Lax completeness for gs-monoidal categories

***Abstract*—Originally introduced in the context of the algebraic approach to term graph rewriting, the notion of gs-monoidal category has surfaced a few times under different monikers in the last decades. They can be thought of as symmetric monoidal categories whose arrows are generalised relations, with enough structure to facilitate talk of domains and partial functions, but less structure than that required of cartesian bicategories. The aim of this paper is threefold. The first goal is to extend the original definition of gs-monoidality in order to enrich it with a preorder on arrows, ending up with what we call oplax cartesian categories. Second, we show that (preorder-enriched) gs-monoidal categories naturally arise both as Kleisli categories and as span categories, and the relation between the resulting formalisms is thoroughly explored, resulting in the introduction of the novel concept of weakly affine monad. Finally, we present two theorems concerning Yoneda embeddings on the one hand and functorial completeness on the one other, the latter inducing a completeness result also for lax functors from oplax cartesian categories to Rel.**

1. Introduction

The notion of *gs-monoidal category* was originally introduced in the context of the algebraic approach to term graph rewriting [1], and their study was pursued in a series of papers (see [2], [3] among others), including their use for the functorial semantics of relational and partial algebras [4].

Briefly, gs-monoidal categories are symmetric monoidal categories equipped with two families of arrows, a duplicator $\nabla_A : A \to A \otimes A$ and a discharger $!_A : A \to 1$ for each object A, subject to a few coherence axioms. It is known that if duplicators and dischargers satisfy naturality, then the monoidal product is the categorical product, and thus the category is cartesian monoidal [5]: a structure used for example in the context of Lawvere theories [6] to represent abstractly algebraic operations and their compositions. But since no form of naturality is required for duplicators and dischargers in gs-monoidal categories in general, one may think of the arrows of such categories as abstract counterparts of *relations* and *partial functions*, instead.

Conceptually, gs-monoidal categories can be considered as a weaker version of cartesian bicategories [7], [8], lacking the dual arrows for duplicators and dischargers. The relevance of cartesian bicategories to both mathematics and computer science has increased in the last years, see for example [9], [10], [11] and similarly the notion of gs-monoidality has surfaced a few times under different monikers. In fact, the simplicity of the notion and its pervasiveness in many applications has led several authors to investigate such structures, in some cases independently developed. This is the case, for example, for the work by Golubtsov [12], whose *categories of information transformers* are very similar to gs-monoidal categories. More recently, gs-monoidal categories appeared as *CD categories* [13], and their *affine* variant is the basis for a recent promising approach to categorical probability, where these categories are dubbed *Markov categories* [14] based on the interpretation of the arrows as generalised Markov kernels.

There is an obvious conceptual hierarchy of categories, spanning from symmetric monoidal to cartesian ones, sketched in the following diagram, with forgetful functors going upwards

$$\begin{tikzcd}[column sep=tiny]
& \text{symmetric monoidal} \arrow[d, dash] & \\
& \text{gs-monoidal} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{Markov} \arrow[dr, dash] & & \text{restriction} \arrow[dl, dash] \\
& \text{cartesian monoidal} &
\end{tikzcd}$$

As indicated, a *gs-monoidal* category is a symmetric monoidal category with a duplicator ∇_A and a discharger $!_A$ arrows for each object A, subject to a few coherence axioms but not to naturality; *Markov* categories are gs-monoidal categories where the discharger is natural; and dually *restriction categories with restriction products* [15] are those where the duplicator is natural, as shown later in Section 2. Finally, if both dischargers and duplicators are natural, the monoidal product is the categorical product and thus we get *cartesian monoidal* categories.

Motivated by the current interest in these categorical structures, in this work we aim to explore more in depth the original notion in several directions. We first provide an overview of the main characteristics of gs-monoidality and of its preorder-enriched version, in particular introducing *oplax cartesian categories*. The presentation adopts the graphical formalism of string diagrams, and it highlights the main properties underlying gs-monoidal categories. This also allows for making precise the connection with a well-established proposal for the categorical modelling of partiality, restriction categories [15], [16], [17].

We then explore two different settings in which the gs-monoidal and oplax cartesian structures naturally arise, namely, Kleisli categories and span categories. Similarly to what was observed in [14] in the context of Markov categories and affine monads, the Kleisli category of a commutative monad on a cartesian category $\mathcal{A}$ is shown to be gs-monoidal, and this holds more generally if $\mathcal{A}$ is merely gs-monoidal itself. We moreover recall an almost folklore result, namely, that the category $\mathbf{PSpan}(\mathcal{A})$, obtained from the bicategory of spans of a category with finite limits $\mathcal{A}$ by identifying two arrows whenever they are isomorphic as spans and by taking the preorder reflection of the 2-cells, is gs-monoidal and in fact admits a canonical oplax cartesian structure. More importantly, we also establish a "bridge" between the Kleisli and span formalisms via a suitable functor. For this purpose, we introduce a generalisation of the notion of *affine monad* [18], which we call *weakly affine*. We show that if T is a weakly affine monad on a category with finite limits $\mathcal{A}$, then the Kleisli category $\mathcal{A}_T$ admits a Frobenius lax functor [19], [20] to the category of spans $\mathbf{PSpan}(\mathcal{A})$ that preserves the oplax cartesian structure. This is of interest as this lax functor is often faithful, and thus embeds the Kleisli category $\mathcal{A}_T$ into $\mathbf{PSpan}(\mathcal{A})$.

Finally, we turn our attention to completeness for preorder-enriched gs-monoidal categories. Our first step is to describe a gs-monoidal Yoneda embedding. Our second, and more important, step is to obtain a functorial completeness theorem with respect to suitable bilax functors [21], which induces completeness also for models of oplax cartesian categories in $\mathbf{Rel}$. Our completeness results offer a tool for exploiting gs-monoidal and oplax cartesian categories in the setting of functorial semantics for relational and partial algebras, in the spirit of [4].

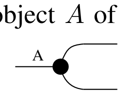
Section 2 recalls the basic notions of gs-monoidal categories and overviews their main properties. Section 3 introduces preorder-enriched gs-monoidal categories and oplax cartesian categories, setting the stage for our later results. Section 4 presents a characterisation of Kleisli and span categories having the gs-monoidal and oplax cartesian structure, respectively. Then, it introduces the notion of weakly affine monad and it provides a formal link between those categories via a suitable Frobenius functor. Section 5 presents a gs-monoidal Yoneda embedding and a completeness result for oplax cartesian categories. Section 6 concludes and discusses future works.

The reader may find some categorical background in Appendices A–B, while the remaining appendices collect the proofs omitted from the main text.

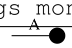
2. Background on gs-monoidal categories

Originally introduced in the context of algebraic approaches to term graph rewriting [22], the notion of *gs-monoidal category* has been developed in a series of papers [2], [3], [4]. We recall here the basic definitions adopting the graphical formalism of string diagrams, referring to [23] for an overview of various notions of monoidal categories and their associated string diagrams.

Definition 2.1 (gs-monoidal category)**.** A **gs-monoidal category** $\mathcal{C}$ is a symmetric monoidal category, where we denote by $\otimes$ the tensor product and by I the unit, such that every object A of $\mathcal{C}$ is equipped with arrows

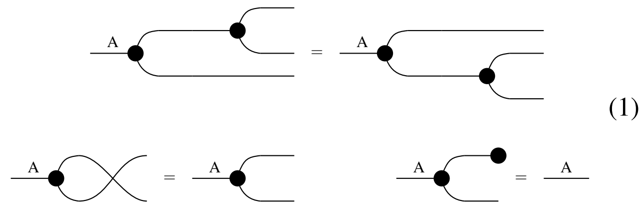


$: A \to A \otimes A$



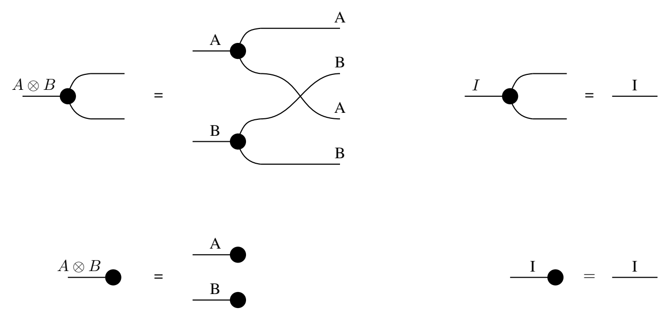
$: A \to I$

which satisfy the commutative comonoid axioms



(1)

and the monoidal multiplicativity axioms



(2)

Symbolically, we also write $\nabla_A : A \to A \otimes A$ for the first structure arrow above and call it **duplicator**, and similarly $!_A : A \to I$ for the **discharger**.

Remark 2.2. The monoidal multiplicativity equations come in two pairs, one pair specifying the duplicator and discharger on a tensor product object and one for doing so on the monoidal unit I. However, the two equations for I imply each other upon using the counitality axiom (bottom-right of (1)), thus one of them could be omitted from the definition.

Remark 2.3. Following the style of presentation of string diagrams, the axioms are given for *strict* monoidal categories, i.e. where the coherence isomorphisms for associativity and unitality are identities. This is without loss of generality, either by considering the strictification of the categories at hand [24, Proposition 3.28] or by adding the additional coherence isomorphisms to the axioms.

Example 2.4. If $\mathcal{C}$ is the category of commutative comonoids in a symmetric monoidal category $\mathcal{A}$, with arrows given by the arrows of $\mathcal{A}$ without any further conditions, then $\mathcal{C}$ is a gs-monoidal category in a canonical way: the duplicators and dischargers are given by the comultiplications and counits of the comonoids, respectively. Then the commutative comonoid equations (1) hold by definition of commutative comonoid, and the monoidal multiplicativity equations (2) hold by definition of the monoidal product of commutative comonoids.

As for functors between symmetric monoidal categories, also functors between gs-monoidal categories come in several variants. The definitions that follow refer to lax, oplax, strong and strict monoidal functors (see e.g. [21]) that are recalled in Appendix A.

Definition 2.5. For gs-monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is

1) **lax gs-monoidal** if it is equipped with a lax symmetric monoidal structure $\psi\colon \otimes\circ(F\times F)\to F\circ\otimes$, $\psi_0\colon I\to F(I)$ such that the following diagrams

$$\begin{tikzcd} F(A) \arrow[rr, "F(\nabla_A)"] \arrow[dr, "\nabla_{FA}"'] & & F(A\otimes A) \\ & F(A)\otimes F(A) \arrow[ur, "\psi_{A,A}"'] & \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[rr, "F(!_A)"] \arrow[dr, "!_{FA}"'] & & F(I) \\ & I \arrow[ur, "\psi_0"'] & \end{tikzcd}$$

commute for all A

2) **oplax gs-monoidal** if it is equipped with an oplax symmetric monoidal structure $\phi\colon F\circ\otimes\to\otimes\circ(F\times F)$, $\phi_0\colon F(I)\to I$ such that the following diagrams

$$\begin{tikzcd} F(A) \arrow[rr, "F(\nabla_A)"] \arrow[dr, "\nabla_{FA}"'] & & F(A\otimes A) \arrow[dl, "\phi_{A,A}"] \\ & F(A)\otimes F(A) & \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[rr, "F(!_A)"] \arrow[dr, "!_{FA}"'] & & F(I) \arrow[dl, "\phi_0"] \\ & I & \end{tikzcd}$$

commute for all A;

3) **strong gs-monoidal** if it is strong symmetric monoidal and the above diagrams commute;

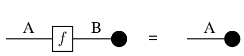
4) **strict gs-monoidal** if it is strict symmetric monoidal, that is in particular $F(A\otimes B)=F(A)\otimes F(B)$ and $F(I)=I$, and it satisfies

$$F(\nabla_A)=\nabla_{F(A)} \qquad F(!_A)=!_{F(A)}$$

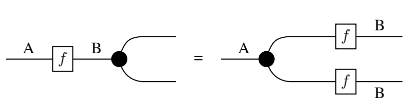
As in the purely monoidal case, "gs-monoidal functor" without further qualification refers to the strong version.

Recall also the notions of *bilax symmetric monoidal* and *Frobenius symmetric monoidal* functor from Definitions A.3 and A.4. We then obtain definitions of **bilax gs-monoidal** and **Frobenius gs-monoidal functors** upon adding the four commutative triangles above to these definitions.

Definition 2.6. Let $\mathcal{C}$ be a gs-monoidal category. An arrow $f\colon A\to B$ is $\mathcal{C}$-**total** if the following equation holds



and $\mathcal{C}$-**functional** if the following equation holds



We denote the subcategory of $\mathcal{C}$-functional arrows by $\mathcal{C}$-**Fun**, the one of $\mathcal{C}$-total arrows by $\mathcal{C}$-$\mathbf{Total}$, and the one of $\mathcal{C}$-total and $\mathcal{C}$-functional arrows by $\mathcal{C}$-**TFun**.

Example 2.7. If $\mathcal{C}$ is the gs-monoidal category of commutative comonoids in a symmetric monoidal category $\mathcal{A}$, then the total arrows are the counital ones, the functional arrows the comultiplicative ones, and therefore $\mathcal{C}$-**TFun** is the category of commutative comonoids and comonoid homomorphisms in $\mathcal{A}$.

Example 2.8. The category $\mathbf{Rel}$ of sets and relations with the operation $\times\colon \mathbf{Rel}\times\mathbf{Rel}\to\mathbf{Rel}$ given by the direct product of sets is a gs-monoidal category [4]. In particular, $\mathbf{Rel}$-functional arrows are precisely partial functions, $\mathbf{Rel}$-total arrows are total relations, and $\mathbf{Rel}$-total, $\mathbf{Rel}$-functional arrows are functions.

Example 2.9. Every Markov category $\mathcal{M}$ in the sense of [14], [25] is gs-monoidal. In fact, Markov categories are exactly those gs-monoidal categories whose monoidal unit I is terminal, or equivalently those for which every arrow is $\mathcal{M}$-total. The $\mathcal{M}$-functional arrows in the sense of Definition 2.6 are precisely those called *deterministic* there.

We note a useful property of $\mathcal{C}$-total and $\mathcal{C}$-functional arrows, which generalise corresponding observations for Markov categories [14, Lemma 10.12].

Proposition 2.10. *Let $\mathcal{C}$ be a gs-monoidal category. Then $\mathcal{C}$-**Fun** and $\mathcal{C}$-$\mathbf{Total}$ are gs-monoidal sub-categories of $\mathcal{C}$.*

Proof: See Appendix C.1. □

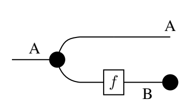
The proof boils down to the fact that the arrows $!_A$ and ∇_A are both $\mathcal{C}$-total and $\mathcal{C}$-functional for every object A.

2.1. Substructures of a gs-monoidal category

In this section we investigate the categorical structure of functional and total arrows. In particular, we are going to explore their connections with restriction categories [16], p-categories [26], and cartesian categories. In this way we will introduce by suitable examples some notions that are used in the following sections.

To achieve this goal, we start by observing that gs-monoidal categories have enough structure to properly express a notion of *domain* of arrows.

