Weakly-affine monads

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Abstract

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1 Introduction

For context:

▶ **Proposition 1.1.** *A monoid (M, m, e) is a group if and only if the associativity square*

$$\begin{tikzcd} M \times M \times M \arrow[r, "m\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"] & M \end{tikzcd} \tag{1}$$

is a pullback.

The same can be said more generally for a monoid object in a cartesian monoidal category.

Proof. The square (1) is a pullback, both of sets and of groups, if and only if given $a, g, h, c \in M$ such that $ag = hc$, there exists a unique $b \in M$ such that $g = bc$ and $h = ab$. First, suppose that g is a group. The only possible choice of b is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (1) is a pullback. We can set $g, h = e$ and $c = a$ so that $ae = ea = a$. Instantiating the pullback property, there is a unique b such that $ab = e$ and $ba = e$, that is, $b = a^{-1}$. ◀

▶ **Definition 1.1.** *Let X be a set. Denote by MX the set of* finitely supported measures *on X, i.e. functions $m : X \to [0, \infty)$ which are zero for all but a finite number of $x \in X$. Given a function $f : X \to Y$, denote by $Mf : MX \to MY$ the function sending $m \in MX$ to the assignment*

$$f_*m : y \mapsto \sum_{x\in f^{-1}(y)} p(x).$$

This way M is a functor, and even a monad with the following unit and multiplication maps.

$$\begin{aligned} X &\xrightarrow{\delta} MX & \qquad MMX &\xrightarrow{E} MX \\ x &\longmapsto \delta_x, & \xi &\longmapsto E\xi, \end{aligned}$$

where

$$\delta_x(x') = \begin{cases} 1 & x = x'; \\ 0 & x \neq x', \end{cases} \qquad E\xi(x) = \sum_{m \in MX} \xi(m)\, m(x).$$

Call M the measure monad *on* **Set**.

Denote also by $DX \subseteq MX$ the subset of probability measures*, i.e. those finitely supported $p : X \to [0, \infty)$ such that*

$$\sum_{x \in X} p(x) = 1.$$

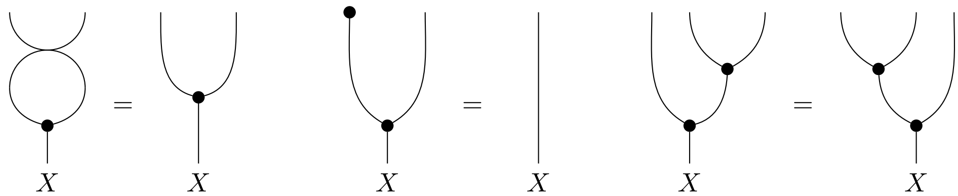
D forms a submonad of M called the distribution monad.

1.1 GS and Markov categories

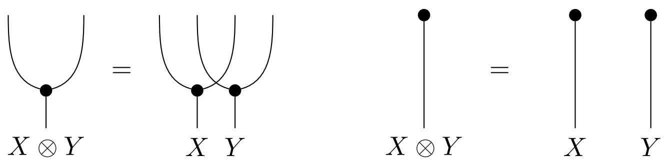
The notion of *gs-monoidal category* has been originally introduced in the context of algebraic approaches to term graph rewriting [?], and then developed in a series of papers [?, ?, ?]. We recall here the basic definitions adopting the graphical formalism of string diagrams, referring to [?] for an overview of various notions of monoidal categories and their associated diagrammatic calculus.

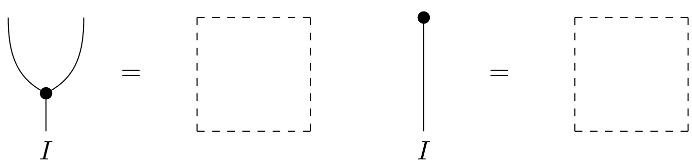
▶ **Definition 1.2.** *A **gs-monoidal category** is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ with a commutative comonoid structure on each object X, consisting of a comultiplication and counit,*

which satisfy the commutative comonoid equations:

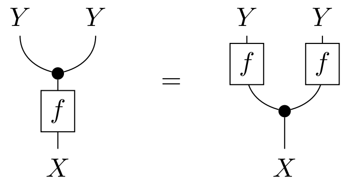


These comonoid structures must be multiplicative with respect to the monoidal structure:

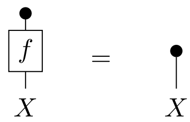




▶ **Definition 1.3.** *A morphism $f : X \to Y$ in a gs-monoidal category is called **copyable** or **functional** if and only if*



*It is called **discardable** or **full** if*



▶ **Example 1.4.** The category **Rel** of sets and relations with the monoidal operation $\otimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ given by the given by the direct product of sets is a gs-monoidal category [**?**]. In this gs-monoidal category, copyble arrows are precisely partial functions, and discardable arrows are total relations.

