Weakly-affine monads

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Abstract

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1 Introduction

For context:

▶ **Proposition 1.1.** *A monoid $(M, \cdot, 1)$ is a group if and only if the associativity square*

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\cdot \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \cdot"] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"] & M
\end{tikzcd} \tag{1}$$

is a pullback.

Proof. The square (1) is a pullback, both of sets and of groups, if and only if given $a, g, h, c \in M$ such that $ag = hc$, there exists a unique $b \in M$ such that $g = bc$ and $h = ab$. First, suppose that g is a group. The only possible choice of b is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (1) is a pullback. We can set $g, h = e$ and $c = a$ so that $ae = ea = a$. Instantiating the pullback property, there is a unique b such that $ab = e$ and $ba = e$, that is, $b = a^{-1}$. ◀

Recall that a monoidal functor generalizes a monoid object (in turn, generalizing a monoid). Similarly, a *weakly affine monoidal functor* generalizes a group in the sense of the proposition above.

2 Weakly-affine monads

Nome da scegliere e valutare se dare la def per una arbitraria gs

▶ **Definition 2.1.** Let T be a commutative monad on a category $\mathcal{A}$ with finite products. A triple (X, Y, Z) of objects of $\mathcal{A}$ is said to be **TBA** if the commutative square

$$\begin{tikzcd}
T(X) \times T(Y) \times T(Z) \arrow[r, "\mathrm{id} \times c_{Y,Z}"] \arrow[d, "c_{X,Y} \times \mathrm{id}"'] & T(X) \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
T(X \times Y) \times T(Z) \arrow[r, "c_{X \times Y, Z}"] & T(X \times Y \times Z)
\end{tikzcd}$$

esempi?

is a pullback.

▶ **Definition 2.2.** Let T be a commutative monad on a category $\mathcal{A}$ with finite products. We say that the monad T is **weakly affine** if the following associativity diagram is a pullback for every X, Y, Z in $\mathcal{A}$:

$$\begin{tikzcd}
T(X)\times T(Y)\times T(Z) \arrow[r, "\mathrm{id}\times c_{Y,Z}"] \arrow[d, "c_{X,Y}\times\mathrm{id}"'] & T(X)\times T(Y\times Z) \arrow[d, "c_{X,Y\times Z}"] \\
T(X\times Y)\times T(Z) \arrow[r, "c_{X\times Y,Z}"] & T(X\times Y\times Z)
\end{tikzcd} \tag{2}$$

Let us derive some general properties before looking at examples. It is a standard fact that for any commutative monad T, the composite arrow

$$T(1)\times T(1) \xrightarrow{c_{1,1}} T(1\times 1) \xrightarrow{\cong} T(1)$$

equips $T(1)$ with the structure of a commutative monoid internal to $\mathcal{A}$ with unit $\eta_1 \colon 1 \to T(1)$.

▶ **Lemma 2.3.** *If T is weakly affine, then $T(1)$ is a group.*

Proof. If T is weakly affine, then taking $X = Y = Z = 1$ in (2) shows that this monoid must be an abelian group: assuming that $\times$ is a strict monoidal structure for simplicity, we obtain a unique arrow $\iota \colon T(1) \to T(1)$ such that the diagram

$$\begin{tikzcd}
T(1) \arrow[drr, bend left, "(\mathrm{id},\eta_1!)"] \arrow[dr, dashed, "(\mathrm{id},\iota,\mathrm{id})"] \arrow[ddr, bend right, "(\eta_1!,\mathrm{id})"'] & & \\
& T(1)\times T(1)\times T(1) \arrow[r, "\mathrm{id}\times c_{1,1}"] \arrow[d, "c_{1,1}\times\mathrm{id}"'] & T(1)\times T(1) \arrow[d, "c_{1,1}"] \\
& T(1)\times T(1) \arrow[r, "c_{1,1}"] & T(1)
\end{tikzcd}$$

and the commutativity shows that ι satisfies the equations making it the inversion map for a group structure. ◀

ovviamente abbiamo anche un'azione destra.... devo finire di controllare se γ viene l'identità o in generale no

▶ **Proposition 2.4.** *If T is weakly affine, then for every object X there exists a unique morphism $\gamma : T(X) \to T(X)$ such that the composition $c_{1,X} \circ (\mathrm{id}_{T1} \times \gamma) : T(1)\times T(X) \to T(X)$ is a (left) group action.*