Definition 2.11. Let $\mathcal{C}$ be a gs-monoidal category and $f\colon A\to B$ in $\mathcal{C}$. The **domain** of f is the arrow $\mathrm{dom}(f):=(\mathrm{id}_A\otimes !_B f)\nabla_A\colon A\to A$, graphically



The motivation behind the choice of the arrow $\mathrm{dom}(f):=(\mathrm{id}_A\otimes !_B f)\nabla_A$ is that in $\mathbf{Rel}$, for a relation R from A to B, the arrow $\mathrm{dom}(R)=(\mathrm{id}_A\times !_B R)\nabla_A$ is the relation representing the domain of definition of R, i.e.

$$a\,\mathrm{dom}(R)a' \quad\Longleftrightarrow\quad a=a' \text{ and } \exists b\in B:\ aRb.$$

As we will show, the arrow $\mathrm{dom}(f)$ in a gs-monoidal category enjoys several algebraic properties that generalise those of the usual notion of domain of a relation. A first property we expect to hold from a morphism that wants to abstract the notion of domain of a relation is that the domain of a total arrow has to be the identity. This is what we prove in the next proposition.

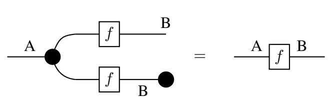
Proposition 2.12. *Let $\mathcal{C}$ be a gs-monoidal category and $f\colon A \to B$. Then*

1) *if f is $\mathcal{C}$-functional, then $\mathrm{dom}(f)$ is $\mathcal{C}$-functional;*
2) *f is $\mathcal{C}$-total if and only if $\mathrm{dom}(f) = \mathrm{id}_A$.*

Proof: See Appendix C.2. □

Remark 2.13. The notion of domain allows us to state the precise link between gs-monoidal and restriction categories [16] and p-categories [26]. We summarise the relevant result of Cockett and Lack [15, Thm 5.2]. When $\mathcal{C}$ is a gs-monoidal category, the subcategory $\mathcal{C}$-**Fun** of $\mathcal{C}$-functional arrows is a restriction category, with the restriction structure given by $\mathrm{dom}(-)$. Moreover, this category has restriction products, which in particular implies the equation $\mathrm{dom}(f \otimes g) = \mathrm{dom}(f) \otimes \mathrm{dom}(g)$. Furthermore, $\mathcal{C}$-**Fun** also is a p-category with one-element object (given by I), where the diagonal is given by ∇ and the two projections by the families of arrows of the form $\mathrm{id} \otimes !$ and $! \otimes \mathrm{id}$, respectively.

Remark 2.14. In **Rel**, domains have an additional property given by the following equation: $f\,\mathrm{dom}(f) = f$, namely



(3)

even for non-functional arrows f. However, this equation does not hold for arrows in gs-monoidal categories in general: we will address this issue in Section 3 after introducing preorder-enriched gs-monoidal categories. For example, the Kleisli category of the multiset monad on **FinSet** is a gs-monoidal category in a canonical way (by Proposition 4.4), and its arrows $A \to B$ can be identified with functions $A \times B \to \mathbb{N}$ that compose via convolution. The above (3) does not hold: already with $A = B = I$ a one-element set, where f is determined by a natural number, we have the number itself on the right but its square on the left.

We now recall a few, almost folklore properties (see e.g. [5]) that are needed later on.

Proposition 2.15. *Let $\mathcal{C}$ be a gs-monoidal category. Then*

1) *the subcategory $\mathcal{C}$-**Total** of $\mathcal{C}$-total arrows has weak binary products;*
2) *the subcategory $\mathcal{C}$-**TFun** of $\mathcal{C}$-functional and $\mathcal{C}$-total arrows has binary products.*

Proof: See Appendix C.3. □

Example 2.16. Let us consider a gs-monoidal category $\mathcal{C}$ and a $\mathcal{C}$-total arrow $f\colon B \to A$. Notice that the diagram

$$\begin{tikzcd} A & A \otimes A \arrow[l, "\mathrm{id}_A \otimes !_A"'] \arrow[r, "!_A \otimes \mathrm{id}_A"] & A \\ & B \arrow[ul, "f"] \arrow[u, "h"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes when $h := \nabla_A f$, but also when we take $h := (f \otimes f)\nabla_B$. If f is non-functional, then these two arrows could be different. Hence we can conclude that the subcategory $\mathcal{C}$-**Total** has weak binary products by Proposition 2.15, but in general these are not categorical products.

3. Oplax cartesian categories

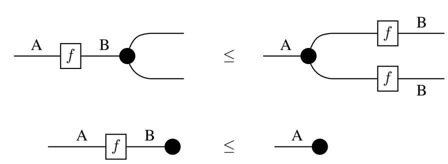
We have seen that gs-monoidal categories enjoy some features of **Rel** with respect to total and functional arrows. Our next step is to show how to build on the notion of gs-monoidality, in order to recover the usual preorder-enrichment given for relations. This extension will be pivotal later on, e.g. for our completeness theorem. As a direct consequence, in this section we show that even if the notion of domain we discussed in the previous section lacks some of its usual axioms, as observed in Remark 2.14, these can be recovered in terms of preorder equivalence.

Definition 3.1. A **preorder-enriched gs-monoidal category** $\mathcal{C}$ is a gs-monoidal category $\mathcal{C}$ that is at the same time a preorder-enriched monoidal category.

Recall that a preorder-enriched monoidal category consists of preorder-enriched category $\mathcal{C}$, an object I of $\mathcal{C}$, a preorder-enriched functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and enriched natural isomorphisms $\lambda\colon I \otimes - \to \mathrm{id}_{\mathcal{C}}$, $\rho\colon - \otimes I \to \mathrm{id}_{\mathcal{C}}$ and $\alpha\colon (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ such that the underlying category equipped with the underlying functor $\otimes$, the object I and the natural isomorphisms λ, ρ and α is an ordinary monoidal category (see [27] for more details).

Since a preorder-enriched functor is just an ordinary functor that is in addition monotone, the preorder structure and the gs-monoidal structure are required to interact by the monotonicity of the tensor product $\otimes$. In a general preorder-enriched gs-monoidal category, no further compatibility is required. However, in many (but not all) examples, the following compatibility holds in addition.

Definition 3.2. An **oplax cartesian category** $\mathcal{C}$ is a preorder-enriched gs-monoidal category $\mathcal{C}$ such that the following inequalities hold for every arrow $f\colon A \to B$:[1]



1. Viewing a preorder-enriched category as a 2-category, these inequalities state that the families of arrows ∇_A and $!_A$ are the components of an *oplax natural transformation*. Their link to rewriting is shown in [3].

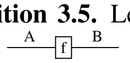
Notice that the notion of oplax cartesian category is reminiscent of cartesian bicategories in the sense of [7, Def. 1.2]. The main difference is that in Definition 3.2, the existence of right adjoints for ∇ and ! is not required.

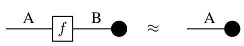
Example 3.3. $\mathbf{Rel}$ has a natural preorder-enriched structure with the preorder given by the set-theoretic inclusions between relations. Moreover, for every relation $R\colon A \to B$ we trivially have the inequalities discussed in Definition 3.2. Hence $\mathbf{Rel}$ is an oplax cartesian category.

On the other hand, reversing the preorder on every hom-set of $\mathbf{Rel}$ gives a preorder-enriched gs-monoidal category that is clearly not oplax cartesian. As for a trivial non-example, any gs-monoidal category $\mathcal{C}$ is made preorder-enriched with the trivial preorder, but this is not oplax cartesian unless $\mathcal{C}$ is cartesian.

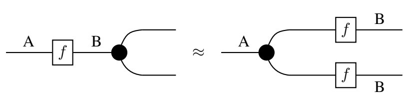
Definition 3.4. Let $\mathcal{C}$ be a preorder-enriched category and $f, g\colon A \to B$. Then f and g are **preorder equivalent**, denoted by $f \approx g$, if $f \leq g$ and $g \leq f$.

Definition 2.6 can be generalised in the natural way.

Definition 3.5. Let $\mathcal{C}$ be an oplax cartesian category. An arrow  is **weakly $\mathcal{C}$-total** if the following equivalence holds

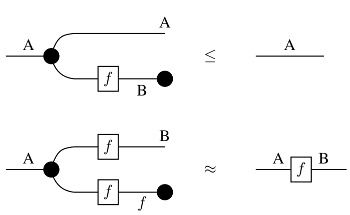


and **weakly $\mathcal{C}$-functional** if the following equivalence holds



Dafinition 3.4 lets us prove that, even if $f\operatorname{dom}(f)$ is different from f in general, these two arrows are preorder equivalent in any oplax cartesian category.

Proposition 3.6. *In any oplax cartesian category, for every arrow $f\colon A \to B$ we have that $\operatorname{dom}(f) \leq \mathrm{id}_A$ and $f\operatorname{dom}(f) \approx f$, graphically*



Proof: See Appendix C.4. □

We show next that for a fixed preorder-enriched category $\mathcal{C}$ and for a fixed symmetric monoidal structure on $\mathcal{C}$, the families of arrows ∇ and ! that equip $\mathcal{C}$ with the structure of oplax cartesian category are unique (when they exist) up to preorder equivalence.

Proposition 3.7. *Let $\mathcal{C}$ be an oplax cartesian category with structure arrows ∇ and !. If $\mathcal{C}$ admits a structure of oplax cartesian category given by the same monoidal structure and the same preorder, but with structure arrows ∇' and $!'$, then $\nabla_A \approx \nabla'_A$ and $!_A \approx !'_A$ for every object A of $\mathcal{C}$.*

Proof: See Appendix C.5. □

Example 3.8. Given a (plain) gs-monoidal category $\mathcal{C}$, one can consider the monoidal preorder enrichment *generated* by the inequalities of Definition 3.2, i.e. the smallest preorder on every hom-set which makes both composition and $\otimes$ monotone and satisfies those inequalities. In this way, $\mathcal{C}$ becomes oplax cartesian in a canonical way.

An interesting case of the construction hinted at above is $\mathbf{FinStoch}$, the Markov category of finite sets and stochastic maps. In this case, it was recently shown by Dario Stein (personal communication) that the preorder enrichment generates by the inequalities recovers the *support* of a distribution: for stochastic matrices $f, g : X \to Y$ we have $f \leq g$ if and only if the support of f is contained in the support of g, meaning that $f(y|x) > 0$ implies $g(y|x) > 0$ for all $x \in X$ and $y \in Y$. It follows that the quotient by preorder equivalence $\mathbf{FinStoch}/\approx$ is a gs-monoidal category isomorphic to $\mathbf{FinRel}$.

We conclude this section with a discussion of functors between preorder-enriched gs-monoidal categories and oplax cartesian categories. The various notions of gs-monoidal functor introduced in Definition 2.5 can be used in the context of preorder-enriched gs-monoidal categories, with the only difference that F is additionally required to be a preorder-enriched functor, which amounts to monotonicity on hom-sets. Between oplax cartesian categories, one often has additional inequalities taking the following form.

Definition 3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be oplax cartesian categories and $F\colon \mathcal{C} \to \mathcal{D}$ a preorder-enriched functor. Then

1) F is **colax cartesian** if it is a lax symmetric monoidal functor with structure arrows ψ, ψ_0 such that the following inequalities

$$\begin{tikzcd} F(A) \arrow[rr, "F(\nabla_A)"] \arrow[dr, "\nabla_{FA}"'] & \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & F(A \otimes A) \\ & F(A) \otimes F(A) \arrow[ur, "\psi_{A,A}"'] & \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[rr, "F(!_A)"] \arrow[dr, "!_{FA}"'] & \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & F(I) \\ & I \arrow[ur, "\psi_0"'] & \end{tikzcd}$$

hold for all A in $\mathcal{C}$;

2) F is **colax opcartesian** if it is an oplax symmetric monoidal functor with structure arrows ϕ, ϕ_0 such that

the following inequalities

$$\begin{tikzcd} F(A) \arrow[rr, "F(\nabla_A)"] \arrow[dr, "\nabla_{FA}"'] & \arrow[d, phantom, "\mid\wedge"] & F(A\otimes A) \arrow[dl, "\phi_{A,A}"] \\ & F(A)\otimes F(A) & \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[rr, "F(!_A)"] \arrow[dr, "!_{FA}"'] & \arrow[d, phantom, "\mid\wedge"] & F(I) \arrow[dl, "\phi_0"] \\ & I & \end{tikzcd}$$

hold for all A in $\mathcal{C}$;

3) F is **colax bicartesian** if it is colax cartesian and colax opcartesian in such a way as to become bilax monoidal (Definition A.3);
4) F is **colax Frobenius cartesian** if it is colax cartesian and colax opcartesian in such as way as to become Frobenius monoidal (Definition A.4).

These definitions can be straightforwardly adapted to the case that F is a *lax functor*, i.e. such that identity and composition are not preserved on the nose, that is

$$\mathrm{id}_{FA} \leq F(\mathrm{id}_A), \qquad F(g)\circ F(f) \leq F(g\circ F).$$

4. Kleisli categories are gs-monoidal

In recent years, strong monads and Kleisli categories have been used to provide categorical models in several branches of computer science. The leading example is Moggi's work [28], [29] on an abstract approach to the notion of computation. We refer to [30], [31], [32] for introductions to the theory of monads and to [33], [34] for more details. Appendix B offers a short recap with the main definitions. We start by recalling some relevant and known examples of Kleisli categories.

Example 4.1. $\mathbf{Rel}$ is the Kleisli category of the powerset monad, where its underlying functor $P\colon \mathbf{Set}\to\mathbf{Set}$ sends every set X to its powerset $P(X)$.

Example 4.2. The category of sets and stochastic maps[2] is the Kleisli category of the finite distribution monad $\mathcal{D}\colon \mathbf{Set}\to\mathbf{Set}$, sending a set X to the set $\mathcal{D}(X)$ of finitely supported probability measures on X.

Example 4.3. The category of measurable spaces and Markov kernels is the Kleisli category of the Giry monad $\mathcal{G}\colon \mathbf{Meas}\to\mathbf{Meas}$, where $\mathbf{Meas}$ denotes the category of measurable spaces and measurable functions, see [35], [36].

Notation. To avoid confusion between the arrows of $\mathcal{A}$ and the arrows of a Kleisli category $\mathcal{A}_T$, we adopt the notation $f^\sharp\colon A\to T(B)$ for the representative of an arrow $f\colon A\to B$ in $\mathcal{A}_T$ [25]. We often define a Kleisli arrow f by specifying its representative $f^\sharp$. The definition of Kleisli composition then amounts to the equation $(g\circ f)^\sharp = \mu\circ T(g^\sharp)\circ f^\sharp$.

Given a monad (T,μ,η) on a symmetric monoidal category $\mathcal{A}$, it is well known (see e.g. [37]) that the Kleisli category $\mathcal{A}_T$ inherits a symmetric monoidal structure precisely when the monad is strong and commutative, see Appendix B. On the other hand, it has been observed that if the base category $\mathcal{A}$ is cartesian monoidal, then the induced monoidal product in $\mathcal{A}_T$ is not cartesian in general. A simple counterexample is the powerset monad $P\colon\mathbf{Set}\to\mathbf{Set}$ on the category of sets and functions of Example 4.1. In fact the Kleisli category $\mathbf{Set}_P$ is the category $\mathbf{Rel}$ of sets and relations, and the categorical product on $\mathbf{Set}$ induces a monoidal product on $\mathbf{Rel}$, given by the direct product. However, the direct product of sets is not a categorical product in $\mathbf{Rel}$, but just a monoidal product.