▶ Remark 1.5. It is well-known that if duplicators and dischargers of a given gs-monoidal category $\mathcal{C}$ satisfy naturality, then the monoidal product is the categorical product, and thus the category is cartesian monoidal [**?**], i.e. the following conditions are equivalent for a gs-monoidal category $\mathcal{C}$:

- $\mathcal{C}$ is cartesian monoidal;
- every morphism is copyable and discardable;
- the copy and discard maps are natural.

In recent works [**?**] it has been proved that the gs-monoidal structure naturally arises in several situations, such as when Kleisli categories of commutative monads or span categories. In the following proposition, we recall the result regarding Kleisli categories:

▶ **Proposition 1.6.** *Let T be a monoidal (equivalently, commutative) monad on a cartesian monoidal category $\mathcal{D}$. Then* $\mathrm{Kl}(T)$ *is canonically a gs-monoidal category with copy and discard structure induced by those of $\mathcal{D}$.*

▶ **Definition 1.7.** *A gs-monoidal category is called* Markov *if any (hence all) of the following equivalent conditions are satisfied:*

- *The monoidal unit is terminal;*
- *The discard map is natural;*
- *Every morphism is discardable.*

▶ **Definition 1.8.** *A monad T on a cartesian monoidal category is called* affine *if $T1 \cong 1$.*

▶ **Proposition 1.9.** *In the hypotheses of Proposition 1.6, we have that* $\mathrm{Kl}(T)$ *is Markov if and only if T is affine.*

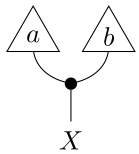
In this work we want to introduce an intermediate level between GS and Markov, which we call *weakly Markov*, and its corresponding notion for monads, which we call *weakly affine*.

2 Weakly Markov categories and weakly affine monads

2.1 The monoid of effects

In a gs-monoidal category $\mathcal{C}$ we call a *state* a morphism from the monoidal unit $p : I \to X$, and *effect* or *co-state* a morphism to the monoidal unit $a : X \to I$. We represent them as triangles as follows.

Effects, i.e. elements of the set $\mathcal{C}(X, I)$, form canonically a commutative monoid as follows: the monoidal unit is the discard map $X \to I$, and given $a, b : X \to I$, their product ab is given by copying, as follows.



If a morphism $f : X \to Y$ is copyable and discardable, precomposition with f induces a morphism of monoids $\mathcal{C}(Y, I) \to \mathcal{C}(X, I)$.

Let's now consider the case where the GS structure comes from a commutative monad on a cartesian monoidal category $\mathcal{D}$. In this case, the monoid structure of Kleisli morphisms $X \to 1$ comes from the following canonical internal monoid structure of $T1$ in $\mathcal{D}$, given by

$$1 \xrightarrow{\eta} T1, \qquad T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

For example, for the monad of measures M, we have that $M1 = [0, \infty)$ with its usual multiplication.

The monoid structure of Kleisli morphisms $X \to 1$ is now given as follows. The unit is given by

$$X \xrightarrow{!} 1 \xrightarrow{\eta} T1,$$

and the multiplication of the morphisms $f^\sharp, g^\sharp : X \to T1$ is

$$X \xrightarrow{\text{copy}} X \times X \xrightarrow{f^\sharp \times g^\sharp} T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

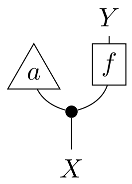
For the monad of measures M, Kleisli morphisms $X \to 1$ are functions $X \to [0, \infty)$, and their monoidal structure is their pointwise product.

2.2 Main definitions

▶ **Definition 2.1.** A GS-category $\mathcal{C}$ is called *weakly Markov* if for every object X, the monoid $\mathcal{C}(X, I)$ is a group.

Every Markov category is weakly Markov: for each X, the monoid $\mathcal{C}(X, I)$ is the trivial group.