Proof. Let $\gamma \colon T(X) \to T(X)$ be the unique morphism such that the diagram

$$\begin{tikzcd}
T(X) \arrow[drr, bend left, "(\eta_1!,\mathrm{id})"] \arrow[dr, dashed, "(\eta_1!,\gamma,\eta_1!)"] \arrow[ddr, bend right, "(\mathrm{id},\eta_1!)"'] & & \\
& T(1)\times T(X)\times T(1) \arrow[r, "\mathrm{id}\times c_{X,1}"] \arrow[d, "c_{1,X}\times\mathrm{id}"'] & T(1)\times T(X) \arrow[d, "c_{1,X}"] \\
& T(X)\times T(1) \arrow[r, "c_{X,1}"] & T(X)
\end{tikzcd}$$

commutes. Now we show that the morphism $c_{1,X} \circ (\mathrm{id}_{T1} \times \gamma) : T(1) \times T(X) \to T(X)$ provides an action of the group $T(1)$ on $T(X)$. First, notice that identity axiom, i.e. $(c_{1,X}(\mathrm{id}_{T1} \times \gamma))(\eta_1!, \mathrm{id}) = \mathrm{id}$, is satisfied by definition of γ, because $c_{1,X}(\eta_1!, \gamma) = \mathrm{id}$. The compatibility axiom follows from the fact that the diagram

$$\begin{tikzcd}
T(1) \times T(1) \times T(X) \arrow[r, "\mathrm{id} \times c_{1,X}"] \arrow[d, "c_{1,1} \times \mathrm{id}"'] & T(1) \times T(X) \arrow[d, "c_{1,X}"] \\
T(1) \times T(X) \arrow[r, "c_{1,X}"'] & T(X)
\end{tikzcd}$$

commutes. ◀

▶ **Proposition 2.5.** *Let T be a weakly affine monad. If the diagram*

$$\begin{tikzcd}
T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\
T(1) \arrow[r, "T(\eta_1)"'] & T^2(1)
\end{tikzcd}$$

commutes, then $T^2(1) \cong T(1)$ in $\mathcal{A}$.

Proof. To prove the result it is enough to show that $T(1) \cong 1$ in the Kleisli category $\mathcal{A}_T$. We know from Lemma that $T(1)$ is a group in $\mathcal{A}$, where the arrow $\eta_1 \colon 1 \to T(1)$ is the unit of the group, and $\iota \colon T(1) \to T(1)$ is the inversion map. Therefore, we have that the composition $\iota\eta_1 \colon 1 \to T(1)$ has to be equal to η_1. Therefore, we can consider the arrows $1 \to T(1)$ and $T(1) \to 1$ in the Kleisli category $\mathcal{A}_T$ given by $T(\eta_1)\eta_1$ and ι respectively. The composition $T(\eta_1)\eta_1$ with ι in $\mathcal{A}_T$ is given by $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$. Employing the naturality of η_1 and the fact that $\iota\eta_1 = \eta_1$, it is direct to check that $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$, that is the identity $1 \to 1$ in $\mathcal{A}_T$. Now to show that the other composition gives the identity on $T(1)$ in $\mathcal{A}_T$, it is enough to show that $T(\eta_1)\iota = \eta_{T(1)}$, but this follows by hypothesis. ◀

(Paolo) Credo che $T(\eta_1)\iota \neq \eta_{T(1)}$ nell'esempio delle misure nonzero. Per ogni x in $(0, \infty) = T1$ abbiamo che $\eta_{T(1)}(x) = \delta_x$ (delta di Dirac), mentre $T\eta_1(\iota(x)) = T\eta_1(1/x) = 1/x\,\delta_1$.

▶ **Corollary 2.6.** *Let T be a weakly affine monad. If the diagram*

$$\begin{tikzcd}
T(1) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & T(1) \arrow[d, "\eta_{T1}"] \\
T(1) \arrow[r, "T(\eta_1)"'] & T^2(1)
\end{tikzcd}$$

commutes, then $T(1)$ is an idempotent group, namely $\iota = \mathrm{id}_{T1}$.

Proof. By Lemma 2.3 we have that $T(1)$ is a group. If $\eta_{T1} = T(\eta_1)\iota$, then we can apply the multiplication of the monad to both sides, obtaining $\iota = \mathrm{id}_{T1}$. ◀

The following result shows that weak affinity occurs frequently. Recall that a strong monad $T \colon \mathcal{A} \to \mathcal{A}$ on a category $\mathcal{A}$ with finite products is **affine** if $T(1) \cong 1$ (see also Remark **??**). Three relevant examples of affine monads are the distribution monad on **Set** (for discrete probability), the Giry monad on the category of measurable spaces (for measure-theoretic probability, see Examples **??** and **??**), and the expectation monad, see [**?**].

▶ **Proposition 2.7.** *Let T be a commutative monad on a category $\mathcal{A}$ with finite limits. If T is affine, then it is weakly affine.*

Proof. Let $m_{X,Y} : T(X \times Y) \longrightarrow TX \times TY$ be the arrow defined as the pairing of $T(\pi_1)$ and $T(\pi_2)$. Then it is known that T is affine if and only if $m_{X,Y} c_{X,Y} = \mathrm{id}_{TX \times TY}$ [**?**, Lemma 4.2(i)].[1] In particular, $c_{X,Y}$ is a split mono and therefore mono.