Thus a natural question is: what is the algebraic structure that is inherited from the base category by the Kleisli category of a commutative monad? Looking at the hierarchy of categories sketched in the introduction, spanning from symmetric monoidal to cartesian ones and including gs-monoidal, Markov and restriction categories (with restriction products), we answer that gs-monoidality is inherited, while the naturality of dischargers or duplicators is not.

Proposition 4.4. *Let (T,μ,η) be a commutative monad on a gs-monoidal category $\mathcal{A}$. Then the Kleisli category $\mathcal{A}_T$ is a gs-monoidal category with duplicators and dischargers given for every object A by*

$$\nabla_A^\sharp := \eta_{A\otimes A}\nabla_A, \qquad !_A^\sharp := \eta_I !_A.$$

Proof: See Appendix D.1. □

Remark 4.5. An alternative choice for the arrow $\nabla_A^\sharp$ in Proposition 4.4 could be $\nabla_A^\sharp := c_{A,A}\nabla_{T(A)}\eta_A$, for $c_{A,A}\colon T(A)\otimes T(A)\to T(A\otimes A)$ the canonical arrow defined via the commutative structure of the monad T, meaning the diagonal of the diagram in Definition B.5. However, it is straightforward to check that $c_{A,A}\nabla_{T(A)}\eta_A = \eta_{A\otimes A}\nabla_A$, so that the two alternative choices coincide.

The above discussed example of the powerset monad on $\mathbf{Set}$ shows that for a commutative monad, the naturality of the discharger is not preserved in general by the Kleisli category construction (therefore $\mathcal{A}_T$ is not a restriction category with restriction products in general, even if $\mathcal{A}$ is). Similarly, the same example shows that $\mathcal{A}_T$ is not a Markov category in general, even if $\mathcal{A}$ is.

Remark 4.6. If $\mathcal{A}$ is a Markov category (Example 2.9) and (T,μ,η) is a commutative monad, then 1 is terminal in $\mathcal{A}_T$ if and only if $T(1)\cong 1$ in $\mathcal{A}$, a property known as T being an *affine* monad [38], [39]. Thus, if the monad T preserves the terminal object, every arrow of $\mathcal{A}_T$ is $\mathcal{A}_T$-total and this makes $\mathcal{A}_T$ into a Markov category [14, Corollary 3.2]. As an example, consider the non-empty powerset monad $P^*\colon\mathbf{Set}\to\mathbf{Set}$, i.e., associating with a set X the family of its non-empty subsets $P(X)\setminus\emptyset$: the arrows of the Kleisli

2. Or to be more precise, stochastic maps with pointwise finite support.

categories are total relations, thus $\mathbf{Rel\text{-}Total} \cong \mathbf{Set}_{P^*}$, and indeed we have that $T(I) \cong I$.

Remark 4.7. For representing functional relations in terms of Kleisli categories, it suffices to consider the lifting monad (also called "maybe monad"), associating with a set X the pointed set $X_\perp := X + 1$. Its Kleisli category is exactly $\mathbf{Rel\text{-}Fun}$, the category of sets and partial functions.

Example 4.8. The category of measurable spaces and Markov kernels of Example 4.3 is gs-monoidal, and actually a Markov category, since the Giry monad $\mathcal{G}\colon \mathbf{Meas} \to \mathbf{Meas}$ is an affine commutative monad with respect to the cartesian monoidal structure of $\mathbf{Meas}$. This Kleisli category is often denoted by $\mathbf{Stoch}$.

Similarly, the category of quasi-Borel spaces $\mathbf{QBS}$ is cartesian and the monad $P\colon \mathbf{QBS} \to \mathbf{QBS}$ of probability measures on it is an affine commutative monad, see [40] for all the details. Therefore the Kleisli category $\mathbf{QBS}_P$ is a gs-monoidal category, and in fact a Markov category.

4.1. The oplax cartesianity of Kleisli categories

Recall that given an arbitrary monad $T\colon \mathcal{A} \to \mathcal{A}$, we can define a pair of functors $F_T\colon \mathcal{A} \to \mathcal{A}_T$ and $G_T\colon \mathcal{A}_T \to \mathcal{A}$ such that $F_T \dashv G_T$, for

- $F_T(X) := X$ and $F_T(f)^\sharp := \eta_B f$ for $f\colon A \to B$ in $\mathcal{A}$;
- $G_T(X) := TX$ and $G_T(f) := \mu_B T(f^\sharp)$ for $f\colon A \to B$ in $\mathcal{A}_T$;

In the hypotheses of Proposition 4.4 it is easy to see that the functor $F_T\colon \mathcal{A} \to \mathcal{A}_T$ is a strict gs-monoidal functor, while the functor $G_T\colon \mathcal{A}_T \to \mathcal{A}$ is not strict monoidal in general.

In fact, G_T can be only proved to be lax-monoidal:

Proposition 4.9. *Let T be a commutative monad on a gs-monoidal category $\mathcal{A}$. Then the functor $G_T\colon \mathcal{A}_T \to \mathcal{A}$ is a lax-monoidal functor, with structure morphisms*

$$\psi_{X,Y} := c_{X,Y}\colon T(X) \times T(Y) \to T(X \times Y)$$

and

$$\psi_0 := \eta_I\colon I \to T(I).$$

Proof: See Appendix **??**. ☐

Now we investigate when the functor G_T preserves the gs-monoidal structure. First, observe that every strong commutative monad on a symmetric monoidal category provides a lax monoidal functor, when it is considered with the canonical morphisms $\psi_{X,Y} := c_{X,Y}$ and $\psi_0 := \eta_I$. What we have is that the functor G_T happens to be a lax gs-monoidal functor precisely when the monad T is a lax gs-monoidal functor:

Corollary 4.10. *Let T be a commutative monad on a gs-monoidal category $\mathcal{A}$. Then $G_T\colon \mathcal{A}_T \to \mathcal{A}$ is a lax gs-monoidal functor if and only if T is a lax gs-monoidal functor.*

Now we consider the case of oplax cartesian categories and preorder-enriched monads. In this setting, we introduce the following definition:

Definition 4.11. Let $\mathcal{A}$ be a oplax cartesian category. A preorder-enriched strong commutative monad $T\colon \mathcal{A} \to \mathcal{A}$ is called **oplax cartesian monad** if $T\colon \mathcal{A} \to \mathcal{A}$ is a colax cartesian functor with respect to the structure morphisms

$$\psi_{X,Y} := c_{X,Y}\colon T(X) \times T(Y) \to T(X \times Y)$$

and

$$\psi_0 := \eta_I\colon I \to T(I).$$

In particular, when we consider a preorder-enriched monad $T\colon \mathcal{A} \to \mathcal{A}$ on an oplax cartesian category $\mathcal{A}$, it is direct to check that the preoredered structure of hom-sets of $\mathcal{A}$ induces a natural preoredered structure on the hom-sets of the Kleisli category $\mathcal{A}_T$, i.e. we have that $f \le g$ in $\mathcal{A}_T$ if $f^\sharp \le g^\sharp$ in $\mathcal{A}$. Since T is a preorder-enriched functor, this makes the Kleisli category $\mathcal{A}_T$ preoreder-enriched. For sake of clarity we will denote by $\le_{\mathcal{A}}$ the preorder of $\mathcal{A}$ and by $\le_{\mathcal{A}_T}$ that of $\mathcal{A}_T$.

Proposition 4.12. *Let $\mathcal{A}$ be an oplax cartesian category, and let $T\colon \mathcal{A} \to \mathcal{A}$ be an oplax cartesian monad. Then the Kleisli category $\mathcal{A}_T$ equipped with the preorder given by $f \le_{\mathcal{A}_T} g \iff f^\sharp \le_{\mathcal{A}} g^\sharp$ is oplax cartesian.*

Proof: See Appendix D.2. ☐

5. From Kleisli to spans

Proposition 4.4 shows that the Kleisli category $\mathcal{A}_T$ of a commutative monad T on a gs-monoidal category $\mathcal{A}$ is gs-monoidal. Generalising previous results holding on specific categories (see e.g. [41]), in this subsection we first show that for any $\mathcal{A}$ with finite limits the category $\mathbf{PSpan}(\mathcal{A})$, obtained by taking the preorder reflection of the 2-cells of the bicategory of spans $\mathbf{Span}(\mathcal{A})$ and identifying two arrows when they are isomorphic spans, is oplax cartesian. Subsequently, we construct a lax functor $\mathcal{A}_T \to \mathbf{PSpan}(\mathcal{A})$, provided that the monad satisfies a novel condition we define called *weak affinity*. This establishes a precise "bridge" between two natural mechanisms for obtaining gs-monoidal categories. Furthermore, in many examples this lax functor is faithful, which seems of interest insofar as it shows that the single category $\mathbf{PSpan}(\mathcal{A})$ in some sense contains the Kleisli category for potentially many different commutative monads T.

The following definitions are standard.

Definition 5.1. Let $\mathcal{A}$ be a category with pullbacks. The bicategory of spans $\mathbf{Span}(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its arrows are defined as

- an arrow from X to Y is a *span* $(X \leftarrow A \to Y)$ of $\mathcal{A}$;
- the identity of X is the *identity span* $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$;
- the composition of spans $X \leftarrow A \xrightarrow{f} Y$ and $Y \xleftarrow{g} B \to Z$ is given by the span $X \leftarrow A \times_{f,g} B \to Z$ obtained by taking the pullback of f and g;

- a 2-cell $\alpha : (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$ is an arrow $\alpha : A \rightarrow B$ in $\mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[d, "\alpha"] & \\ X & B \arrow[l] \arrow[r] & Y \end{tikzcd}$$

- vertical composition of 2-cells is given by composition in $\mathcal{A}$;
- horizontal composition of 2-cells as well as associators and unitors are induced by the universal property of pullbacks.

Definition 5.2. Let $\mathcal{A}$ be a category with pullbacks and let $\mathbf{Span}(\mathcal{A})$ be its bicategory of spans. The preorder-enriched category $\mathbf{PSpan}(\mathcal{A})$ has

- the same objects as $\mathcal{A}$;
- isomorphism classes of arrows of $\mathbf{Span}(\mathcal{A})$ as arrows: spans $(X \xleftarrow{f} A \xrightarrow{g} Y)$ and $(X \xleftarrow{f'} A' \xrightarrow{g'} Y)$ are isomorphic if there is an iso $i \colon A \rightarrow A'$ such that $f' \circ i = f$ and $g' \circ i = g$;
- a preorder enrichment defined as $[(X \xleftarrow{f} A \xrightarrow{g} Y)] \leq [(X \xleftarrow{f'} A' \xrightarrow{g'} Y)]$ if there is 2-cell $\alpha : (X \xleftarrow{f} A \xrightarrow{g} Y) \Rightarrow (X \xleftarrow{f'} A' \xrightarrow{g'} Y)$ in $\mathbf{Span}(\mathcal{A})$.

Note that $\mathbf{PSpan}(\mathcal{A})$ is a locally small category as soon as $\mathcal{A}$ is small.

Now, when $\mathcal{A}$ is also a cartesian category, it is straightforward to check that the categorical product $\times$ of $\mathcal{A}$ induces a monoidal product $\otimes$ on $\mathbf{PSpan}(\mathcal{A})$. However, we have actually more structure, as witnessed by the result below.

Proposition 5.3. *Let $\mathcal{A}$ be a category with finite limits. Then $\mathbf{PSpan}(\mathcal{A})$ is an oplax cartesian category with*

$$\nabla^s_X = (X \xleftarrow{\mathrm{id}} X \xrightarrow{\nabla_X} X \times X), \qquad !^s_X = (X \xleftarrow{\mathrm{id}} X \xrightarrow{!} 1).$$

Proof: See Appendix D.3. □

D: New

We may characterise weak functionality and weak totality in terms of properties of the components of a span.

Proposition 5.4. *Let $(X \xleftarrow{f} Z \xrightarrow{g} Y)$ be a span in $\mathbf{PSpan}(\mathcal{A})$. Then*

1) *it is weakly $\mathbf{PSpan}(\mathcal{A})$-functional if and only if f is a monomorphism;*
2) *it is weakly $\mathbf{PSpan}(\mathcal{A})$-total if and only if f is a split epimorphism.*

Proof: See Appendix D.4. □

While the first item is hardly surprising, since it reflects the intuition at the basis of the use of spans with mono left leg for modelling partial functions, the second seems new. An immediate corollary of Proposition 5.4 follows.

Corollary 5.5. *A span $(X \xleftarrow{f} Z \xrightarrow{g} Y)$ of $\mathbf{PSpan}(\mathcal{A})$ is weakly $\mathbf{PSpan}(\mathcal{A})$-functional and weakly $\mathbf{PSpan}(\mathcal{A})$-total if and only if f is an isomorphism.*

Thus these morphisms are precisely those that are in the image of the canonical inclusion functor $\mathcal{A} \rightarrow \mathbf{PSpan}(\mathcal{A})$.

Recall that, by Proposition 4.4, a Kleisli category $\mathcal{A}_T$ is gs-monoidal (for $\mathcal{A}$ cartesian and T strong and commutative). Moreover, note that $\mathcal{A}_T$ comes equipped with the free oplax cartesian structure as in Remark 3.8. Therefore, for the rest of this section we consider $\mathcal{A}_T$ as an oplax cartesian category in this way.

The next step is to establish a relation between $\mathcal{A}_T$ and $\mathbf{PSpan}(\mathcal{A})$ whenever $\mathcal{A}$ has finite limits and the commutative monad T satisfies a suitable condition. In fact, it suffices to consider the assignment $S_T \colon \mathcal{A}_T \rightarrow \mathbf{PSpan}(\mathcal{A})$ given on objects and arrows by[3]

$$S_T \;:\; \begin{array}{ccc} X & \longmapsto & TX \\ (f \colon X \rightarrow Y) & \longmapsto & (T(X) \xleftarrow{\eta_X} X \xrightarrow{f^\sharp} T(Y)) \end{array}$$

Our goal is to show that S_T is a colax cartesian lax functor. To formulate the relevant additional condition, we denote by

$$c_{X,Y} \colon T(X) \times T(Y) \rightarrow T(X \times Y)$$

the canonical lax symmetric monoidal structure of T, see Appendix B.

T: start weakly affine stuff, to be moved to right place (also proofs can go to App)

Definition 5.6. Let T be a commutative monad on a category $\mathcal{A}$ with finite products. We say that the monad T is **weakly affine** if the following associativity diagram is a pullback for every X, Y, Z in $\mathcal{A}$:

$$\begin{tikzcd} T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"] \arrow[d, "c_{X,Y} \times \mathrm{id}"'] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\ T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"] & T(X \times Y \times Z) \end{tikzcd} \tag{4}$$

Let us derive some general properties before looking at examples. It is a standard fact that for any commutative monad T, the composite arrow

$$T(1) \times T(1) \xrightarrow{c_{1,1}} T(1 \times 1) \xrightarrow{\cong} T(1)$$

equips $T(1)$ with the structure of a commutative monoid internal to $\mathcal{A}$ with unit $\eta_1 \colon 1 \rightarrow T(1)$.

