1) \rightarrow T(1)$  making the following

288 diagram commute,

$$\begin{array}{ccccc}
 T1 & & \xrightarrow{(id, \eta_1!)} & & \\
 & \searrow (id, \iota, id) & & & \\
 & T1 \times T1 \times T1 & \xrightarrow{id \times c_{1,1}} & T1 \times T(1 \times 1) & \xrightarrow{\cong} T1 \times T1 \\
 & \downarrow c_{1,1} \times id & & \downarrow c_{1,1} \times 1 & \downarrow c_{1,1} \\
 & T(1 \times 1) \times T1 & \xrightarrow{c_{1 \times 1, 1}} & T(1 \times 1 \times 1) & \xrightarrow{\cong} T(1 \times 1) \\
 & \downarrow \cong & & \downarrow \cong & \downarrow \cong \\
 & T1 \times T1 & \xrightarrow{c_{1,1}} & T(1 \times 1) & \xrightarrow{\cong} T1
 \end{array}$$

290 and the commutativity shows that  $\iota$  satisfies the equations making it the inversion map for a  
 291 group structure. ◀

292 ▶ **Example 3.9.** In the Kleisli category of the measure monad  $Kl_M$  (which is not weakly  
 293 affine) consider the following diagram.

$$\begin{array}{ccc}
 MX \times MY \times MZ & \xrightarrow{id \times c_{Y,Z}} & MX \times M(Y \times Z) \\
 \downarrow c_{X,Y} \times id & & \downarrow c_{X,Y \times Z} \\
 M(X \times Y) \times MZ & \xrightarrow{c_{X \times Y, Z}} & M(X \times Y \times Z)
 \end{array}$$

295 In the top-right corner  $MX \times M(Y \times Z)$ , take the pair  $(0, p)$  where  $p$  is a nonzero measure  
 296 on  $Y \times Z$ , and similarly, in the bottom-left corner take the pair  $(q, 0)$  where  $q$  is a nonzero  
 297 measure on  $X \times Y$ . Following the diagram, both pairs are mapped to the zero measure in  
 298 the bottom-right corner. If the diagram was a pullback, we would be able to express the  
 299 top-right and bottom-left corners as coming from the same triple in  $MX \times MY \times MZ$ , that  
 300 is, there would exist a measure  $m$  on  $Y$  such that  $m \cdot 0 = p$  and  $0 \cdot m = q$ . Since  $p$  and  $q$  are  
 301 nonzero, this is not possible.

## 302 4 Further results

303 ▶ **Proposition 4.1.** *Let  $T$  be a weakly affine monad. If the diagram*

$$\begin{array}{ccc}
 T(1) & \xrightarrow{id} & T(1) \\
 \downarrow \iota & & \downarrow \eta_{T1} \\
 T(1) & \xrightarrow{T(\eta_1)} & T^2(1)
 \end{array}$$

305 commutes, then:

- 306 1.  $T^2(1) \cong T(1)$  in  $\mathcal{D}$ .
- 307 2.  $T(1)$  is an idempotent group, namely  $\iota = id_{T1}$ .

308 *T: Idempotency usually means that  $xx = x$  for every group/monoid element  $x$ , right? The correct term here would be exponent 2, see [math.stackexchange.com/questions/1228513/what-is-the-exponent-of-a-group](https://math.stackexchange.com/questions/1228513/what-is-the-exponent-of-a-group)*

309 *T: Having a nontrivial example of this statement would help to motivate and illustrate it. Like this, its meaning and significance remains quite unclear*

**Proof.** To prove the first claim, it is enough to show that  $T(1) \cong 1$  in the Kleisli category  $\mathbf{Kl}_T$ . By weak affinity,  $T(1)$  is a group in  $\mathcal{D}$ , where the arrow  $\eta_1: 1 \rightarrow T(1)$  is the unit of the group and  $\iota: T(1) \rightarrow T(1)$  is the inversion map. Therefore, we have that the composition  $\iota\eta_1: 1 \rightarrow T(1)$  has to be equal to  $\eta_1$ . Hence we can consider the arrows  $1 \rightarrow T(1)$  and  $T(1) \rightarrow 1$  in the Kleisli category  $\mathbf{Kl}_T$  represented by  $T(\eta_1)\eta_1$  and  $\iota$ , respectively. The composition  $T(\eta_1)\eta_1$  with  $\iota$  in  $\mathbf{Kl}_T$  is given by  $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$ . Employing the naturality of  $\eta_1$  and the fact that  $\iota\eta_1 = \eta_1$ , it is direct to check that  $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$ , that is the identity  $1 \rightarrow 1$  in  $\mathbf{Kl}_T$ . Now to show that the other composition gives the identity on  $T(1)$  in  $\mathbf{Kl}_T$ , it is enough to show that  $T(\eta_1)\iota = \eta_{T(1)}$ , but this follows by hypothesis.

For the second claim, we can compose the diagram with the monad multiplication, obtaining  $\iota = \text{id}_{T1}$ . ◀

► **Remark 4.2.** Bart Jacobs calls a strong monad  $T$  on a cartesian monoidal category *strongly affine* [11] if for every pair of objects  $X$  and  $Y$ , the following diagram is a pullback,

$$\begin{array}{ccc} X \times TY & \xrightarrow{s} & T(X \times Y) \\ \downarrow \pi_1 & & \downarrow T\pi_1 \\ X & \xrightarrow{\eta} & TX \end{array}$$

where  $s$  denotes the strength and  $\eta$  denotes the unit of the monad. Every strongly affine monad is affine. The corresponding condition on the (Markov) category  $\mathbf{Kl}_T$  is called *positivity* [9, Section 2].

