

# Weakly-affine monads

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## Abstract

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## 1 Introduction

For context:

► **Proposition 1.1.** *A monoid  $(M, \cdot, 1)$  is a group if and only if the associativity square*

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\cdot \times \text{id}} & M \times M \\ \downarrow \text{id} \times \cdot & & \downarrow \cdot \\ M \times M & \xrightarrow{\cdot} & M \end{array} \quad (1)$$

*is a pullback.*

**Proof.** The square (1) is a pullback, both of sets and of groups, if and only if given  $a, g, h, c \in M$  such that  $ag = hc$ , there exists a unique  $b \in M$  such that  $g = bc$  and  $h = ab$ . First, suppose that  $g$  is a group. The only possible choice of  $b$  is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (1) is a pullback. We can set  $g, h = e$  and  $c = a$  so that  $ae = ea = a$ . Instantiating the pullback property, there is a unique  $b$  such that  $ab = e$  and  $ba = e$ , that is,  $b = a^{-1}$ . ◀

Recall that a monoidal functor generalizes a monoid object (in turn, generalizing a monoid). Similarly, a *weakly affine monoidal functor* generalizes a group in the sense of the proposition above.

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► **Definition 2.1.** Let  $T$  be a commutative monad on a category  $\mathcal{A}$  with finite products. A triple  $(X, Y, Z)$  of objects of  $\mathcal{A}$  is said to be **TBA** if the commutative square

$$\begin{array}{ccc} T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\ c_{X,Y} \times \text{id} \downarrow & & \downarrow c_{X,Y \times Z} \\ T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z) \end{array}$$

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is a pullback.

► **Definition 2.2.** Let  $T$  be a commutative monad on a category  $\mathcal{A}$  with finite products. We say that the monad  $T$  is **weakly affine** if the following associativity diagram is a pullback for every  $X, Y, Z$  in  $\mathcal{A}$ :

$$\begin{array}{ccc} T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\ c_{X,Y} \times \text{id} \downarrow & & \downarrow c_{X,Y \times Z} \\ T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z) \end{array} \quad (2)$$

Let us derive some general properties before looking at examples. It is a standard fact that for any commutative monad  $T$ , the composite arrow

$$T(1) \times T(1) \xrightarrow{c_{1,1}} T(1 \times 1) \xrightarrow{\cong} T(1)$$

equips  $T(1)$  with the structure of a commutative monoid internal to  $\mathcal{A}$  with unit  $\eta_1 : 1 \rightarrow T(1)$ .

► **Lemma 2.3.** If  $T$  is weakly affine, then  $T(1)$  is a group.

**Proof.** If  $T$  is weakly affine, then taking  $X = Y = Z = 1$  in (2) shows that this monoid must be an abelian group: assuming that  $\times$  is a strict monoidal structure for simplicity, we obtain a unique arrow  $\iota : T(1) \rightarrow T(1)$  such that the diagram

$$\begin{array}{ccc} T(1) & \xrightarrow{(\text{id}, \eta_1!)} & T(1) \times T(1) \\ \downarrow (\text{id}, \iota, \text{id}) & & \downarrow c_{1,1} \times \text{id} \\ T(1) \times T(1) \times T(1) & \xrightarrow{\text{id} \times c_{1,1}} & T(1) \times T(1) \\ \downarrow c_{1,1} \times \text{id} & & \downarrow c_{1,1} \\ T(1) \times T(1) & \xrightarrow{c_{1,1}} & T(1) \end{array}$$

and the commutativity shows that  $\iota$  satisfies the equations making it the inversion map for a group structure. ◀

► **Proposition 2.4.** If  $T$  is weakly affine, then for every object  $X$  there exists a unique morphism  $\gamma : T(X) \rightarrow T(X)$  such that the composition  $c_{1,X} \circ (\text{id}_{T(1)} \times \gamma) : T(1) \times T(X) \rightarrow T(1)$  is a (left) group action.