Lemma 5.7. *If T is weakly affine, then $T(1)$ is a group.*

T: It's possible to generalize the definition of weakly affine monad to the case where $\mathcal{A}$ is merely gs-monoidal or even just symmetric monoidal. However, I have not implemented this (yet) since already the proof of this lemma does not generalize straightforwardly

3. This definition of $S_T(X)$ is reminiscent of the construction of the bicategory of T-spans for a cartesian monad T [42, Section II.4.2], yet it is more general since T is not required to be a cartesian monad.

Proof: If T is weakly affine, then taking $X = Y = Z = 1$ in (4) shows that this monoid must be an abelian group: assuming that $\times$ is a strict monoidal structure for simplicity, we obtain a unique arrow $\iota\colon T(1) \to T(1)$ such that the diagram

$$\begin{tikzcd}
T(1) \arrow[dr, "{(\mathrm{id},\iota,\mathrm{id})}"] \arrow[drr, bend left, "{(\mathrm{id},\eta_1!)}"] \arrow[ddr, bend right, "{(\eta_1!,\mathrm{id})}"'] & & \\
& T(1)\times T(1)\times T(1) \arrow[r, "\mathrm{id}\times c_{1,1}"] \arrow[d, "c_{1,1}\times\mathrm{id}"'] & T(1)\times T(1) \arrow[d, "c_{1,1}"] \\
& T(1)\times T(1) \arrow[r, "c_{1,1}"] & T(1)
\end{tikzcd}$$

and the commutativity shows that ι satisfies the equations making it the inversion map for a group structure. □

The following result shows that weak affinity occurs frequently. Recall that a strong monad $T\colon \mathcal{A} \to \mathcal{A}$ on a category $\mathcal{A}$ with finite products is **affine** if $T(1) \cong 1$ (see also Remark 4.6). Three relevant examples of affine monads are the distribution monad on $\mathbf{Set}$ (for discrete probability), the Giry monad on the category of measurable spaces (for measure-theoretic probability, see Examples 4.3 and 4.8), and the expectation monad, see [18].

Proposition 5.8. *Let T be a commutative monad on a category $\mathcal{A}$ with finite limits. If T is affine, then it is weakly affine.*

Proof: See Appendix D.5. □

Remark 5.9. We are not aware of any relation between weakly affine monads in our sense and Jacobs' *strongly affine* monads [18], other than the fact that strongly affine implies affine implies weakly affine.

Example 5.10. We present a family of examples of commutative monads that are weakly affine but not affine. Let A be an abelian group (written multiplicatively). Then the functor $T_A := A \times -$ on $\mathbf{Set}$ has a canonical structure of commutative monad, where the lax structure components $c_{X,Y}$ are given by multiplying elements in A while carrying the elements of X and Y along.

Since $T_A \cong A$, the monad T_A is clearly not affine unless A is the trivial group. However, T_A is always weakly affine. Indeed, in order to show that (4) is a pullback, it suffices to show that the associativity square of A

$$\begin{tikzcd}
A\times A\times A \arrow[r, "\mathrm{id}\times\cdot"] \arrow[d, "\cdot\times\mathrm{id}"'] & A\times A \arrow[d, "\cdot"] \\
A\times A \arrow[r, "\cdot"] & A
\end{tikzcd}$$

is a pullback. Using element-wise reasoning, this amounts to showing that the system of equations $ax = c$ and $xb = d$ has a solution for $x \in A$ if and only if $cb = ad$, and in this case the solution is unique. But this is indeed the case with $x = a^{-1}c = db^{-1}$. (Note that this argument does not even require A to be abelian, but we need to require this in order for T_A to be commutative.)

Example 5.11. Many monads in categorical measure theory are weakly affine but not affine. Let e.g. $M^* : \mathbf{Set} \to \mathbf{Set}$ be the monad assigning to every set the set of finitely supported discrete *nonzero* measures on M^*, or equivalently let $M^*(X)$ for any set X be the set of nonzero finitely supported functions $X \to [0, \infty)$. The monad structure is defined in terms of the same formulas as for the distribution monad on $\mathbf{Set}$ and the components $c_{X,Y}$ are also given by the formation of product measures, or equivalently pointwise products of functions $X \to [0, \infty)$.

Since $M^*1 \cong (0, \infty)$, this monad is clearly not affine. However, it is weakly affine, and we limit ourselves to a sketch of the proof. Indeed to prove that (4) is a pullback, we again reason in terms of elements. If all measures are normalised, then we are back in the situation of the distribution monad, which is affine and the claim follows. In the general case, one can reduce to the normalised case by showing that the normalisation of the desired element of $M^*(Y)$ is uniquely determined. This works in the same way as in Example 5.10 with $A = (0, \infty)$.

On the other hand, if the zero measure is included, then we obtain a commutative monad M which can be seen as the monad of semimodules for the semiring of nonnegative reals. Since $M1 \cong [0, \infty)$ is not a group under multiplication, M is not weakly affine.

The previous two examples and Lemma 5.7 suggest the following problem.

Problem 5.12. Let T be a commutative monoid such that $T(1)$ is an abelian group. Does it follow that T is weakly affine?

We continue with another characterization of weak affinity involving $\mathbf{PSpan}$. Recall that there is a canonical inclusion functor $\mathcal{A} \to \mathbf{PSpan}(\mathcal{A})$ obtained by mapping every morphism in $\mathcal{A}$ to a span with identity left leg. This functor is clearly strict monoidal. Let us then write $\hat{T}$ for the composite functor

$$\mathcal{A} \xrightarrow{T} \mathcal{A} \longrightarrow \mathbf{PSpan}(\mathcal{A}),$$

and $\hat{c}$ for its induced monoidal structure, which has components

$$\begin{tikzcd}
& T(X)\times T(Y) \arrow[dl, equal] \arrow[dr, "c_{X,Y}"] & \\
T(X)\times T(Y) & & T(X\times Y)
\end{tikzcd} \tag{5}$$

Finally, it is well-known that swapping the legs of a span makes $\mathbf{PSpan}(\mathcal{A})$ into a *dagger category*. Let us say that T is *dagger Frobenius monoidal* if its lax structure components

$c_{X,Y}$ make following diagrams commute:

thm: S is colax Frobenius opcartesian lax-functor

$$\begin{tikzcd}
T(X\otimes Y)\otimes T(Z) \arrow[r, "\hat{c}_{X\otimes Y,Z}"] \arrow[d, "\hat{c}^{\dagger}_{X,Y}\otimes \mathrm{id}"'] & T(X\otimes Y\otimes Z) \arrow[d, "\hat{c}^{\dagger}_{X,Y\otimes Z}"] \\
T(X)\otimes T(Y)\otimes T(Z) \arrow[r, "\mathrm{id}\otimes \hat{c}_{Y,Z}"'] & T(X)\otimes T(Y\otimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
T(X)\otimes T(Y\otimes Z) \arrow[r, "\hat{c}_{X,Y\otimes Z}"] \arrow[d, "\mathrm{id}\otimes \hat{c}^{\dagger}_{Y,Z}"'] & T(X\otimes Y\otimes Z) \arrow[d, "\hat{c}^{\dagger}_{X\otimes Y,Z}"] \\
T(X)\otimes T(Y)\otimes T(Z) \arrow[r, "\hat{c}_{X\otimes Y,Z}\otimes \mathrm{id}"'] & T(X\otimes Y)\otimes T(Z)
\end{tikzcd}$$

Note that we do *not* require naturality for the transformation $\hat{c}^{\dagger}$.

T: This is exactly the naturality that fails in the proposition below (unless c is a cartesian natural transformation, which is quite rare)

With these things in mind, we can state our next result.

Proposition 5.13. *A commutative monad T on a symmetric monoidal category $\mathcal{A}$ is weakly affine if and only if $\hat{c}$ makes $\hat{T}$ into a dagger Frobenius monoidal functor.*

T: this works for symmetric monoidal $\mathcal{A}$ in general

Proof. By symmetry, it is enough to consider commutativity of the first diagram above. The lower composite is represented by the span

$$\begin{tikzcd}
& T(X)\otimes T(Y)\otimes T(Z) \arrow[dl, "c_{X,Y}\otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes c_{Y,Z}"] & \\
T(X\otimes Y)\otimes T(Z) & & T(X)\otimes T(Y\otimes Z)
\end{tikzcd}$$

sec:completeness

The upper composite in turn is given by the pullback of the cospan

$$\begin{tikzcd}
T(X\otimes Y)\otimes T(Z) \arrow[dr, "c_{X\otimes Y,Z}"'] & & T(X)\otimes T(Y\otimes Z) \arrow[dl, "c_{X,Y\otimes Z}"] \\
& T(X\otimes Y\otimes Z) &
\end{tikzcd}$$

This is the span above (up to unique iso) precisely if and only if T is weakly affine. ☐

T: end new weakly affine stuff

pullback

Lemma 5.14. *Let T be a commutative monad on a category $\mathcal{A}$ with finite limits. Then the following diagram*

$$\begin{tikzcd}
X\times Y \arrow[r, "\mathrm{id}"] \arrow[d, "\eta_X\times\eta_Y"'] & X\times Y \arrow[d, "\eta_{X\times Y}"] \\
T(X)\times T(Y) \arrow[r, "c_{X,Y}"'] & T(X\times Y)
\end{tikzcd} \tag{6}$$

commutes for every X,Y in $\mathcal{A}$, and it is a pullback if T is weakly affine.

proof:lem:weakly affine monad useful

Proof: See Appendix D.6. ☐

Theorem 5.15. *Let T be a commutative monad that is weakly affine on a category $\mathcal{A}$ with finite limits. Then*

$$S_T\colon \mathcal{A}_T \to \mathbf{PSpan}(\mathcal{A})$$

is a colax Frobenius opcartesian lax functor, with the lax monoidal structure given by the spans

$$\psi_0 := (1 \xleftarrow{\mathrm{id}} 1 \xrightarrow{\eta_1} T(1)),$$

$$\psi_{X,Y} := (T(X)\times T(Y) \xleftarrow{\mathrm{id}} T(X)\times T(Y) \xrightarrow{c_{X,Y}} T(X\times Y))$$

and the oplax monoidal structure given by the spans

$$\phi_0 := (T(1) \xleftarrow{\eta_1} 1 \xrightarrow{\mathrm{id}} 1)$$

$$\phi_{X,Y} := (T(X\times Y) \xleftarrow{c_{X,Y}} T(X)\times T(Y) \xrightarrow{\mathrm{id}} T(X)\times T(Y))$$

proof:thm: S is colax Frobenius opca

Proof: See Appendix D.7. ☐

Remark 5.16 (NEW)**.**

T: This looks a bit unmotivated and out of place now

The reader may wonder about the connection between weak affinity for a monad T and the mono condition on T, i.e. the requirement that the unit components are monomorphisms (hence, that all the spans in the image of S_T are weakly $\mathbf{PSpan}(\mathcal{A})$-functional, per Proposition 5.4). In fact, there exist weakly affine monads that do not satisfy the mono condition. A trivial example is the terminal monad on $\mathbf{FinSet}$, i.e. such that $T(X) = 1$ for all X. A less trivial example is given by the lower Vietoris monad on $\mathbf{Top}$ [43], which is affine if one restricts to *nonempty* closed subsets, and where a unit component η_X is injective if and only if X is a sober space.

prop_functional_and_to

6. On completeness

In this section we first present a gs-monoidal Yoneda embedding and then a key outcome of our work, namely, a completeness result for oplax cartesian categories. The latter theorem offers a general tool for using oplax cartesian categories in the setting of functorial semantics of relational and partial algebras, in the spirit of [4], [9], [10].

6.1. A gs-monoidal Yoneda embedding

For $\mathcal{C}$ a symmetric monoidal category, let us consider functors $F : \mathcal{C} \to \mathbf{Set}$. Assuming that $\mathcal{C}$ is small, these functors form a symmetric monoidal category with respect to **Day convolution** $\boxtimes$ [44], [45], where for $X \in \mathcal{C}$,

$$(F\boxtimes G)(X) := \int^{A,B\in\mathcal{C}} \mathcal{C}(A\otimes B, X)\times F(A)\times G(B),$$

weakly_affine

and $F\boxtimes G$ is defined on arrows in terms of the universal property of the coend. $F\boxtimes G$ has the universal property that natural transformations $F\boxtimes G \to H$ for any functor $H : \mathcal{C} \to \mathbf{Set}$ are in natural bijection with transformations

$$F(X)\times G(Y) \longrightarrow H(X\otimes Y)$$

natural in $X, Y \in \mathcal{C}$. This Day convolution turns the category of functors $\mathcal{C} \to \mathbf{Set}$ into a symmetric monoidal category. The associator is obvious, and with the monoidal unit given by the hom-functor $\mathcal{C}(I, -)$, the left unitor component $\mathcal{C}(I, -)\boxtimes F \to F$ at any F corresponds to the transformation with components

$$\begin{aligned} \mathcal{C}(I, X) \times F(Y) &\longrightarrow F(X \otimes Y), \\ (f, \alpha) &\longmapsto F(f \otimes \mathrm{id}_Y)(\alpha). \end{aligned}$$

The braidings are inherited from $\mathcal{C}$.

We denote by $\mathbf{LaxSymMon}(\mathcal{C}, \mathbf{Set})$ the category of lax symmetric monoidal functors $\mathcal{C} \to \mathbf{Set}$ and *all* natural transformations.

Lemma 6.1. *Let $\mathcal{C}$ be a small monoidal category. Then the category of lax symmetric monoidal functors $\mathbf{LaxSymMon}(\mathcal{C}, \mathbf{Set})$ is a co-gs-monoidal category in a canonical way.*

The monoidal natural transformations are the co-total and co-functional arrows in this co-gs-monoidal category.

Proof: See Appendix E.1. □

By Lemma 6.1, the opposite category $\mathbf{LaxSymMon}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ is gs-monoidal in a canonical way. We think of these categories as gs-monoidal analogues of the functor categories in the usual Yoneda lemma. The gs-monoidal Yoneda embedding then reads as follows.

Proposition 6.2. *Let $\mathcal{C}$ be a small gs-monoidal category. Then there is a fully faithful oplax gs-monoidal functor*

$$\mathcal{Y} : \mathcal{C} \longrightarrow \mathbf{LaxSymMon}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$$

Proof: See Appendix E.2. □

6.2. Completeness for oplax cartesian categories

In this section, we consider the category $\mathbf{Preord}$ of preordered sets and monotone maps as a preorder-enriched cartesian monoidal category, and therefore in particular an oplax cartesian category.