Note that the group $\mathcal{C}(X, I)$ (or monoid, in the general case) acts on the set $\mathcal{C}(X, Y)$: given $a : X \to I$ and $f : X \to Y$, $a \cdot f$ is given as follows,



and the product $(f, g) \mapsto f \cdot g \coloneqq (f \otimes g) \circ \mathrm{copy}_X$ is equivariant for this action in both variables (separately).

▶ **Definition 2.2.** Given two parallel morphisms $f, g : X \to Y$ in a weakly Markov GS-category $\mathcal{C}$, we say that f and g are *equivalent*, and write $f \sim g$, if they lie in the same orbit for the action of $\mathcal{C}(X, I)$, i.e. if there is $a \in \mathcal{C}(X, I)$ such that $a \cdot f = g$. We say they are *uniquely equivalent* if there is a unique $a \in \mathcal{C}(X, I)$ such that $a \cdot f = g$.

Let's now look at the Kleisli case.

▶ **Definition 2.3.** A commutative monad T on a cartesian monoidal category is called *weakly affine* if $T1$ with its canonical internal monoid structure is a group.

This choice of terminology is motivated by the following proposition, which can be seen as a weak version of Proposition 1.9.

▶ **Proposition 2.4.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. The Kleisli category of T is weakly Markov if and only if T is weakly affine.*

Proof. First, suppose that $T1$ is an internal group, and denote by $\iota : T1 \to T1$ its inversion map. The inverse of the morphism $f^\sharp : X \to T1$ in $\mathrm{Kl}_T(X, 1)$ is given by $\iota \circ f$: indeed, the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r, "\mathrm{copy}"] \arrow[d, "f^\sharp"'] & X \times X \arrow[d, "f^\sharp \times f^\sharp"'] \arrow[dr, "f \times (\iota \circ f)"] & & \\
T1 \arrow[r, "\mathrm{copy}"] \arrow[d, "!"'] & T1 \times T1 \arrow[r, "\mathrm{id} \times \iota"'] & T1 \times T1 \arrow[r, "c"] & T(1 \times 1) \arrow[d, "\cong"] \\
1 \arrow[rrr, "\eta"] & & & T1
\end{tikzcd}$$

where the bottom rectangle commutes since ι is the inversion map for $T1$. The analogous diagram with $\iota \times \mathrm{id}$ in place of $\mathrm{id} \times \iota$ commutes analogously.

Conversely, suppose that for every X, the monoid structure on $\mathrm{Kl}_T(X, 1)$ has inverses. Then in particular we can take $X = T1$, and the inverse of the Kleisli morphism $\mathrm{id} : T1 \to T1$ is an inversion map for $T1$. ◀

This result can be seen in terms of the Yoneda embedding, see the details in Appendix A.

2.3 Examples of weakly affine monads

Every affine monad is a weakly affine monad. Here are less trivial examples.

▶ **Example 2.5.** Let $M^* : \mathbf{Set} \to \mathbf{Set}$ be the monad assigning to every set the set of finitely supported discrete *nonzero* measures on M^*, or equivalently let $M^*(X)$ for any set X be the set of nonzero finitely supported functions $X \to [0, \infty)$. The monad structure is defined in terms of the same formulas as for the monad of measures M (Definition 1.1) and the components $c_{X,Y}$ are also given by the formation of product measures, or equivalently point-wise products of functions $X \to [0, \infty)$.

Since $M^*1 \cong (0,\infty) \ncong 1$, this monad is not affine. However the monoid structure of $(0,\infty)$ induced by M^* is the usual multiplication of positive real numbers, which form a group. Therefore M^* is weakly affine, and its Kleisli category is weakly Markov.

On the other hand, if the zero measure is included, we have $M1 \cong [0,\infty)$ which is not a group under multiplication, so M is not weakly affine.

▶ **Example 2.6.** Let A be a commutative monoid. Then the functor $T_A \coloneqq A \times -$ on **Set** has a canonical structure of commutative monad, where the lax structure components $c_{X,Y}$ are given by multiplying elements in A while carrying the elements of X and Y along.

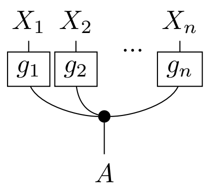
Since $T_A(1) \cong A$, the monad T_A is weakly affine if and only if A is a group, and affine if and only if $A \cong 1$.