**Proof.** Let  $\gamma : T(X) \rightarrow T(X)$  be the unique morphism such that the diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{(\eta_1!, \text{id})} & T(1) \times T(X) \\ \downarrow (\eta_1!, \gamma, \eta_1!) & & \downarrow c_{1,X} \\ T(1) \times T(X) \times T(1) & \xrightarrow{\text{id} \times c_{X,1}} & T(1) \times T(X) \\ \downarrow c_{1,X} \times \text{id} & & \downarrow c_{1,X} \\ T(X) \times T(1) & \xrightarrow{c_{X,1}} & T(X) \end{array}$$

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67 commutes. Now we show that the morphism  $c_{1,X} \circ (\text{id}_{T1} \times \gamma) : T(1) \times T(X) \rightarrow T(1)$   
 68 provides an action of the group  $T(1)$  on  $T(X)$ . First, notice that identity axiom, i.e.  
 69  $(c_{1,X}(\text{id}_{T1} \times \gamma))(\eta_1!, \text{id}) = \text{id}$ , is satisfied by definition of  $\gamma$ , because  $c_{1,X}(\eta_1!, \gamma) = \text{id}$ . The  
 60 compatibility axiom follows from the fact that the diagram

$$\begin{array}{ccc}
 T(1) \times T(1) \times T(X) & \xrightarrow{\text{id} \times c_{1,X}} & T(1) \times T(X) \\
 c_{1,1} \times \text{id} \downarrow & & \downarrow c_{1,X} \\
 T(1) \times T(X) & \xrightarrow{c_{1,X}} & T(X)
 \end{array}$$

62 commutes. ◀

63 ► **Proposition 2.5.** *Let  $T$  be a weakly affine monad. If the diagram*

$$\begin{array}{ccc}
 T(1) & \xrightarrow{\text{id}} & T(1) \\
 \iota \downarrow & & \downarrow \eta_{T1} \\
 T(1) & \xrightarrow{T(\eta_1)} & T^2(1)
 \end{array}$$

65 commutes, then  $T^2(1) \cong T(1)$  in  $\mathcal{A}$ .

66 **Proof.** To prove the result it is enough to show that  $T(1) \cong 1$  in the Kleisli category  $\mathcal{A}_T$ .  
 67 We know from Lemma that  $T(1)$  is a group in  $\mathcal{A}$ , where the arrow  $\eta_1 : 1 \rightarrow T(1)$  is the  
 68 unit of the group, and  $\iota : T(1) \rightarrow T(1)$  is the inversion map. Therefore, we have that the  
 69 composition  $\iota\eta_1 : 1 \rightarrow T(1)$  has to be equal to  $\eta_1$ . Therefore, we can consider the arrows  
 70  $1 \rightarrow T(1)$  and  $T(1) \rightarrow 1$  in the Kleisli category  $\mathcal{A}_T$  given by  $T(\eta_1)\eta_1$  and  $\iota$  respectively. The  
 71 composition  $T(\eta_1)\eta_1$  with  $\iota$  in  $\mathcal{A}_T$  is given by  $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$ . Employing the naturality  
 72 of  $\eta_1$  and the fact that  $\iota\eta_1 = \eta_1$ , it is direct to check that  $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$ , that is the  
 73 identity  $1 \rightarrow 1$  in  $\mathcal{A}_T$ . Now to show that the other composition gives the identity on  $T(1)$  in  
 74  $\mathcal{A}_T$ , it is enough to show that  $T(\eta_1)\iota = \eta_{T(1)}$ , but this follows by hypothesis. ◀

(Paolo) Credo che  $T(\eta_1)\iota \neq \eta_{T(1)}$  nell'esempio delle misure non zero. Per ogni  $x$  in  $(0, \infty) = T1$  abbiamo che  $\eta_{T(1)}(x) = \delta_x$  (delta di Dirac), mentre  $T\eta_1(\iota(x)) = T\eta_1(1/x) = 1/x \delta_1$ .

76 ► **Corollary 2.6.** *Let  $T$  be a weakly affine monad. If the diagram*

$$\begin{array}{ccc}
 T(1) & \xrightarrow{\text{id}} & T(1) \\
 \iota \downarrow & & \downarrow \eta_{T1} \\
 T(1) & \xrightarrow{T(\eta_1)} & T^2(1)
 \end{array}$$

78 commutes, then  $T(1)$  is an idempotent group, namely  $\iota = \text{id}_{T1}$ .