Theorem 6.3 (bilax completeness to $\mathbf{Preord}$). *Let $\mathcal{C}$ be a locally small oplax cartesian category $\mathcal{C}$ and $f, g : X \to Y$ arrows in $\mathcal{C}$. Then we have that*

1) *$f \leq g$ if and only if $F(f) \leq F(g)$ for every colax bicartesian functor $F : \mathcal{C} \to \mathbf{Preord}$;*
2) *$f \approx g$ if and only if $F(f) \approx F(g)$ for every colax bicartesian functor $F : \mathcal{C} \to \mathbf{Preord}$.*

Since every colax bicartesian functor is in particular colax cartesian and colax opcartesian, the same statements hold for these classes of functors as well.

Proof: See Appendix E.3. □

Remark 6.4. By the usual Yoneda lemma, the proof of Theorem 6.3 boils down to showing that any hom-functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Preord}$ has a canonical colax bicartesian structure. However, the same proof *cannot* be adapted to a completeness result for gs-monoidal categories and bilax gs-monoidal functors $\mathcal{C} \to \mathbf{Set}$. Indeed, for a gs-monoidal category $\mathcal{C}$, the hom-functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ is a lax gs-monoidal functor with respect to the laxator of (10) and (11) if and only if $\mathcal{C}$ is cartesian, since for gs-monoidality the diagrams (12) and (13) would have to commute on the nose, implying that every arrow is total and functional.

The proof of Theorem 6.3 amounts to considering hom-functors of $\mathcal{C}$ as a category enriched in $\mathbf{Preord}$. As it turns out, one can also consider $\mathcal{C}$ as a category "almost enriched" in $\mathbf{Rel}$, where the composition relation

$$\mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is defined to be the relation that relates (f, g) on the left to h on the right if and only if $h \leq gf$, and with identity arrow

$$1 \to \mathcal{C}(X, X)$$

the relation that picks out id_X. We say that this makes $\mathcal{C}$ "almost" into a category enriched in $\mathbf{Rel}$, since although the associativity of composition holds on the nose, the unitality only holds laxly: with the obvious arrows, we have

$$\begin{tikzcd}
 & \mathcal{C}(X,X) \otimes \mathcal{C}(X,Y) \arrow[d] \\
\mathcal{C}(X,Y) \arrow[ur] \arrow[r, equal] \arrow[dr] \arrow[ur, phantom, "\leq", shift right=3ex] \arrow[dr, phantom, "\geq", shift left=3ex] & \mathcal{C}(X,Y) \\
 & \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Y) \arrow[u]
\end{tikzcd}$$

So although a hom-functor $\mathcal{C}(A, -) \colon \mathcal{C} \to \mathbf{Rel}$ preserves binary composition strictly (by strict associativity), it preserves identities only laxly (by the lower triangle), In general, let us say that $F : \mathcal{C} \to \mathbf{Rel}$ is *identity-lax*, if it is a lax functor that preserves binary composition on the nose, and identities at least laxly,

$$\mathrm{id}_{F(X)} \subseteq F(\mathrm{id}_X).$$

We then obtain the following.

Theorem 6.5. *Let $\mathcal{C}$ be a locally small oplax cartesian category $\mathcal{C}$ and $f, g : X \to Y$ arrows in $\mathcal{C}$. Then we have that*

1) *$f \leq g$ if and only if $F(f) \leq F(g)$ for every colax bicartesian identity-lax functor $F : \mathcal{C} \to \mathbf{Rel}$;*
2) *$f \approx g$ if and only if $F(f) \approx F(g)$ for every colax bicartesian identity-lax functor $F : \mathcal{C} \to \mathbf{Rel}$.*

Proof: See Appendix E.4. □

7. Conclusion and future works

Our paper opened with a a string-diagrammatic presentation of gs-monoidal categories and a few related structures, We then introduced their preorder-enriched extensions, in particular oplax cartesian categories. Our aim is to show how such categories fit in the current interest for the use of graphical formalisms in system description, and in fact represent a core language for discussing relations and partial functions. To this end, we propose two mechanisms showing how such categories naturally arise, in terms of Kleisli categories and span categories, thus providing a large number of potential case studies and systematising part of an admittedly large thread of current research. More importantly, we show how the two approaches can be related by means of a suitable lax functor (introducing along the way the novel notion of weak affinity), thus putting their connection on a firm ground.

Finally, we turn to considering completeness, showing how it works for oplax cartesian categories over $\mathbf{Rel}$, thus generalising [4]. The latter result can be considered as closely related to recent results on functorial semantics for (relational and partial) algebras, opening the way to further establish connections with e.g. [46] and [4] on the equational presentation of partial theories.

Future work will keep the focus on completeness with respect to $\mathbf{Rel}$, in order to strengthen it via a proper functor instead of a lax one, following preliminary ideas in [47]. We will also try and establish a stronger connection between the presentations using Kleisli and span categories, also addressing the cyclic case, and discussing the whole 2-categorical structure of $\mathbf{Span}(\mathcal{C})$ for a presentation of graph rewriting and of inequational deduction for relational algebras, investigated in a set-theoretical flavour in [3], [48] and recently (at least for the diagrammatic presentation of the monoidal closed case) in [49].

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Appendix A. Lax/colax/bilax/Frobenius monoidal functors

This section recalls the definitions of lax, colax, bilax, and Frobenius monoidal functors, see e.g. [21]. Throughout, $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories with tensor functor $\otimes$ and monoidal unit I, and we assume that $\otimes$ strictly associates without loss of generality in order to keep the diagrams simple. Left and right unitors are denoted by λ and ρ, respectively[4], and braidings by γ.

Definition A.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **lax monoidal** if it is equipped with a natural transformation

$$\psi\colon \otimes \circ (F \times F) \to F \circ \otimes$$

and an arrow $\psi_0\colon I \to F(I)$ such that the associativity diagrams

$$\begin{tikzcd} F(A) \otimes F(B) \otimes FC \arrow[r, "\mathrm{id} \otimes \psi_{B,C}"] \arrow[d, "\psi_{A,B} \otimes \mathrm{id}"'] & F(A) \otimes F(B \otimes C) \arrow[d, "\psi_{A,B\otimes C}"] \\ F(A \otimes B) \otimes F(C) \arrow[r, "\psi_{A\otimes B,C}"'] & F(A \otimes B \otimes C) \end{tikzcd}$$

and unitality diagrams commute

$$\begin{tikzcd} I \otimes F(A) \arrow[d, "\psi_0 \otimes \mathrm{id}"'] & F(A) \arrow[l, "\lambda_{FA}"'] \arrow[d, "F(\lambda_A)"] \\ F(I) \otimes F(A) \arrow[r, "\psi_{I,A}"'] & F(I \otimes A) \end{tikzcd} \qquad \begin{tikzcd} F(A) \otimes I \arrow[d, "\mathrm{id} \otimes \psi_0"'] & F(A) \arrow[l, "\rho_{FA}"'] \arrow[d, "F(\rho_A)"] \\ F(A) \otimes F(I) & F(A \otimes I) \arrow[l, "\psi_{A,I}"'] \end{tikzcd}.$$

In addition, F is said to be **lax symmetric monoidal** when the following diagram commutes

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "F(\gamma_{A,B})"] \arrow[d, "\psi_{A,B}"'] & F(B \otimes A) \arrow[d, "\psi_{B,A}"] \\ F(A) \otimes F(B) \arrow[r, "\gamma_{FA,FB}"'] & F(B) \otimes F(A) \end{tikzcd}$$

For example, if $\mathcal{C}$ is the terminal monoidal category with only one object I and id_I as the only arrow, then F is simply a monoid in $\mathcal{D}$. We do not spell out the following dual version in full detail.

Definition A.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **oplax monoidal** if it is equipped with a natural transformation

$$\phi\colon F \circ \otimes \to \otimes \circ (F \otimes F)$$

and a map $\phi_0\colon F(I) \to I$ satisfying axioms dual to those in Definition A.1. Similarly, a **oplax symmetric monoidal functor** is a oplax monoidal functor such that ϕ commutes with the braiding γ.

4. Strict unitality could also be assumed, but such a choice would make some diagrams potentially confusing.

We also have the notion of **strong symmetric monoidal functor**, which is a lax symmetric monoidal functor with invertible structure arrows, or equivalently an oplax monoidal functor with invertible structure arrows; and that of **strict symmetric monoidal functor**, in which the structure arrows are identities.

A monoid and comonoid structure on an object in a symmetric monoidal category often interact in a nice way, either such that they form a *bimonoid* or a *Frobenius monoid* (and sometimes both). The following definitions (see [21]) generalise these notions to functors.

lfunctor

Definition A.3. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bilax monoidal** if it is equipped with a lax monoidal structure ψ, ψ_0 and an oplax monoidal structure ϕ, ϕ_0 such that the following compatibility conditions hold

- **Braiding.** The following hexagon commutes

$$\begin{tikzcd}[column sep=tiny] & F(A\otimes B)\otimes F(C\otimes D) \arrow[dl, "\psi_{A\otimes B,C\otimes D}"'] \arrow[dr, "\phi_{A,B}\otimes\phi_{C,D}"] & \\ F(A\otimes B\otimes C\otimes D) \arrow[d, "F(\mathrm{id}\otimes\gamma\otimes\mathrm{id})"'] & & F(A)\otimes F(B)\otimes F(C)\otimes F(D) \arrow[d, "id\otimes\gamma\otimes\mathrm{id}"] \\ F(A\otimes C\otimes B\otimes D) \arrow[dr, "\phi_{A\otimes C,B\otimes D}"'] & & F(A)\otimes F(C)\otimes F(B)\otimes F(D) \arrow[dl, "\psi_{A,C}\otimes\psi_{B,D}"] \\ & F(A\otimes C)\otimes F(B\otimes D) & \end{tikzcd}$$

sec: strong and commutative monad

- **Unitality.** The following diagrams commute

$$\begin{tikzcd} I \arrow[r, "\psi_0"] \arrow[d, "\lambda_I"'] & F(I) \arrow[r, "F(\lambda_I)"] & F(I\otimes I) \arrow[d, "\phi_{I,I}"] \\ I\otimes I \arrow[rr, "\psi_0\otimes\psi_0"'] & & F(I)\otimes F(I) \end{tikzcd}$$

$$\begin{tikzcd} I & F(I) \arrow[l, "\phi_0"'] & F(I\otimes I) \arrow[l, "F(\lambda_I^{-1})"'] \\ I\otimes I \arrow[u, "\lambda_I^{-1}"] & & F(I\otimes I) \arrow[ll, "\phi_0\otimes\phi_0"] \arrow[u, "\psi_{I,I}"'] \end{tikzcd}$$

$$\begin{tikzcd} & F(I) \arrow[dr, "\phi_0"] & \\ I \arrow[ur, "\psi_0"] \arrow[rr, equal] & & I \end{tikzcd}$$

We also say that F is **bilax symmetric monoidal** if in addition both the lax and oplax structures are symmetric.

The following notion is due to Szlachányi [19, Definition 1.7], see also [20].

lfunctor

Definition A.4 (Frobenius monoidal functor)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **Frobenius monoidal** if it is equipped with a monoidal structure ψ, ψ_0 and an oplax monoidal structure ϕ, ϕ_0 such that the following diagrams commute

$$\begin{tikzcd} F(A\otimes B)\otimes F(C) \arrow[r, "\psi_{A\otimes B,C}"] \arrow[d, "\phi_{A,B}\otimes\mathrm{id}"'] & F(A\otimes B\otimes C) \arrow[d, "\phi_{A,B\otimes C}"] \\ F(A)\otimes F(B)\otimes F(C) \arrow[r, "\mathrm{id}\otimes\psi_{B,C}"'] & F(A)\otimes F(B\otimes C) \end{tikzcd}$$

$$\begin{tikzcd} F(A)\otimes F(B\otimes C) \arrow[r, "\psi_{A,B\otimes C}"] \arrow[d, "\mathrm{id}\otimes\phi_{B,C}"'] & F(A\otimes B\otimes C) \arrow[d, "\phi_{A\otimes B,C}"] \\ F(A)\otimes F(B)\otimes F(C) \arrow[r, "\psi_{A\otimes B,C}\otimes\mathrm{id}"'] & F(A\otimes B)\otimes F(C) \end{tikzcd}$$

There is again the obvious notion of **Frobenius symmetric monoidal** functor. In the main text, we also use all of these definitions for lax functors, to which they apply without further modifications.

Appendix B. Kleisli categories and commutative monads

Here we start by recalling the definition of Kleisli category of a monad.

Definition B.1. The **Kleisli category** $\mathcal{C}_T$ of a monad (T, μ, η) on a category $\mathcal{C}$ has as objects those of $\mathcal{C}$, and for every X and Y in $\mathcal{C}_T$ the hom-set

$$\mathcal{C}_T(X, Y) := \mathcal{C}(X, TY).$$

Writing $f^\sharp\colon X \to TY$ for the representative in $\mathcal{C}$ of an arrow $f\colon X \to Y$ in $\mathcal{C}_T$, composition of $f \in \mathcal{C}_T(X, Y)$ and $g \in \mathcal{C}_T(Y, Z)$ is defined as

$$(g \circ f)^\sharp := \mu_Z \circ T(g^\sharp) \circ f^\sharp.$$

In particular, we have $\mathrm{id}_X^\sharp = \eta_X$, since the naturality of η and the unitality equations for T imply

$$\mu_Y \circ T(f^\sharp) \circ \eta_X = \mu_Y \circ T(\eta_Y) \circ f^\sharp = f^\sharp,$$

and this shows the required equations $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$.

A *strength* and a *costrength* for a monad on a monoidal category are structures relating the monad with the tensor product of the category at least *in one direction*. A monad equipped with a strength is called a strong monad. This notion was introduced by Kock in [50], [51] as an alternative description of enriched monads. Strong monads have been successfully used in computer science, playing a fundamental role in Moggi's theory of computation [28], [29].

We recall these concepts in the following definitions.

Definition B.2. A **strong monad** (T, μ, η) on a symmetric monoidal category $\mathcal{C}$ is a monad (T, μ, η) on $\mathcal{C}$ together with a natural transformation

$$t_{X,Y}\colon X \otimes TY \to T(X \otimes Y),$$

called **strength**, such that the following diagrams commute for all objects X, Y and Z of $\mathcal{C}$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "id_X \otimes \eta_Y"] \arrow[dr, "\eta_{X\otimes Y}"'] & X \otimes T(Y) \arrow[d, "t_{X,Y}"] \\
& T(X \otimes Y)
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes T(X) \arrow[r, "t_{I,X}"] \arrow[dr, "\lambda^{-1}_{T(X)}"'] & T(I \otimes X) \arrow[d, "T(\lambda_X^{-1})"] \\
& T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \otimes T(Z) \arrow[rr, "t_{X\otimes Y,Z}"] \arrow[dr, "id_X \otimes t_{Y,Z}"'] & & T(X \otimes Y \otimes Z) \\
& X \otimes T(Y \otimes Z) \arrow[ur, "t_{X,Y\otimes Z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes T^2(Y) \arrow[r, "t_{X,T(Y)}"] \arrow[d, "id_X \otimes \mu_Y"'] & T(X \otimes T(Y)) \arrow[r, "T(t_{X,Y})"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
X \otimes T(Y) \arrow[rr, "t_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}$$

Example B.3. The list monad $T_{\text{list}} \colon \mathbf{Set} \to \mathbf{Set}$ is strong. Given two sets X and Y, the strength component

$$t_{X,Y} \colon X \times T_{\text{list}}(Y) \to T_{\text{list}}(X \times Y)$$

is given by the function assigning to an element $(x, [y_1, \dots, y_m]))$ of $X \times T_{\text{list}}(Y)$ the element $[(x, y_1), \dots, (x, y_m)]$ of $T_{\text{list}}(X \times Y)$.