▶ **Example 2.7.** Here is a negative example. Consider the free abelian group monad F on **Set**. Its functor takes a set X and forms the set FX of finite multisets (with repetition, where order does not matter) of elements of X and their formal inverses. We have that $F1 \cong \mathbb{Z}$, which is an abelian group under addition. However, the monoid structure on $F1$ induced by the monoidal structure of the monad corresponds to the *multiplication* in $\mathbb{Z}$, which does not have inverses. Therefore F is not weakly affine.

3 Conditional independence in weakly Markov categories

Markov categories have a rich theory of conditional dependence and independence [**?**]. Some of those ideas can be translated and generalized to the setting of weakly Markov categories.

▶ **Definition 3.1.** A morphism $f : A \to X_1 \otimes \cdots \otimes X_n$ in a A GS-category $\mathcal{C}$ is said to exhibit *conditional independence of the X_i given A* if and only if it can be expressed as a product of the following form.



Note that this is slightly different from [**?**, Definition 6.6], although it is equivalent for the case of Markov categories.

Here is what conditional independence looks like in the Kleisli case.

▶ **Proposition 3.2.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. A Kleisli morphism $f^\sharp : A \to T(X_1 \times \cdots \times X_n)$ exhibits conditional independence of the X_i given A if and only if it factors as follows*

$$\begin{tikzcd} A \arrow[d, "{(g_1^\sharp,\dots,g_n^\sharp)}"'] \arrow[dr, "f^\sharp"] & \\ TX_1 \times \cdots \times TX_n \arrow[r, "c"'] & T(X_1 \times \cdots \times X_n), \end{tikzcd}$$

for some Kleisli maps $g_i^\sharp : A \to TX_i$, where the map c above is the one obtained by iterating the multiplication of the monoidal structure (such a map is unique by associativity).

Proof. In terms of the base category $\mathcal{D}$, a Kleisli morphism in the form of Definition 3.1 reads as follows.

$$A \xrightarrow{\text{copy}} A \times \cdots \times A \xrightarrow{g_1^\sharp \times \cdots \times g_n^\sharp} TX_1 \times \cdots \times TX_n \xrightarrow{c} T(X_1 \times \cdots \times X_n).$$

Therefore $f^\sharp : A \to T(X_1 \times \cdots \times X_n)$ is exhibiting conditional independence if and only if it is in the form above. ◀

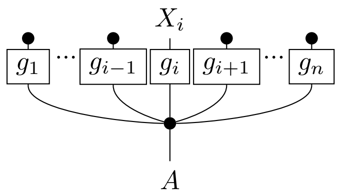
▶ **Example 3.3.** In the Kleisli category of the distribution monad D, which is Markov, a morphism $f : A \to X \otimes Y$ exhibits conditional independence if and only if it is is the product of its marginals [?, Section 12].

▶ **Example 3.4.** In the Kleisli category of the measure monad M, the zero measure always displays conditional independence of its outputs given its inputs: for example, for $A = 1$, the zero measure on $X \times Y$ is the product of the zero measure on X and the zero (or any other) measure on Y. Notice that both marginals of the zero measure are zero measures—therefore, the factors appearing in the product are not necessarily related to the marginals.

In a weakly Markov category, the situation is similar to the Markov case, but up to equivalence.

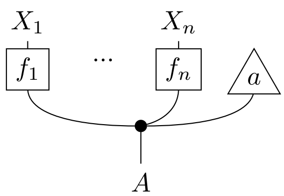
▶ **Proposition 3.5.** *Let $f : A \to X_1 \otimes \cdots \otimes X_n$ be a morphism in a weakly Markov GS-category $\mathcal{C}$. Then f exhibits conditional independence of the X_i given A if and only if it is equivalent to the product of all its marginals. Moreover, in that case f is* uniquely *equivalent to the product of its marginals.*

Proof. Denote the marginals of f by $f_1, \ldots, f_n$. Suppose that f is a product as in Definition 3.1. For each $i = 1, \ldots, n$, by marginalizing, we get that f_i is equal to the following.

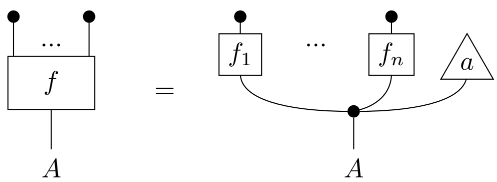


Therefore for each i we have that $f_i \sim g_i$.