Note that for a generic commutative monad, the diagram above may even fail to commute (take for example the measure monad  $M$ , and start with  $(x, 0)$  in the top left corner). One can however consider the following diagram, which reduces to the one above (up to isomorphism) in the affine case,

$$\begin{array}{ccc} X \times TY & \xrightarrow{s} & T(X \times Y) \\ \downarrow \text{id} \times T! & & \downarrow T(\text{id} \times !) \\ X \times T1 & \xrightarrow{s} & T(X \times 1) \cong TX \end{array}$$

and which always commutes by naturality of the strength.

T: Oh yes! Now Bart's diagram makes a lot more sense

One can then call the monad  $T$  *positive* if this second diagram is a pullback (and possibly define *positive gs-monoidal categories* analogously to positive Markov categories). All the examples of weakly affine monads that we have are positive in this sense, so one may wonder if every weakly affine monad is positive. For now, this remains an open question.

T: Isn't the answer clearly negative since we have affine monads that are not strongly affine?

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## 369 A Yoneda embedding interpretation of Proposition 2.4

370 We can interpret Proposition 2.4 more abstractly in terms of presheaves. Let  $\mathcal{D}$  be a  
 371 cartesian monoidal category. Consider the presheaf category  $[\mathcal{D}^{\text{op}}, \mathbf{Set}]$ , equipped with the  
 372 Day convolution product,

$$373 \quad F \boxtimes G \cong \int^{A, B \in \mathcal{D}} \mathcal{D}(-, A \times B) \times F(A) \times G(B).$$

374 The Yoneda embedding  $\mathcal{D} \rightarrow [\mathcal{D}^{\text{op}}, \mathbf{Set}]$  is strong monoidal: indeed, for each  $X$ ,

$$375 \quad 1 \cong \mathcal{D}(X, 1),$$

376 since 1 is terminal, and for each  $X$  and  $Y$ , by Yoneda reduction,

$$377 \quad \mathcal{D}(-, X) \boxtimes \mathcal{D}(-, Y) \cong \int^{A, B \in \mathcal{D}} \mathcal{D}(-, A \times B) \times \mathcal{D}(-, X) \times \mathcal{D}(-, Y) \\ 378 \quad \cong \mathcal{D}(-, X \times Y). \\ 379$$

380 Therefore, and by the universal property of products, at the level of individual hom-sets the  
 381 Day convolution product of representable presheaves just takes the cartesian products of  
 382 sets:

$$383 \quad (\mathcal{D}(-, X) \boxtimes \mathcal{D}(-, Y))(A) \cong \mathcal{D}(A, X \times Y) \cong \mathcal{D}(A, X) \times \mathcal{D}(A, Y).$$

384 Take now an object  $M$  of  $\mathcal{D}$ . Since the Yoneda embedding is fully faithful and strong  
 385 monoidal, a monoid structure  $(M, m, e)$  on  $M$  is equivalently a monoid structure on the

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representable presheaf  $\mathcal{D}(-, M)$ . This makes the individual hom-sets monoids, with unit and multiplication as follows for each object  $X$ :

$$\begin{aligned} 1 &\xrightarrow{\cong} \mathcal{D}(X, 1) \xrightarrow{e_*} \mathcal{D}(X, M) \\ \mathcal{D}(X, M) \times \mathcal{D}(X, M) &\xrightarrow{\cong} \mathcal{D}(X, M \times M) \xrightarrow{m_*} \mathcal{D}(X, M) \end{aligned}$$

T: Using this doesn't require Day convolution though, so perhaps we can get rid of that to simplify?

This is precisely the monoid structure that we have defined in Section 2.1 for  $M = T1$ .

► **Proposition A.1.**  *$M$  is an internal group if and only if all the monoids  $\mathcal{D}(X, M)$  are groups.*

**Proof.** By Proposition 1.1,  $M$  is a group object if and only if its associativity square (1) is a pullback. Since the hom-functor preserves and reflects all limits in its second argument, we have that (1) is a pullback if and only if for each object  $X$ , the following diagram (or equivalently, its bottom right square) is a pullback,

$$\begin{array}{ccccc} \mathcal{D}(X, M) \times \mathcal{D}(X, M) \times \mathcal{D}(X, M) & \xrightarrow{\quad\quad\quad} & \mathcal{D}(X, M) \times \mathcal{D}(X, M) & & \\ \downarrow & \searrow \cong & \downarrow \cong & & \\ & \mathcal{D}(X, M \times M \times M) & \xrightarrow{(m \times \text{id})_*} & \mathcal{D}(X, M \times M) & \\ & \downarrow (\text{id} \times m)_* & & \downarrow m_* & \\ \mathcal{D}(X, M) \times \mathcal{D}(X, M) & \xrightarrow{\cong} & \mathcal{D}(X, M \times M) & \xrightarrow{m_*} & \mathcal{D}(X, M) \end{array}$$

where the unlabelled arrows are the unique ones that make the diagram commute. Again by Proposition 1.1, the diagram above is a pullback if and only if  $\mathcal{D}(X, M)$  is a group. ◀