In fact, *any* monad on the cartesian category $\mathbf{Set}$ is strong in a unique way, where the strength can be defined similarly to the strength of the list monad. We refer to [50], [51] for more details.

Remark B.4. The braiding γ of $\mathcal{C}$ let us define a **costrength** with components

$$t'_{X,Y} \colon T(X) \otimes Y \to T(X \otimes Y)$$

given by

$$t'_{X,Y} := T(\gamma_{Y,X}) \circ t_{Y,X} \circ \gamma_{X,TY}.$$

It satisfies axioms that are analogous to those of strength.

Definition B.5. A strong monad (T, μ, η) on a symmetric monoidal category $\mathcal{C}$ is said to be **commutative** if the following diagram commutes for every object X and Y

$$\begin{tikzcd}
T(X) \otimes T(Y) \arrow[r, "t_{T(X),Y}"] \arrow[d, "t'_{X,T(Y)}"'] & T(T(X) \otimes Y) \arrow[r, "T(t'_{X,Y})"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
T(X \otimes T(Y)) \arrow[r, "T(t_{X,Y})"'] & T^2(X \otimes Y) \arrow[r, "\mu_{X\otimes Y}"'] & T(X \otimes Y)
\end{tikzcd} \tag{7}$$

Remark B.6. It is well-known that on a symmetric monoidal category, commutative monads are equivalent to symmetric monoidal monads [50, Theorem 2.3]. Indeed, the diagonal of (7) equips the functor T with a lax symmetric monoidal structure, whose components we denote by $c_{X,Y} : TX \otimes TY \to T(X \otimes Y)$.

Appendix C. Omitted proofs, Section 2

For simplicity of notation, we assume without loss of generality that $\mathcal{C}$ is strict.

C.1. Proof of Proposition 2.10

The proofs of these statements are straightforward. We just show that both ! and ∇ are $\mathcal{C}$-total and $\mathcal{C}$-functional: every arrow $!_A$ is $\mathcal{C}$-total because $!_I !_A = \mathrm{id}_I !_A$ since $!_I = \mathrm{id}_I$ by Definition 2.1. Moreover $!_A$ is $\mathcal{C}$-functional because

$$(!_A \otimes !_A)\nabla_A = (!_A \otimes \mathrm{id}_I)(\mathrm{id}_A \otimes !_A)\nabla_A = !_A = \nabla_I !_A$$

by strictness of $\mathcal{C}$ and the axioms of Definition 2.1. The same calculation also shows that every duplicator ∇_A is $\mathcal{C}$-total. It is $\mathcal{C}$-functional by the first monoidal multiplicativity axiom combined with the commutative comonoid equations.

C.2. Proof of Proposition 2.12

1) By Proposition 2.10, $\mathcal{C}$-functional arrows are closed under composition and tensor product. Since ! and ∇ are $\mathcal{C}$-functional arrows and f is $\mathcal{C}$-functional by hypothesis, we have that $\mathrm{dom}(f) = (\mathrm{id}_A \otimes !_B f)\nabla_A$ is $\mathcal{C}$-functional.
2) If f is $\mathcal{C}$-total, then $!_B f = !_A$, hence
$$\mathrm{dom}(f) = (\mathrm{id}_A \otimes !_B f)\nabla_A = (\mathrm{id}_A \otimes !_A)\nabla_A = \mathrm{id}_A$$
by the second axiom of gs-monoidal categories. Conversely, if $\mathrm{dom}(f) = \mathrm{id}_A$, then $!_A \,\mathrm{dom}(f) = !_A$, hence $!_A(\mathrm{id}_A \otimes !_B f)\nabla_A = !_A$. Since $!_A(\mathrm{id}_A \otimes !_B f)\nabla_A = !_B f(!_A \otimes \mathrm{id}_A)\nabla_A = !_B f$, we can conclude that $!_B f = !_A$, i.e., that f is $\mathcal{C}$-total.

C.3. Proof of Proposition 2.15

1) Let us consider two objects A and B of $\mathcal{C}$ and show that $A \otimes B$ is a weak product in $\mathcal{C}$-**Total**, with projections $\mathrm{id}_A \otimes !_B \colon A \otimes B \to A$ and $!_A \otimes \mathrm{id}_B \colon A \otimes B \to B$. First notice that by Proposition 2.10 these projections are $\mathcal{C}$-total arrows, since they are monoidal products of $\mathcal{C}$-total arrows.
Now let us consider two $\mathcal{C}$-total arrows $f \colon C \to A$ and $g \colon C \to B$. We claim that the following diagram commutes
$$\begin{tikzcd}
A & A \otimes B \arrow[l, "\mathrm{id}_A \otimes !_B"'] \arrow[r, "!_A \otimes \mathrm{id}_B"] & B \\
& C \arrow[ul, "f"] \arrow[u, "(f\otimes g)\nabla_C"] \arrow[ur, "g"'] &
\end{tikzcd}$$

In particular, consider the left triangle. In this case

$$(\mathrm{id}_A \otimes !_B)(f \otimes g)\nabla_C = (f \otimes !_B g)\nabla_C,$$

and since g is total, that is $!_B g = !_C$, we get

$$(\mathrm{id}_A \otimes !_B)(f \otimes g)\nabla_C = (f \otimes !_C)\nabla_C = f.$$

Similarly one can check that $g = (!_A \otimes \mathrm{id}_B)(f \otimes g)\nabla_C$. The claim now follows since $(f \otimes g)\nabla_C$ is itself $\mathcal{C}$-total by Proposition 2.10.

2) For any two objects A and B of $\mathcal{C}$, we show that $A \otimes B$ is the categorical product of A and B in $\mathcal{C}$-**TFun** with projections $\mathrm{id}_A \otimes !_B \colon A \otimes B \to A$ and $!_A \otimes \mathrm{id}_B \colon A \otimes B \to B$. First, by the previous point, we know that this defines a weak product in $\mathcal{C}$-**Total**, and thus in particular in $\mathcal{C}$-**TFun**, since if f and g are $\mathcal{C}$-functional and $\mathcal{C}$-total, then again so is $(f \otimes g)\nabla_C$ by Proposition 2.10.
We still have to prove the uniqueness part of the universal property. So suppose that there exists another $\mathcal{C}$-total, $\mathcal{C}$-functional arrow $h \colon C \to A \otimes B$ such that $(\mathrm{id}_A \otimes !_B)h = f$ and $(!_A \otimes \mathrm{id}_B)h = g$. Then we have

$$(f \otimes g)\nabla_C = [(\mathrm{id}_A \otimes !_B)h \otimes (!_A \otimes \mathrm{id}_B)h]\,\nabla_C$$

and this is equal to

$$(\mathrm{id}_A \otimes !_B \otimes !_A \otimes \mathrm{id}_B)(h \otimes h)\nabla_C$$

Since h is $\mathcal{C}$-functional, we can further evaluate to

$$(f \otimes g)\nabla_C = (\mathrm{id}_A \otimes !_B \otimes !_A \otimes \mathrm{id}_B)\nabla_{A\otimes B} h = h$$

C.4. Proof of Proposition 3.6

1) By definition of oplax cartesian category, we have an inequality $!_B f \leq !_A$, and therefore

$$\mathrm{dom}(f) = (\mathrm{id}_A \otimes !_B f)\nabla_A \leq (\mathrm{id}_A \otimes !_A)\nabla_A = \mathrm{id}_A$$

2) The inequality $f\,\mathrm{dom}(f) \leq f$ follows by the previous point. We can prove the reverse $f \leq f\,\mathrm{dom}(f)$ by considering the following diagram

$$\begin{tikzcd}
B \arrow[r, "\nabla_B"] & B \otimes B \arrow[r, "id_{B\otimes B}"] & B \otimes B \arrow[r, "id_B \otimes !_B"] & B \\
A \arrow[u, "f"] \arrow[r, "\nabla_A"'] \arrow[ur, phantom, "\leq"] & A \otimes A \arrow[u, "f \otimes f"'] \arrow[r, "id_A \otimes f"'] & A \otimes B \arrow[u, "f \otimes id_B"'] \arrow[r, "id_A \otimes !_B"'] & A. \arrow[u, "f"']
\end{tikzcd}$$

The second and the third squares commute, while in the first one we have the first defining inequality of oplax cartesian category. Since $(\mathrm{id}_B \otimes !_B)\nabla_B = \mathrm{id}_B$ by the definition of gs-monoidal category, the upper composite is just f and we obtain the required $f \leq f\,\mathrm{dom}(f)$.

C.5. Proof of Proposition 3.7

Let us consider the two operators $!$ and $!'$. By definition of oplax cartesian category and, in particular, employing the second axiom combined with the gs-monoidal axiom $!_I = \mathrm{id}_I$, we have that since $\mathcal{C}$ is oplax cartesian with respct to ∇ and $!$ we can conclude that for every A object of $\mathcal{C}$, $!'_A \leq !_A$. We then conclude $!'_A \approx !_A$ by symmetry. With a similar argument we can prove that $\nabla_A \approx \nabla'_A$.

Appendix D. Omitted proofs, Section 4

D.1. Proof of Proposition 4.4

It is know that, under the current assumptions, the Kleisli category $\mathcal{A}_T$ is a symmetric monoidal category. Now let us consider an object A of the Kleisli category $\mathcal{A}_T$. As mentioned in the statement, we define the arrow $\nabla^\sharp_A \colon A \to A \otimes A$ of $\mathcal{A}_T$ as represented by the arrow

$$A \xrightarrow{\nabla_A} A \otimes A \xrightarrow{\eta_{A\otimes A}} T(A \otimes A)$$

of $\mathcal{A}$. Similarly, we define the arrow $!^\sharp_A \colon A \to I$ of $\mathcal{A}_T$ as represented by the arrow

$$A \xrightarrow{!_A} I \xrightarrow{\eta_I} TI.$$

Although it is possible now to verify directly that the axioms required of a gs-monoidal category are satisfied, there is a more concise and more insightful argument that works as follows.[5] By definition, duplicators and dischargers in $\mathcal{A}_T$ are defined as the images of those in $\mathcal{A}$ under the inclusion functor $\mathcal{A} \to \mathcal{A}_T$, which is strict symmetric monoidal. It now suffices to note that a strict symmetric monoidal functor maps commutative comonoids to commutative comonoids, and the monoidal multiplicativity conditions (2) transfer from $\mathcal{A}$ to $\mathcal{A}_T$ for the same reason. Therefore $\mathcal{A}_T$ is a gs-monoidal category.

D.2. Proof of Proposition 4.12

Let us consider an arrow $f \colon A \to B$ of $\mathcal{A}_T$. We first show that $\nabla_B \circ f \leq_{\mathcal{A}_T} (f \otimes f) \circ \nabla_A$. By definition of $\leq_{\mathcal{A}_T}$ this holds if $(\nabla_B \circ f)^\sharp \leq_{\mathcal{A}} ((f \otimes f) \circ \nabla_A)^\sharp$. Observe that

$$(\nabla_B \circ f)^\sharp = \mu_{B\otimes B} T(\eta_{B\otimes B}\nabla_B) f^\sharp = T(\nabla_B) f^\sharp$$

and applying the naturality of η it is direct to check that

$$((f \otimes f) \circ \nabla_A)^\sharp = c_{B,B}(f^\sharp \otimes f^\sharp)\nabla_A.$$

Employing first the assumption that T is a colax cartesian functor and then that $\mathcal{A}$ is oplax cartesian, we have that

$$T(\nabla_B) f^\sharp \leq c_{B,B} \nabla_{TB} f^\sharp \leq c_{B,B}(f^\sharp \otimes f^\sharp)\nabla_A.$$

Therefore we can conclude that $\nabla_B \circ f \leq_{\mathcal{A}_T} (f \otimes f) \circ \nabla_A$. Similarly, one can check that $!_B f \leq !_A$ in $\mathcal{A}_T$.

5. See [14, Corollary 3.2]: It was previously used for Markov categories.

D.3. Proof of Proposition 5.3

It is easy to check and well-known [41, Section 3.1][6] that $\mathbf{PSpan}(\mathcal{A})$ is gs-monoidal with respect to the above duplicators and dischargers. So, we just check that the axioms for oplax cartesianity hold in addition. Let us consider an arrow from X to Y in $\mathbf{PSpan}(\mathcal{A})$, i.e. $(X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)$. We have to show that the $\nabla^s_Y \circ (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)$ is less or equal to $((X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y) \otimes (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)) \circ \nabla^s_X$. First, note that by definition of composition in $\mathbf{PSpan}(\mathcal{A})$ we have that

$$\nabla_Y \circ (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y) = (X \overset{f}{\leftarrow} A \xrightarrow{\nabla_A g} Y \times Y)$$

and since $\mathcal{A}$ is cartesian, and hence $\nabla_Y g = (g \times g)\nabla_A f$ we have that

$$\nabla^s_Y \circ (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y) = (X \overset{f}{\leftarrow} A \xrightarrow{(g\times g)\nabla_Y} Y \times Y)$$

Now, employing the universal property of pullbacks, it is direct to check that $(X \overset{f}{\leftarrow} A \xrightarrow{(g\times g)\nabla_A} Y \times Y) \leq ((X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y) \otimes (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)) \circ \nabla_X$, using the naturality of ∇ in $\mathcal{A}$. Similarly we have the inequality

$$!^s_Y \circ (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y) \leq \, !^s_X$$

via the 2-cell obtained via f, since $!^s_Y \circ (X \overset{f}{\leftarrow} A \overset{g}{\rightarrow} Y)$ is equal to $(X \overset{f}{\leftarrow} A \xrightarrow{!_A} 1)$.

D: new I am checking again point 1) of Prop 4.16

D.4. Proof of Proposition 5.4

1) By the first axiom of oplax cartesian categories, it is enough to show that $\nabla^s_Y \circ (X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$ is greater or equal to $(X \times X \xleftarrow{f\times f} Z \times Z \xrightarrow{g\times g} Y \times Y) \circ \nabla^s_X$ if and only if f is mono. First we have that

$$\nabla^s_Y \circ (X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y) = (X \overset{f}{\leftarrow} Z \xrightarrow{\nabla_Y g} Y \times Y),$$

while $(X \times X \xleftarrow{f\times f} Z \times Z \xrightarrow{g\times g} Y \times Y) \circ \nabla^s_X$ is given by the composite span in

$$\begin{tikzcd}
 & & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\nabla_X"] & & Z \times Z \arrow[dl, "f\times f"'] \arrow[dr, "g\times g"] & \\
X & & X \times X & & Y \times Y
\end{tikzcd}$$

Therefore, we have that $(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$ is weakly $\mathbf{PSpan}(\mathcal{A})$-functional if and only if the commutative square (in $\mathcal{A}$)

$$\begin{tikzcd}
Z \arrow[r, "\nabla_Z"] \arrow[d, "f"'] & Z \times Z \arrow[d, "f\times f"] \\
X \arrow[r, "\nabla_X"'] & X \times X
\end{tikzcd} \tag{8}$$

is a pullback. It is direct to check that (8) is a pullback if and only if f is a monomorphism, hence we can conclude that $(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$ is weakly $\mathbf{PSpan}(\mathcal{A})$-functional if and only if f is a monomorphism.