Conversely, suppose that f is equivalent to the product of its marginals, i.e. that there exists $a : X \to I$ such that f is equal to the following.

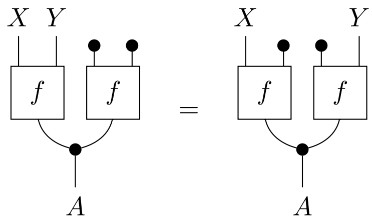


One can then choose $g_i = f_i$ for all $i < n$, and $g_n = a \cdot f_n$, so that f is in the form of Definition 3.1. Moreover, by marginalizing over all the X_i at once, we see that



so that a is uniquely determined. ◀

▶ **Remark 3.6.** For $n = 2$, a morphism $f : A \to X \otimes Y$ in a weakly Markov GS-category $\mathcal{C}$ exhibits conditional independence of X and Y given A if and only if the following equation holds.



3.1 Main result

The concept of conditional independence for general weakly Markov categories allow us to give an equivalent characterization of affine monads. The condition is in terms of a pullback condition on the associativity diagram, and can be seen as a generalization of Proposition 1.1.

▶ **Theorem 3.7.** *Let $\mathcal{D}$ be a cartesian monoidal category, and let T be a commutative monad on $\mathcal{D}$. The following conditions are equivalent.*

1. *T is weakly affine;*
2. $\mathrm{Kl}(T)$ *is weakly Markov;*
3. *For all objects X, Y, and Z, the following associativity diagram is a pullback.*

$$\begin{tikzcd}
T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"] \arrow[d, "c_{X,Y} \times \mathrm{id}"'] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"] & T(X \times Y \times Z)
\end{tikzcd} \tag{2}$$

We prove the theorem by means of the following property of weakly Markov categories.

▶ **Lemma 3.8** (localized independence property)**.** *Let $\mathcal{C}$ be a weakly Markov GS-category. Whenever a morphism $f : A \to X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A, as well as conditional independence of X and $Y \otimes Z$ given A, then it exhibits conditional independence of X, Y and Z given A.*

Proof of Lemma 3.8. Suppose $f : A \to X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A, as well as conditional independence of X and $Y \otimes Z$ given A. By marginalizing out X, we have that f_{YZ} exhibits conditional independence of Y and Z given A. Since by hypothesis f exhibits conditional independence of X and $Y \otimes Z$ given A, by Proposition 3.5 we have that f is equivalent to the product of f_X and f_{YZ}. But, again by Proposition 3.5, f_{YZ} is equivalent to the product of f_Y and f_Z, so we have that f is equivalent to the product of all its marginals. Using Proposition 3.5 in the other direction, this means that f exhibits conditional independence of X, Y and Z given A. ◀

We are now ready to prove the theorem.

Proof of Theorem 3.7. $1 \Leftrightarrow 2$: see Proposition 2.4.

$1 \Rightarrow 3$: By the universal property of products, a cone over the cospan in (2) consists of maps $g_1^\sharp : A \to TX$, $g_{23}^\sharp : A \to T(Y \times Z)$, $g_{12}^\sharp : A \to T(X \times Y)$ and $g_3^\sharp : A \to TZ$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[drr, bend left, "(g_1^\sharp, g_{23}^\sharp)"] \arrow[dr, bend right, "(g_{12}^\sharp, g_3^\sharp)"'] & & \\
& T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"'] \arrow[d, "c_{X,Y} \times \mathrm{id}"] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
& T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"'] & T(X \times Y \times Z)
\end{tikzcd}$$

By Proposition 3.2, this amounts to a Kleisli map $f^\sharp : A \to T(X \times Y \times Z)$ exhibiting conditional independence of X and $Y \otimes Z$ given A, as well as of $X \otimes Y$ and Z given A. By the localized independence property (Lemma 3.8), we then have that f exhibits conditional independence of all X, Y and Z given A, and so, again by Proposition 3.2, $f^\sharp$ factors through the product $TX \times TY \times TZ$. More specifically, by marginalizing over Z, we have that $g^\sharp_{12}$ factors through $TX \times TY$, i.e. the following diagram on the left commutes for some $h^\sharp_1 : A \to TX$ and $h^\sharp_2 : A \to TY$, and similarly, by marginalizing over X, the diagram on the right commutes for some $\ell^\sharp_2 : A \to TY$ and $\ell^\sharp_3 : A \to TZ$.