79 **Proof.** By Lemma 2.3 we have that  $T(1)$  is a group. If  $\eta_{T1} = T(\eta_1)\iota$ , then we can apply the  
 80 multiplication of the monad to both sides, obtaining  $\iota = \text{id}_{T1}$ . ◀

81 The following result shows that weak affinity occurs frequently. Recall that a strong monad  
 82  $T : \mathcal{A} \rightarrow \mathcal{A}$  on a category  $\mathcal{A}$  with finite products is **affine** if  $T(1) \cong 1$  (see also Remark ??).  
 83 Three relevant examples of affine monads are the distribution monad on **Set** (for discrete  
 84 probability), the Giry monad on the category of measurable spaces (for measure-theoretic  
 85 probability, see Examples ?? and ??), and the expectation monad, see [?].

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► **Proposition 2.7.** *Let  $T$  be a commutative monad on a category  $\mathcal{A}$  with finite limits. If  $T$  is affine, then it is weakly affine.*

**Proof.** Let  $m_{X,Y} : T(X \times Y) \rightarrow TX \times TY$  be the arrow defined as the pairing of  $T(\pi_1)$  and  $T(\pi_2)$ . Then it is known that  $T$  is affine if and only if  $m_{X,Y}c_{X,Y} = \text{id}_{TX \times TY}$  [?, Lemma 4.2(i)].<sup>1</sup> In particular,  $c_{X,Y}$  is a split mono and therefore mono.

To show that (2) is a pullback, we prove the universal property starting with a diagram

$$\begin{array}{ccc}
 A & \xrightarrow{(f_1, f_2)} & TX \times T(Y \times Z) \\
 \text{\scriptsize $\exists !$} \searrow & & \downarrow c_{X,Y \times Z} \\
 TX \times TY \times TZ & \xrightarrow{\text{id} \times c_{Y,Z}} & TX \times T(Y \times Z) \\
 \downarrow c_{X,Y} \times \text{id} & & \downarrow c_{X,Y \times Z} \\
 T(X \times Y) \times TZ & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z)
 \end{array} \tag{3}$$

where the dashed arrow will be constructed; its uniqueness is clear since  $\text{id} \times c_{Y,Z}$  and  $c_{X,Y} \times \text{id}$  are mono, so it remains to prove existence. Taking the unlabelled arrows to be (induced by) product projections, we have the commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{(g_1, g_2)} & T(X \times Y) \times TZ & \rightarrow & T(X \times Y \times Z) \\
 (f_1, f_2) \downarrow & & \nearrow c_{X, Y \times Z} & & \downarrow \\
 TX \times T(Y \times Z) & \xrightarrow{\quad \quad \quad} & & \rightarrow & T(Y \times Z)
 \end{array}$$

where the upper left triangle commutes by assumption, and the lower right triangle commutes by naturality of  $c$  with respect to the unique arrow  $X \rightarrow 1$  together with  $T1 \cong 1$  and the fact that  $c_{1, Y \times Z}$  is a coherence isomorphism. By the naturality of  $c$ ,  $f_2$  can be written as the composite

$$A \xrightarrow{(g_1, g_2)} T(X \times Y) \times TZ \rightarrow TY \times TZ \xrightarrow{c_{Y,Z}} T(Y \times Z).$$

By analogous reasoning, we identify  $g_1$  with the composite

$$A \xrightarrow{(f_1, f_2)} TX \times T(Y \times Z) \rightarrow TX \times TY \xrightarrow{c_{X,Y}} T(X \times Y).$$

Getting back to (3), we take the dashed arrow to be the arrow whose component on  $TX$  is given by  $f_1$ , on  $TZ$  by  $g_2$ , and on  $TY$  by the diagonal in the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f_2} & T(Y \times Z) \\
 g_1 \downarrow & & \downarrow \\
 T(X \times Y) & \xrightarrow{\quad \quad \quad} & TY
 \end{array}$$

which commutes for similar reasons as above. The fact that this arrow recovers the  $f_2$  component after composition with  $\text{id} \times c_{Y,Z}$  and the  $g_1$  component after composition with  $c_{X,Y} \times \text{id}$  follows by the expressions for  $f_2$  and  $g_1$  derived above. The fact that it recovers  $f_1$  and  $g_2$  is by construction.