2) Notice that $!^s_Y \circ (X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y) = (X \overset{f}{\leftarrow} Z \xrightarrow{!_Z} 1)$. Hence we have that $!^s_Y \circ (X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$ is greater or equal (and hence, by the second axiom of oplax cartesian categories, preorder equivalent) to $!^s_X$ if and only if there exists an arrow $h \colon X \to Z$ such that $fh = \mathrm{id}_X$, i.e. if and only if f is a split epimorphism.

D.5. Proof of Proposition 5.8

Let $m_{X,Y} : T(X \times Y) \longrightarrow TX \times TY$ be the arrow defined as the pairing of $T(\pi_1)$ and $T(\pi_2)$. Then it is known that T is affine if and only if $m_{X,Y} c_{X,Y} = \mathrm{id}_{TX \times TY}$ [39, Lemma 4.2(i)].[7] In particular, $c_{X,Y}$ is a split mono and therefore mono.

To show that (4) is a pullback, we prove the universal property starting with a diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "(f_1,f_2)"] \arrow[ddr, bend right, "(g_1,g_2)"'] & & \\
 & TX \times TY \times TZ \arrow[r, "\mathrm{id}\times c_{Y,Z}"] \arrow[d, "c_{X,Y}\times \mathrm{id}"'] & TX \times T(Y \times Z) \arrow[d, "c_{X,Y\times Z}"] \\
 & T(X \times Y) \times TZ \arrow[r, "c_{X\times Y,Z}"] & T(X \times Y \times Z)
\end{tikzcd} \tag{9}$$

where the dashed arrow will be constructed; its uniqueness is clear since $\mathrm{id} \times c_{Y,Z}$ and $c_{X,Y} \times \mathrm{id}$ are mono, so it remains to prove existence. Taking the unlabelled arrows to be (induced by) product projections, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "(g_1,g_2)"] \arrow[d, "(f_1,f_2)"'] & T(X \times Y) \times TZ \arrow[r] & T(X \times Y \times Z) \arrow[d] \\
TX \times T(Y \times Z) \arrow[urr, "c_{X,Y\times Z}"] \arrow[rr] & & T(Y \times Z)
\end{tikzcd}$$

where the upper left triangle commutes by assumption, and the lower right triangle commutes by naturality of c with respect to the unique arrow $X \to 1$ together with $T1 \cong 1$ and the fact that $c_{1,Y\times Z}$ is a coherence isomorphism. By the naturality of c, f_2 can be written as the composite

$$A \xrightarrow{(g_1,g_2)} T(X \times Y) \times TZ \longrightarrow TY \times TZ \xrightarrow{c_{Y,Z}} T(Y \times Z).$$

By analogous reasoning, we identify g_1 with the composite

$$A \xrightarrow{(f_1,f_2)} TX \times T(Y \times Z) \longrightarrow TX \times TY \xrightarrow{c_{X,Y}} T(X \times Y).$$

6. Although that reference only considers spans in $\mathbf{Set}$, the proofs go through for any cartesian category $\mathcal{A}$.

7. For probability monads, this equation can be interpreted as stating that the marginals of a product distribution are the original factors [52].

Getting back to (9), candidate_pullback we take the dashed arrow to be the arrow whose component on TX is given by f_1, on TZ by g_2, and on TY by the diagonal in the diagram

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "g_1"'] & T(Y\times Z) \arrow[d] \\ T(X\times Y) \arrow[r] & TY \end{tikzcd}$$

which commutes for similar reasons as above. The fact that this arrow recovers the f_2 component after composition with $\mathrm{id}\times c_{Y,Z}$ and the g_1 component after composition with $c_{X,Y}\times\mathrm{id}$ follows by the expressions for f_2 and g_1 derived above. The fact that it recovers f_1 and g_2 is by construction.

D.6. Proof of Lemma 5.14

lem:weakly affine monad useful pullback

pullback

Commutativity of the diagram can be seen by unfolding the definition of $c_{X,Y}$ in terms of the strength and multiplication of T and applying the compatibility of these with the unit. In the diagram

$$\begin{tikzcd} X\times Y \arrow[r, "\mathrm{id}"] \arrow[d, "\eta_X\times\eta_Y"'] & X\times Y \arrow[d, "\eta_X\times\eta_Y"] \\ TX\times TY \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & TX\times TY \arrow[d, "c_{X,Y}"] \\ TX\times TY \arrow[r, "c_{X,Y}"] & T(X\times Y) \end{tikzcd}$$

the upper square is a trivial pullback and the lower square is a pullback since $c_{X,Y}$ is mono. Therefore also the composite square is a pullback. This composite square is exactly (6).

D.7. Proof of Theorem 5.15

thm: S is colax Frobenius opcartesian lax-functor

functor

We show that S_T is a *lax functor*: let us consider two arrows $f\colon X\to Y$ and $g\colon Y\to Z$ of $\mathcal{A}_T$, represented by $f^\sharp\colon X\to TY$ and $g^\sharp\colon Y\to TZ$ in $\mathcal{A}$. The span $S_T(g)\circ S_T(f)$ is represented by the composite in

$$\begin{tikzcd} & & X\times_{f^\sharp,\eta_Y} Y \arrow[dl, "\ell"'] \arrow[dr, "r"] & & \\ & X \arrow[dl, "\eta_X"'] \arrow[dr, "f^\sharp"] & & Y \arrow[dl, "\eta_Y"'] \arrow[dr, "g^\sharp"] & \\ TX & & TY & & TZ \end{tikzcd}$$

where the upper square is a pullback. On the other hand, the span $S_T(g\circ f)$ is $(TX \xleftarrow{\eta_X} X \xrightarrow{\mu_Z T(g^\sharp)f^\sharp} TZ)$. Now notice that the diagram

$$\begin{tikzcd} & X\times_{f^\sharp,\eta_Y} Y \arrow[dl, bend right, "\eta_X\ell"'] \arrow[dr, bend left, "g^\sharp r"] \arrow[dd, "\ell"] & \\ TX & & TZ \\ & X \arrow[ul, bend left, "\eta_X"] \arrow[ur, bend right, "\mu_Z T(g^\sharp)f^\sharp"'] & \end{tikzcd}$$

commutes, since $\mu_Z T(g^\sharp)f^\sharp\ell = \mu_Z T(g^\sharp)\eta_Y r$ and by the naturality of η, we can conclude the desired $\mu_Z T(g^\sharp)f^\sharp\ell = (\mu_Z\eta_{TZ})g^\sharp r = g^\sharp r$. Therefore we obtain that $S_T(g)\circ S_T(f)\le S_T(g\circ f)$.

Similarly, it is easy to check that $\mathrm{id}_{S_T(X)}\le S_T(\mathrm{id}_X)$ in $\mathbf{PSpan}(\mathcal{A})$. Thus $S_T\colon\mathcal{A}_T\to\mathbf{PSpan}(\mathcal{A})$ is a lax functor.

Now we show that S_T is *lax symmetric monoidal*, with the structure arrows given by the spans

$$\psi_0 := (1 \xleftarrow{\mathrm{id}} 1 \xrightarrow{\eta_1} T(1)),$$
$$\psi_{X,Y} := (TX\times TY \xleftarrow{\mathrm{id}} TX\times TY \xrightarrow{c_{X,Y}} T(X\times Y)).$$

We first prove that ψ is a natural transformation. Let us consider two arrows $f\colon X\to W$ and $g\colon Y\to Z$ of $\mathcal{A}_T$. We have to check that

$$\psi_{W,Z}\circ(S_T(f)\otimes S_T(g)) = S_T(f\otimes g)\circ\psi_{X,Y}.$$

First notice that $\psi_{W,Z}\circ(S_T(f)\otimes S_T(g))$ is

$$(TX\times TY \xleftarrow{\eta_X\times\eta_Y} X\times Y \xrightarrow{c_{Z,W}(f^\sharp\times g^\sharp)} T(Z\times W)).$$

Now, since the monad T is weakly affine by hypothesis, we have that $S_T(f\otimes g)\circ\psi_{X,Y}$ is given precisely by the composite span in

$$\begin{tikzcd} & & X\times Y \arrow[dl, "\eta_X\times\eta_Y"'] \arrow[dr, "\mathrm{id}"] & & \\ & TX\times TY \arrow[dl, "\mathrm{id}"'] \arrow[dr, "c_{X,Y}"] & & X\times Y \arrow[dl, "\eta_{X\times Y}"'] \arrow[dr, "c_{Z,W}(f^\sharp\times g^\sharp)"] & \\ TX\times TY & & T(X\times Y) & & T(Z\times W) \end{tikzcd}$$

weakly affine ~~and then~~ we can conclude that $\psi_{W,Z}\circ(S_T(f)\otimes S_T(g)) = S_T(f\otimes g)\circ\psi_{X,Y}$. So ψ is a natural transformation.

The associativity and unitality diagrams of a lax monoidal structure (Definition A.1) def:lax monoidal functor follow from the fact that ψ is given by the composition of strength arrows and multiplications. Therefore ψ equips S_T with the structure of a lax symmetric monoidal functor.

Now we show that S_T is *oplax symmetric monoidal*, with the structure arrows given by the spans

$$\phi_0 := (T(1) \xleftarrow{\eta_1} 1 \xrightarrow{\mathrm{id}} 1)$$
$$\phi_{X,Y} := (T(X\times Y) \xleftarrow{c_{X,Y}} TX\times TY \xrightarrow{\mathrm{id}} T(X)\times T(Y)).$$

We first prove that ϕ is a natural transformation. Let us consider two arrows $f\colon X\to W$ and $g\colon Y\to Z$ of $\mathcal{A}_T$. We have to check that

$$\phi_{W,Z}\circ S_T(f\otimes g) = (S_T(f)\otimes S_T(g))\circ\phi_{X,Y}.$$

First notice that, since $c_{X,Y}(\eta_X\times\eta_Y) = \eta_{X\times Y}$, we have that $(S_T(f)\otimes S_T(g))\circ\phi_{X,Y}$ is represented by the span

$$T(X\times Y) \xleftarrow{\eta_{X\times Y}} X\times Y \xrightarrow{f^\sharp\times g^\sharp} TW\times TZ.$$

Now since the monad T is weakly affine, the span $\phi_{W,Z}\circ S_T(f\otimes g)$ is given precisely by the composite span in

$$
\begin{tikzcd}[column sep=small]
& & X\times Y \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f^\sharp\times g^\sharp"] \arrow[dd, phantom, "\lrcorner" rotate=-45, near start] & & \\
& X\times Y \arrow[dl, "\eta_{X\times Y}"'] \arrow[dr, "c_{W,Z}(f^\sharp\times g^\sharp)"'] & & TW\times TZ \arrow[dl, "c_{W,Z}"'] \arrow[dr, "\mathrm{id}"] & \\
T(X\times Y) & & T(W\times Z) & & TW\times TZ
\end{tikzcd}
$$

and then $\phi_{W,Z}\circ S_T(f\otimes g) = (S_T(f)\otimes S_T(g))\circ\phi_{X,Y}$. As for the lax structure, the associativity and unitality diagrams of an oplax monoidal structure follow from the fact that ϕ is given by the composition of strength arrows and multiplications. Therefore ϕ equips S_T with the structure of an oplax symmetric monoidal functor.

Now we show that S_T is a *Frobenius monoidal lax functor*. By the definition of Frobenius monoidality (Definition def:frobenius_monoidalfunctor A.4), we have to prove that the two diagrams

$$
\begin{tikzcd}
S_T(X\times Y)\otimes S_T(Z) \arrow[r, "\psi_{X\otimes Y,Z}"] \arrow[d, "\phi_{X,Y}\otimes\mathrm{id}"'] & S_T(X\times Y\times Z) \arrow[d, "\phi_{X,Y\otimes Z}"] \\
S_T(X)\otimes S_T(Y)\otimes S_T(Z) \arrow[r, "\mathrm{id}\otimes\psi_{Y,Z}"'] & S_T(X)\otimes S_T(Y\times Z)
\end{tikzcd}
$$

$$
\begin{tikzcd}
S_T(X)\otimes S_T(Y\times Z) \arrow[r, "\psi_{X,Y\otimes Z}"] \arrow[d, "\mathrm{id}\otimes\phi_{Y,Z}"'] & S_T(X\times Y\times Z) \arrow[d, "\phi_{X\otimes Y,Z}"] \\
S_T(X)\otimes S_T(Y)\otimes S_T(Z) \arrow[r, "\psi_{X\otimes Y,Z}\otimes\mathrm{id}"'] & S_T(X\times Y)\otimes S_T(Z)
\end{tikzcd}
$$

commute in $\mathbf{PSpan}(\mathcal{A})$. Let us consider the first square. By definition of ψ and ϕ, the arrow $(\mathrm{id}\otimes\psi_{Y,Z})\circ(\phi_{X,Y}\otimes\mathrm{id})$ is given by the span

$$T(X\times Y)\times TZ \xleftarrow{c_{X,Y}\times\mathrm{id}} TX\times TY\times TZ \xrightarrow{\mathrm{id}\times c_{Y,Z}} TX\times T(Y\times Z).$$

Employing the assumption that T is weakly affine, and in particular that the diagram (c_assoc_pullback 4) is a pullback, we can conclude that the composite $\phi_{X,Y\otimes Z}\circ\psi_{X\otimes Y,Z}$ is given by the same span and hence that the first square commutes. By analogous reasoning, we can conclude that also the second square commutes, and hence that S_T is Frobenius monoidal.

Now we show that S_T is a *colax cartesian lax functor*: employing the fact that $\mathcal{A}$ is cartesian, and hence that ∇ is natural, it is straightforward to check that in $\mathbf{PSpan}(\mathcal{A})$ we have the inequality

$$S_T(\nabla_X)\le\psi_{X,X}\circ\nabla_{S_T(X)}$$

Similarly, it is straightforward to check that

$$S_T(!_A)\le\psi_0\circ !_{S_TA}.$$

Consider now the two spans

$$\nabla_{S_T(X)} := (TX \xleftarrow{\mathrm{id}_X} TX \xrightarrow{\nabla_{TX}} TX\times TX),$$
$$S_T(\nabla_X) := (TX \xleftarrow{\eta_X} X \xrightarrow{\eta_{X\times X}\nabla_X} T(X\times X))$$

proof:lax_is_cogs

In $\mathbf{PSpan}(\mathcal{A})$, we will prove the inequality

$$S_T(\nabla_X)\le\psi_{X,X}\circ\nabla_{S_T(X)}$$

as witnessed by the unit η_X. Indeed, we have that

$$\psi_{X,X}\circ\nabla_{S_T(X)} = (TX \xleftarrow{\mathrm{id}_X} TX \xrightarrow{c_{X,X}\nabla_{TX}} TX\times TX)$$

and, employing the naturality of ∇ we have that

$$c_{X,X}\nabla_{TX}\eta_X = c_{X,X}(\eta_X\times\eta_X)\nabla_X$$

And since $c_{X,X}(\eta_X\times\eta_X) = \eta_{X\times X}$, we have that the following diagram commute

$$
\begin{tikzcd}
& X \arrow[dl, bend right, "\eta_X"'] \arrow[dd, "\eta_X"] \arrow[dr, bend left, "\eta_{X\times X}\nabla_X"] & \\
TX & & T(X\times X) \\
& TX \arrow[ul, bend left, "\mathrm{id}"] \arrow[ur, bend right, "c_{X,X}\nabla_{TX}"'] &
\end{tikzcd}
$$

It is immediate to check that

$$S_T(!_X)\le\psi_0\circ !_{S_TX}$$

Thus we can conclude that S_T is a colax cartesian functor.