To show that (2) is a pullback, we prove the universal property starting with a diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "(f_1,f_2)"] \arrow[ddr, bend right, "(g_1,g_2)"'] & & \\
& TX \times TY \times TZ \arrow[r, "\mathrm{id} \times c_{Y,Z}"] \arrow[d, "c_{X,Y} \times \mathrm{id}"'] & TX \times T(Y \times Z) \arrow[d, "c_{X,Y \times Z}"] \\
& T(X \times Y) \times TZ \arrow[r, "c_{X \times Y,Z}"] & T(X \times Y \times Z)
\end{tikzcd} \tag{3}$$

where the dashed arrow will be constructed; its uniqueness is clear since $\mathrm{id} \times c_{Y,Z}$ and $c_{X,Y} \times \mathrm{id}$ are mono, so it remains to prove existence. Taking the unlabelled arrows to be (induced by) product projections, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "(g_1,g_2)"] \arrow[d, "(f_1,f_2)"'] & T(X \times Y) \times TZ \arrow[r] & T(X \times Y \times Z) \arrow[d] \\
TX \times T(Y \times Z) \arrow[urr, "c_{X,Y \times Z}"] \arrow[rr] & & T(Y \times Z)
\end{tikzcd}$$

where the upper left triangle commutes by assumption, and the lower right triangle commutes by naturality of c with respect to the unique arrow $X \to 1$ together with $T1 \cong 1$ and the fact that $c_{1,Y \times Z}$ is a coherence isomorphism. By the naturality of c, f_2 can be written as the composite

$$A \xrightarrow{(g_1,g_2)} T(X \times Y) \times TZ \longrightarrow TY \times TZ \xrightarrow{c_{Y,Z}} T(Y \times Z).$$

By analogous reasoning, we identify g_1 with the composite

$$A \xrightarrow{(f_1,f_2)} TX \times T(Y \times Z) \longrightarrow TX \times TY \xrightarrow{c_{X,Y}} T(X \times Y).$$

Getting back to (3), we take the dashed arrow to be the arrow whose component on TX is given by f_1, on TZ by g_2, and on TY by the diagonal in the diagram

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, "g_1"'] & T(Y \times Z) \arrow[d] \\
T(X \times Y) \arrow[r] & TY
\end{tikzcd}$$

which commutes for similar reasons as above. The fact that this arrow recovers the f_2 component after composition with $\mathrm{id} \times c_{Y,Z}$ and the g_1 component after composition with $c_{X,Y} \times \mathrm{id}$ follows by the expressions for f_2 and g_1 derived above. The fact that it recovers f_1 and g_2 is by construction.

◀

[1] For probability monads, this equation can be interpreted as stating that the marginals of a product distribution are the original factors [**?**].

▶ **Remark 2.8.** We are not aware of any relation between weakly affine monads in our sense and Jacobs' *strongly affine* monads [**?**], other than the fact that strongly affine implies affine implies weakly affine.

▶ **Example 2.9.** We present a family of examples of commutative monads that are weakly affine but not affine. Let A be an abelian group (written multiplicatively). Then the functor $T_A \coloneqq A \times -$ on **Set** has a canonical structure of commutative monad, where the lax structure components $c_{X,Y}$ are given by multiplying elements in A while carrying the elements of X and Y along.

Since $T_A \cong A$, the monad T_A is clearly not affine unless A is the trivial group. However, T_A is always weakly affine. Indeed, in order to show that (2) is a pullback, it suffices to show that the associativity square of A

$$\begin{tikzcd} A \times A \times A \arrow[r, "\mathrm{id} \times \cdot"] \arrow[d, "\cdot \times \mathrm{id}"] & A \times A \arrow[d, "\cdot"] \\ A \times A \arrow[r, "\cdot"] & A \end{tikzcd}$$

is a pullback. Using element-wise reasoning, this amounts to showing that the system of equations $ax = c$ and $xb = d$ has a solution for $x \in A$ if and only if $cb = ad$, and in this case the solution is unique. But this is indeed the case with $x = a^{-1}c = db^{-1}$. (Note that this argument does not even require A to be abelian, but we need to require this in order for T_A to be commutative.)

▶ **Example 2.10.** Many monads in categorical measure theory are weakly affine but not affine. Let e.g. $M^* : \mathbf{Set} \to \mathbf{Set}$ be the monad assigning to every set the set of finitely supported discrete *nonzero* measures on M^*, or equivalently let $M^*(X)$ for any set X be the set of nonzero finitely supported functions $X \to [0, \infty)$. The monad structure is defined in terms of the same formulas as for the distribution monad on **Set** and the components $c_{X,Y}$ are also given by the formation of product measures, or equivalently point-wise products of functions $X \to [0, \infty)$.

Since $M^*1 \cong (0, \infty)$, this monad is clearly not affine. However, it is weakly affine, and we limit ourselves to a sketch of the proof. Indeed to prove that (2) is a pullback, we again reason in terms of elements. If all measures are normalised, then we are back in the situation of the distribution monad, which is affine and the claim follows. In the general case, one can reduce to the normalised case by showing that the normalisation of the desired element of $M^*(Y)$ is uniquely determined. This works in the same way as in Example 2.9 with $A = (0, \infty)$.

On the other hand, if the zero measure is included, then we obtain a commutative monad M which can be seen as the monad of semimodules for the semiring of nonnegative reals. Since $M1 \cong [0, \infty)$ is not a group under multiplication, M is not weakly affine.

The previous two examples and Lemma 2 suggest the following problem.

▶ **Problem 2.11.** Let T be a commutative monoid such that $T(1)$ is an abelian group. Does it follow that T is weakly affine?