$$\begin{tikzcd} & A \arrow[d, "(h^\sharp_1, h^\sharp_2)"'] \arrow[dr, "g^\sharp_{12}"] & \\ & TX \times TY \arrow[r, "c"'] & T(X \times Y) \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "(\ell^\sharp_2, \ell^\sharp_3)"'] \arrow[dr, "g^\sharp_{23}"] & \\ & TY \times TZ \arrow[r, "c"'] & T(Y \times Z) \end{tikzcd}$$

In other words, the upper and the left curved triangles in the following diagram commute.

$$\begin{tikzcd} A \arrow[rrd, bend left, "(g^\sharp_1, g^\sharp_{23})"] \arrow[d, shift left, "(g^\sharp_1, \ell^\sharp_2, \ell^\sharp_3)"] \arrow[d, shift right, "(h^\sharp_1, h^\sharp_2, g^\sharp_3)"'] \arrow[dd, bend right, "(g^\sharp_{12}, g^\sharp_3)"'] & & \\ T(X) \times T(Y) \times T(Z) \arrow[rr, "\mathrm{id} \times c_{Y,Z}"'] \arrow[d, "c_{X,Y} \times \mathrm{id}"] & & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\ T(X \times Y) \times T(Z) \arrow[rr, "c_{X \times Y, Z}"'] & & T(X \times Y \times Z) \end{tikzcd}$$

By marginalizing over Y and Z, and by weak affinity of T, there exists a unique $a^\sharp : A \to T1$ such that $h_1 = a \cdot g_1$. Therefore

$$g_{12} = h_1 \cdot h_2 = (a \cdot g_1) \cdot h_2 = g_1 \cdot (a \cdot h_2),$$

and so in the diagram above we can equivalently replace h_1 and h_2 with g_1 and $a \cdot h_2$. Similarly by marginalizing over X and Y, there exists a unique $c^\sharp : A \to T1$ such that $\ell_3 = c \cdot g_3$, so that

$$g_{23} = \ell_2 \cdot \ell_3 = \ell_2 \cdot (c \cdot g_3) = (c \cdot \ell_2) \cdot g_3$$

and in the diagram above we can replace ℓ_2 and ℓ_3 with $c \cdot \ell_2$ and g_3, as follows.

$$\begin{tikzcd} A \arrow[rrd, bend left, "(g^\sharp_1, g^\sharp_{23})"] \arrow[d, shift left, "(g^\sharp_1, (c \cdot \ell_2)^\sharp, g^\sharp_3)"] \arrow[d, shift right, "(g^\sharp_1, (a \cdot h_2)^\sharp, g^\sharp_3)"'] \arrow[dd, bend right, "(g^\sharp_{12}, g^\sharp_3)"'] & & \\ T(X) \times T(Y) \times T(Z) \arrow[rr, "\mathrm{id} \times c_{Y,Z}"'] \arrow[d, "c_{X,Y} \times \mathrm{id}"] & & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\ T(X \times Y) \times T(Z) \arrow[rr, "c_{X \times Y, Z}"'] & & T(X \times Y \times Z) \end{tikzcd}$$

Now, marginalizing over Z and Z, we see that necessarily $a \cdot h_2 = c \cdot \ell_2$. Therefore there is a unique map $A \to TX \times TY \times TZ$ making the whole diagram commute, which means that (2) is a pullback.

$3 \Rightarrow 1$: If T is weakly affine, then taking $X = Y = Z = 1$ in (2) shows that this monoid must be an abelian group: we obtain a unique arrow $\iota \colon T(1) \to T(1)$ making the following diagram commute,

$$
\begin{tikzcd}
T1 \arrow[drr, bend left, "{(\mathrm{id},\eta_1!)}"] \arrow[dr, "{(\mathrm{id},\iota,\mathrm{id})}"] \arrow[ddr, bend right, "{(\eta_1!,\mathrm{id})}"'] & & & \\
& T1 \times T1 \times T1 \arrow[r, "\mathrm{id}\times c_{1,1}"] \arrow[d, "c_{1,1}\times\mathrm{id}"'] & T1 \times T(1\times 1) \arrow[r, "\cong"] \arrow[d, "c_{1,1\times 1}"] & T1 \times T1 \arrow[d, "c_{1,1}"] \\
& T(1\times 1) \times T1 \arrow[r, "c_{1\times 1,1}"] \arrow[d, "\cong"'] & T(1\times 1\times 1) \arrow[r, "\cong"] \arrow[d, "\cong"] & T(1\times 1) \arrow[d, "\cong"] \\
& T1 \times T1 \arrow[r, "c_{1,1}"'] & T(1\times 1) \arrow[r, "\cong"'] & T1
\end{tikzcd}
$$

and the commutativity shows that ι satisfies the equations making it the inversion map for a group structure. ◀