<sup>1</sup> For probability monads, this equation can be interpreted as stating that the marginals of a product distribution are the original factors [?].

112 ► **Remark 2.8.** We are not aware of any relation between weakly affine monads in our sense  
 113 and Jacobs' *strongly affine* monads [?], other than the fact that strongly affine implies affine  
 114 implies weakly affine.

115 ► **Example 2.9.** We present a family of examples of commutative monads that are weakly  
 116 affine but not affine. Let  $A$  be an abelian group (written multiplicatively). Then the functor  
 117  $T_A := A \times -$  on **Set** has a canonical structure of commutative monad, where the lax structure  
 118 components  $c_{X,Y}$  are given by multiplying elements in  $A$  while carrying the elements of  $X$   
 119 and  $Y$  along.

120 Since  $T_A \cong A$ , the monad  $T_A$  is clearly not affine unless  $A$  is the trivial group. However,  
 121  $T_A$  is always weakly affine. Indeed, in order to show that (2) is a pullback, it suffices to show  
 122 that the associativity square of  $A$

$$\begin{array}{ccc}
 A \times A \times A & \xrightarrow{\text{id} \times \cdot} & A \times A \\
 \downarrow \cdot \times \text{id} & & \downarrow \cdot \\
 A \times A & \xrightarrow{\cdot} & A
 \end{array}$$

124 is a pullback. Using element-wise reasoning, this amounts to showing that the system of  
 125 equations  $ax = c$  and  $xb = d$  has a solution for  $x \in A$  if and only if  $cb = ad$ , and in this case  
 126 the solution is unique. But this is indeed the case with  $x = a^{-1}c = db^{-1}$ . (Note that this  
 127 argument does not even require  $A$  to be abelian, but we need to require this in order for  $T_A$   
 128 to be commutative.)

129 ► **Example 2.10.** Many monads in categorical measure theory are weakly affine but not  
 130 affine. Let e.g.  $M^* : \mathbf{Set} \rightarrow \mathbf{Set}$  be the monad assigning to every set the set of finitely  
 131 supported discrete *nonzero* measures on  $M^*$ , or equivalently let  $M^*(X)$  for any set  $X$  be the  
 132 set of nonzero finitely supported functions  $X \rightarrow [0, \infty)$ . The monad structure is defined in  
 133 terms of the same formulas as for the distribution monad on **Set** and the components  $c_{X,Y}$   
 134 are also given by the formation of product measures, or equivalently point-wise products of  
 135 functions  $X \rightarrow [0, \infty)$ .

136 Since  $M^*1 \cong (0, \infty)$ , this monad is clearly not affine. However, it is weakly affine, and  
 137 we limit ourselves to a sketch of the proof. Indeed to prove that (2) is a pullback, we again  
 138 reason in terms of elements. If all measures are normalised, then we are back in the situation  
 139 of the distribution monad, which is affine and the claim follows. In the general case, one  
 140 can reduce to the normalised case by showing that the normalisation of the desired element  
 141 of  $M^*(Y)$  is uniquely determined. This works in the same way as in Example 2.9 with  
 142  $A = (0, \infty)$ .

143 On the other hand, if the zero measure is included, then we obtain a commutative monad  
 144  $M$  which can be seen as the monad of semimodules for the semiring of nonnegative reals.  
 145 Since  $M1 \cong [0, \infty)$  is not a group under multiplication,  $M$  is not weakly affine.

146 The previous two examples and Lemma 2 suggest the following problem.

147 ► **Problem 2.11.** Let  $T$  be a commutative monoid such that  $T(1)$  is an abelian group. Does  
 148 it follow that  $T$  is weakly affine?