Now we show that S_T is a *colax opcartesian lax functor*: we must check that

$$\phi_{X,X}\circ S_T(\nabla_X)\le\nabla_{S_T(X)},$$

and that

$$\phi_0\circ S_T(!_A)\le !_{S_TA}.$$

First notice that $\phi_{X,X}\circ S_T(\nabla_X)$ is given the span

$$\phi_{X,X}\circ S_T(\nabla_X) := (TX \xleftarrow{\eta_X} X \xrightarrow{(\eta_X\times\eta_X)\nabla_X} TX\times TX)$$

obtained by the composition

$$
\begin{tikzcd}[column sep=small]
& & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "(\eta_X\times\eta_X)\nabla_X"] \arrow[dd, phantom, "\lrcorner" rotate=-45, near start] & & \\
& X \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_{X\times X}\nabla_X"'] & & TX\times TX \arrow[dl, "c_{X,X}"'] \arrow[dr, "\mathrm{id}"] & \\
TX & & T(X\times X) & & TX\times TX
\end{tikzcd}
$$

while $\nabla_{S_T(X)} := (TX \xleftarrow{\mathrm{id}_X} TX \xrightarrow{\nabla_{TX}} TX\times TX)$. Employing the fact that $\mathcal{A}$ is cartesian, and hence every arrow is $\mathcal{A}$-functional, it is direct to check that $(\eta_X\times\eta_X)\nabla_X = \nabla_{TX}\eta_X$, and then that $\phi_{X,X}\circ S_T(\nabla_X)\le\nabla_{S_T(X)}$ via η_X. Similarly, employing the fact that every arrow in a cartesian category is also $\mathcal{A}$-total, it is direct to check that $\phi_0\circ S_T(!_A)\le !_{S_TA}$.

Appendix E. Omitted proofs, Section sec:completeness 6

E.1. Proof of Lemma lax_is_cogs 6.1

For any $F:\mathcal{C}\to\mathbf{Set}$, there is an equivalence between lax symmetric monoidal structures on F and commutative monoid structures with respect to Day convolution, in such a way that the monoidal natural transformations are in natural bijection with the monoid homomorphisms[8] Thus it suffices to show that the category of commutative monoids in the symmetric monoidal category of functors under Day convolution is co-gs-monoidal in a canonical way. But this latter statement is an instance of the fact that the category of commutative monoids in *any* symmetric monoidal category is a co-gs-monoidal category in a canonical way (Example 2.4).

E.2. Proof of Proposition 6.2

On objects, we define $\mathcal{Y}(A) \coloneqq \mathcal{C}(A,-)$, which is a lax monoidal functor $\mathcal{C} \to \mathbf{Set}$: It can be proved directly, but we we will show it in more abstract terms in the proof of Theorem 6.3. The action of $\mathcal{Y}$ on an arrow $f : B \to A$ is given by precomposition, and this defines a natural transformation $\mathcal{C}(B,-) \to \mathcal{C}(A,-)$. The full faithfulness of $\mathcal{Y}$ holds by the standard Yoneda embedding.

It remains to equip $\mathcal{Y}$ with a oplax gs-monoidal structure, recalling that $\mathbf{LaxSymMon}(\mathcal{C},\mathbf{Set})^{\mathrm{op}}$ carries the gs-monoidal structure introduced in Lemma 6.1. For the oplaxator, note that we have a transformation

$$\begin{aligned} \mathcal{C}(A,X)\otimes\mathcal{C}(B,Y) &\longrightarrow \mathcal{C}(A\otimes B, X\otimes Y)\\ (f,g) &\longmapsto f\otimes g \end{aligned}$$

that is natural in all four arguments $A,B,X,Y \in \mathcal{C}$. By the universal property of Day convolution, this can be regarded as an arrow

$$\mathcal{C}(A,-)\boxtimes\mathcal{C}(B,-) \longrightarrow \mathcal{C}(A\otimes B,-)$$

in $\mathbf{LaxSymMon}(\mathcal{C},\mathbf{Set})$. Its naturality in A and B also follows by the universal property. As we will see, considering these transformations in the opposite category defines the comultiplication of the claimed oplax monoidal structure, while the counit $\mathcal{C}(I,-) \to \mathcal{Y}(I)$ is given by the identity transformation $\mathcal{C}(I,-) \to \mathcal{C}(I,-)$. The coassociativity equation for the comultiplication amounts exactly to the associativity of the monoidal structure of $\mathcal{C}$. The left unitality equation holds by the commutativity of the diagram

$$\begin{tikzcd} & \mathcal{C}(I,-)\boxtimes\mathcal{C}(A,-) \arrow[dl, "\text{unitor of Day convolution}"'] \arrow[dr, "\text{laxator}"] & \\ \mathcal{C}(A,-) \arrow[rr, "\text{induced by unitor in } \mathcal{C}"'] & & \mathcal{C}(I\otimes A,-) \end{tikzcd}$$

and similarly for right unitality.

The preservation of the duplicators and dischargers amounts to the diagram

$$\begin{tikzcd} & \mathcal{C}(A,-)\boxtimes\mathcal{C}(A,-) \arrow[dl, "\text{lax structure on } \mathcal{C}(A,-)"'] \arrow[dr, "\text{laxator}"] & \\ \mathcal{C}(A,-) & & \mathcal{C}(A\otimes A,-) \arrow[ll, "\mathcal{C}(\nabla_A,-)"'] \end{tikzcd}$$

which holds by definition of the lax structure on $\mathcal{C}(A,-)$.

8. See e.g. [45, Example 3.2.2], and [53, Proposition 22.1] for a version of the statement for presheaves with values in topological spaces.

E.3. Proof of Theorem 6.3

By the usual Yoneda lemma, it is enough to show that every hom-functor $\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Preord}$ has a canonical colax bicartesian structure.

First we show that every functor $\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Preord}$ is colax cartesian: the lax symmetric monoidal structure is given by, in components,

$$\psi_{X,Y} \ : \ \begin{aligned} \mathcal{C}(A,X)\times\mathcal{C}(A,Y) &\longrightarrow \mathcal{C}(A,X\otimes Y)\\ (f,g) &\longmapsto (f\otimes g)\circ\nabla_A \end{aligned} \tag{10}$$

and

$$\psi_0 \ : \ \begin{aligned} 1 &\longrightarrow \mathcal{C}(A,I)\\ \bullet &\longmapsto\ !_A \end{aligned} \tag{11}$$

It is straightforward to verify that this assignment is natural in X and Y and satisfies the relevant coherences. The preservation of the braiding holds by the commutativity assumption on ∇_A. This makes $\mathcal{C}(A,-)$ lax symmetric monoidal. The claim that it is colax cartesian then amounts to the inequalities

$$\begin{tikzcd} \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(A,\nabla_X)"] \arrow[dr, "\nabla_{\mathcal{C}(A,X)}"'] & \arrow[d, phantom, "\leq"] & \mathcal{C}(A,X\otimes X) \\ & \mathcal{C}(A,X)\times\mathcal{C}(A,X) \arrow[ur, "\psi_{X,Y}"'] & \end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd} \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(A,!_X)"] \arrow[dr, "!_{\mathcal{C}(A,X)}"'] & \arrow[d, phantom, "\leq"] & \mathcal{C}(A,I) \\ & 1 \arrow[ur, "\psi_0"'] & \end{tikzcd} \tag{13}$$

which hold since they are precisely the defining inequalities of colax cartesianity.

Now we show that every functor $\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Preord}$ is colax opcartesian: the oplax symmetric monoidal structure is given by, in components,

$$\phi_{X,Y} \ : \ \begin{aligned} \mathcal{C}(A,X\otimes Y) &\longrightarrow \mathcal{C}(A,X)\times\mathcal{C}(A,Y)\\ f &\longmapsto ((\mathrm{id}_X\otimes !_Y)\circ f, (!_X\otimes\mathrm{id}_Y)\circ f) \end{aligned}$$

and

$$\phi_0 \ : \ \begin{aligned} \mathcal{C}(A,I) &\longrightarrow 1\\ f &\longmapsto \bullet \end{aligned}$$

It is straightforward to check that $\phi_{X,Y}$ is natural in X and Y and that the coherence axioms for colax monoidal functors are satisfied. This makes $\mathcal{C}(A,-)$ colax opcartesian since the inequalities

$$\begin{tikzcd} \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(A,\nabla_X)"] \arrow[dr, "\nabla_{\mathcal{C}(A,X)}"'] & \arrow[d, phantom, "\leq"] & \mathcal{C}(A,X\otimes X) \arrow[dl, "\phi_{X,Y}"] \\ & \mathcal{C}(A,X)\times\mathcal{C}(A,X) & \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(A,!_X)"] \arrow[dr, "!_{\mathcal{C}(A,X)}"'] & \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \mathcal{C}(A,I) \arrow[dl, "\phi_0"] \\
& 1 &
\end{tikzcd}$$

hold even with equality.

Finally, we show that $\mathcal{C}(A,-)$ is a bilax monoidal functor in the sense of Definition A.3. We first check the braiding axiom. Let us consider two arrows $f\colon A \to W \otimes X$ and $g\colon A \to Y \otimes Z$ of $\mathcal{C}$. We have to show that

$$\phi_{W\otimes Y, X\otimes Z} \circ \mathcal{C}(A, \mathrm{id}_W \otimes \gamma \otimes \mathrm{id}_Z) \circ \psi_{W\otimes X, Y\otimes Z}(f,g) \quad (14)$$

is equal to

$$(\psi_{W,Y} \times \psi_{X,Z}) \circ (\mathrm{id}_W \times \gamma \times \mathrm{id}_Z) \circ (\phi_{W,X} \times \phi_{Y,Z})(f,g) \quad (15)$$

Straightforward evaluation of both sides shows that indeed both (14) and (15) are equal to the following element

$$((\mathrm{id}_W \otimes !_X \otimes \mathrm{id}_Y \otimes !_Z)(f \otimes g)\nabla_A, (!_W \otimes \mathrm{id}_X \otimes !_Y \otimes \mathrm{id}_Z)(f \otimes g)\nabla_A)$$

of $\mathcal{C}(A, W \otimes Y) \times \mathcal{C}(A, X \otimes Z)$. Finally, we check the unitality axioms. The first axiom follows from the fact that both the arrows

$$\phi_{I,I} \circ \mathcal{C}(A, \lambda_I) \circ \psi_0 \colon 1 \to \mathcal{C}(A,I) \times \mathcal{C}(A,I)$$

and

$$(\psi_0 \otimes \psi_0) \circ \lambda_I \colon 1 \to \mathcal{C}(A,I) \times \mathcal{C}(A,I)$$

act as $\bullet \mapsto (!_A, !_A)$. Moreover, the second and third axiom hold because 1 is terminal in **Preord**.

E.4. Proof of Theorem 6.5

In order to reduce as much as possible of this proof to the previous one, we construct an identity-lax functor

$$H\colon \mathbf{Preord} \to \mathbf{Rel}$$

given by assigning to every preordered set P its underlying set, and to a monotone map $f : P \to Q$ its *hypograph*,

$$H(f) := \{(p,q) \in P \times Q \mid q \leq f(p)\}.$$

This construction clearly preserves and reflects the order on the hom-set $\mathbf{Preord}(P,Q)$,

$$f \leq g \iff H(f) \subseteq H(g). \quad (16)$$

To see that this defines an identity-lax functor $H : \mathbf{Preord} \to \mathbf{Rel}$, let us first show that for $f\colon P \to Q$ and $g\colon Q \to R$, the composition is indeed preserved strictly, i.e.

$$H(g \circ f) = H(g) \circ H(f).$$

This is easily proved by showing both containments: if $p \in P$ and $r \in R$ are such that $p(H(g) \circ H(f))r$, then this means that there is $q \in Q$ such that

$$q \leq f(p), \qquad r \leq g(q).$$

But then applying monotonicity of g and transitivity of $\leq$ implies $r \leq g(f(p))$, which gives the desired $pH(g \circ f)r$. On the other hand, if $pH(g \circ f)r$ holds, then we can simply take $q := f(p)$ as our witness of $p(H(g) \circ H(f))r$, since then both of the above inequalities are satisfied.

The lax preservation of identities $\mathrm{id}_P \subseteq H(\mathrm{id}_P)$ is easy to see, and it is worth noting that equality does not hold unless P is discrete.

As for other properties, it is also straightforward to see that H is strictly monoidal. Also, we have $H(!_P) = !_{H(P)}$ and[9] $\nabla_{H(P)} \subseteq H(\nabla_P)$.

T: not sure which of those things are needed exactly

However, the most important point is that composing the hom-functor $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathbf{Preord}$ with H produces the hom-functor $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathbf{Rel}$ mentioned in the main text. We can hence define a lax and oplax structure (co)multiplication $\hat{\psi}_{X,Y}$ and $\hat{\phi}_{X,Y}$ on $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathbf{Rel}$ by taking the hypographs of the $\psi_{X,Y}$ and $\phi_{X,Y}$ from the previous proof. By the fact that H preserves binary composition, it follows that this makes all the relevant diagrams commute that involve those (as opposed to the unit and counit), including their naturality diagrams and the braiding diagram.

However, the unit relation $\psi_0\colon 1 \to \mathcal{C}(A,I)$ should *not* be given by the hypograph of ψ_0, since that would *not* make the unitality diagrams commute (as one may suspect from the fact that H does not preserve identities strictly).

T: That's the problem with this approach

T: Alternatively, one could try to keep $\hat{\psi}_{X,Y}$ as above and define $\hat{\psi}_0$ as the functional relation $1 \to \mathcal{C}(A,I)$ which simply picks out $!_A$, but then the unitality diagram still doesn't commute.

T: Another attempt is to define $\hat{\psi}_{X,Y}$ as the functional relation that corresponds to $\psi_{X,Y}$ itself, as proposed in the original version of the paper. This also doesn't work since then the colax cartesianity fails: if you look at the first colax cartesianity diagram, the lower composite is a functional relation, so it's pretty clear that it will not contain the upper non-functional relation. It doesn't satisfy the opposite inequality (that of a "lax cartesian" functor) either, since that would require $(f \otimes f) \circ \nabla \leq \nabla \circ f$, which is the wrong way around

9. This again holds with equality only if and only if X is discrete.