▶ **Example 3.9.** In the Kleisli category of the measure monad $\mathrm{Kl}(M)$ (which is not weakly affine) consider the following diagram.

$$
\begin{tikzcd}
MX \times MY \times MZ \arrow[r, "\mathrm{id}\times c_{Y,Z}"] \arrow[d, "c_{X,Y}\times\mathrm{id}"'] & MX \times M(Y\times Z) \arrow[d, "c_{X,Y\times Z}"] \\
M(X\times Y) \times MZ \arrow[r, "c_{X\times Y,Z}"] & M(X\times Y\times Z)
\end{tikzcd}
$$

In the top-right corner $MX \times M(Y \times Z)$ take the pair $(0, p)$ where p is a nonzero measure on $Y \times Z$, and similarly, in the bottom-left corner take the pair $(q, 0)$ where q is a nonzero measure on $X \times Y$. Following the diagram, both pairs are mapped to the zero measure in the bottom-right corner. If the diagram were a pullback, we would be able to express the top-right and bottom-left corners as coming from the same triplet in $MX \times MY \times MZ$, that is, there would exist a measure m on Y such that $m \cdot 0 = p$ and $0 \cdot m = q$. Since p and q are nonzero, this is not possible.

4 Further results

▶ **Proposition 4.1.** *Let T be a weakly affine monad. If the diagram*

$$
\begin{tikzcd}
T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\
T(1) \arrow[r, "T(\eta_1)"'] & T^2(1)
\end{tikzcd}
$$

commutes, then $T^2(1) \cong T(1)$ in $\mathcal{A}$.

Proof. To prove the result it is enough to show that $T(1) \cong 1$ in the Kleisli category $\mathcal{A}_T$. We know from Lemma that $T(1)$ is a group in $\mathcal{A}$, where the arrow $\eta_1 \colon 1 \to T(1)$ is the unit of the group, and $\iota \colon T(1) \to T(1)$ is the inversion map. Therefore, we have that the composition $\iota\eta_1 \colon 1 \to T(1)$ has to be equal to η_1. Therefore, we can consider the arrows $1 \to T(1)$ and $T(1) \to 1$ in the Kleisli category $\mathcal{A}_T$ given by $T(\eta_1)\eta_1$ and ι respectively. The composition $T(\eta_1)\eta_1$ with ι in $\mathcal{A}_T$ is given by $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$. Employing the naturality of η_1 and the fact that $\iota\eta_1 = \eta_1$, it is direct to check that $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$, that is the identity $1 \to 1$ in $\mathcal{A}_T$. Now to show that the other composition gives the identity on $T(1)$ in $\mathcal{A}_T$, it is enough to show that $T(\eta_1)\iota = \eta_{T(1)}$, but this follows by hypothesis. ◀

▶ **Corollary 4.2.** *Let T be a weakly affine monad. If the diagram*

$$\begin{tikzcd} T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\ T(1) \arrow[r, "T(\eta_1)"'] & T^2(1) \end{tikzcd}$$

commutes, then $T(1)$ is an idempotent group, namely $\iota = \mathrm{id}_{T1}$.

Proof. By weak affineness, $T(1)$ is a group. If $\eta_{T1} = T(\eta_1)\iota$, then we can apply the multiplication of the monad to both sides, obtaining $\iota = \mathrm{id}_{T1}$. ◀

▶ **Remark 4.3.** Bart Jacobs calls a strong monad T on a cartesian monoidal category *strongly affine* [?] if for every pair of objects X and Y, the following diagram is a pullback,

$$\begin{tikzcd} X \times TY \arrow[r, "s"] \arrow[d, "\pi_1"] & T(X \times Y) \arrow[d, "T\pi_1"] \\ X \arrow[r, "\eta"] & TX \end{tikzcd}$$

where s denotes the strength and η denotes the unit of the monad. Every strongly affine monad is affine. The corresponding condition on the (Markov) category $\mathrm{Kl}(T)$ is called *positivity* [?, Section 2].

Note that for a generic commutative monad, the diagram above may even fail to commute (take for example the measure monad M, and start with $(x, 0)$ in the top left corner). One can however consider the following diagram, which reduces to the one above (up to isomorphism) in the affine case,

$$\begin{tikzcd} X \times TY \arrow[r, "s"] \arrow[d, "\mathrm{id} \times T!"] & T(X \times Y) \arrow[d, "T(\mathrm{id} \times !)"] \\ X \times T1 \arrow[r, "s"] & T(X \times 1) \cong TX \end{tikzcd}$$

and which always commutes by naturality of the strength. One can then call the monad T *positive* if this second diagram is a pullback (and possibly define *positive GS categories* analogously to positive Markov categories). All the examples of weakly affine monads that we have are positive in this sense, so one may wonder if every weakly affine monad is positive. For now, this remains an open question.

A Yoneda embedding interpretation of Proposition 2.4

We can interpret Proposition 2.4 more abstractly in terms of presheaves. Let $\mathcal{D}$ be a cartesian monoidal category. Consider the presheaf category $[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$, equipped with the Day convolution product,

$$F \boxtimes G \cong \int^{A,B \in \mathcal{D}} \mathcal{D}(-, A \times B) \times F(A) \times G(B).$$

The Yoneda embedding $\mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is strong monoidal: indeed, for each X,

$$1 \cong \mathcal{D}(X, 1),$$

since 1 is terminal, and for each X and Y, by Yoneda reduction,

$$\begin{aligned}\mathcal{D}(-,X)\boxtimes\mathcal{D}(-,Y) &\cong \int^{A,B\in\mathcal{D}} \mathcal{D}(-,A\times B)\times\mathcal{D}(-,X)\times\mathcal{D}(-,Y)\\ &\cong \mathcal{D}(-,X\times Y).\end{aligned}$$

Therefore, and by the universal property of products, at the level of individual hom-sets the Day convolution product of representable presheaves just takes the cartesian products of sets:

$$\big(\mathcal{D}(-,X)\boxtimes\mathcal{D}(-,Y)\big)(A)\cong\mathcal{D}(A,X\times Y)\cong\mathcal{D}(A,X)\times\mathcal{D}(A,Y).$$

Take now an object M of $\mathcal{D}$. Since the Yoneda embedding is fully faithful and strong monoidal, a monoid structure (M,m,e) on M is equivalently a monoid structure on the representable presheaf $\mathcal{D}(-,M)$. This makes the individual hom-sets monoids, with unit and multiplication as follows for each object X:

$$\begin{gathered} 1 \xrightarrow{\quad\cong\quad} \mathcal{D}(X,1) \xrightarrow{e_*} \mathcal{D}(X,M)\\ \mathcal{D}(X,M)\times\mathcal{D}(X,M) \xrightarrow{\cong} \mathcal{D}(X,M\times M) \xrightarrow{m_*} \mathcal{D}(X,M)\end{gathered}$$

This is precisely the monoid structure that we have defined in Section 2.1 for $M=T1$.

▶ **Proposition A.1.** *M is an internal group if and only if all the monoids $\mathcal{D}(X,M)$ are groups.*

Proof. By Proposition 1.1, M is a group object if and only if its associativity square (1) is a pullback. Since the hom-functor preserves and reflects all limits in its second argument, we have that (1) is a pullback if and only if for each object X, the following diagram (or equivalently, its bottom right square) is a pullback,

$$\begin{tikzcd}
\mathcal{D}(X,M)\times\mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[rr] \arrow[dr, "\cong"'] \arrow[dd] & & \mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[d, "\cong"] \\
& \mathcal{D}(X,M\times M\times M) \arrow[r, "(m\times\mathrm{id})_*"'] \arrow[d, "(\mathrm{id}\times m)_*"] & \mathcal{D}(X,M\times M) \arrow[d, "m_*"] \\
\mathcal{D}(X,M)\times\mathcal{D}(X,M) \arrow[r, "\cong"'] & \mathcal{D}(X,M\times M) \arrow[r, "m_*"'] & \mathcal{D}(X,M)
\end{tikzcd}$$

where the unlabelled arrows are the unique ones that make the diagram commute. Again by Proposition 1.1, the diagram above is a pullback if and only if $\mathcal{D}(X,M)$ is a group. ◀