


Weakly affine monads

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Abstract

Introduced in the 1990s in works on the algebraic approach to graph rewriting, gs-monoidal categories are symmetric monoidal categories where each object has the structure of a commutative comonoid. They arise for example as Kleisli categories of commutative monads on cartesian categories, and as such provide a useful framework for effectful computation. Recently proposed in the context of categorical probability, Markov categories are gs-monoidal categories where the monoidal unit is also terminal, and they arise for example as Kleisli categories of commutative *affine* monads, where affine means that the monad is required to preserve the terminal object.

The aim of this paper is to study a new condition on the gs-monoidal structure, resulting in the concept of *weakly Markov categories*, which are intermediate between general gs-monoidal categories and Markov categories. In a weakly Markov category, the morphisms to the monoidal unit are not necessarily unique, but form a group. As we show, these categories exhibit a rich theory of conditional independence for morphisms, generalising the known theory for Markov categories. We also introduce the corresponding notion for commutative monads, which we call weakly affine, and for which we give two equivalent characterisations.

The paper argues that such monads are relevant to the study of categorical probability. A case at hand is the monad of non-negative, non-zero measures, which is weakly affine but not affine. With these structures, one can investigate probability without normalisation within a fruitful categorical framework.

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1 Introduction

2 Background

In this section, we develop some relevant background material for later reference. To begin, the following categorical characterization of groups will be useful to keep in mind.

► **Proposition 2.1.** *A monoid (M, m, e) in **Set** is a group if and only if the associativity square*

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times \text{id}} & M \times M \\ \downarrow \text{id} \times m & & \downarrow m \\ M \times M & \xrightarrow{m} & M \end{array} \quad (1)$$

is a pullback.

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Proof. The square (1) is a pullback of sets if and only if given $a, g, h, c \in M$ such that $ag = hc$, there exists a unique $b \in M$ such that $g = bc$ and $h = ab$. First, suppose that G is a group. Then the only possible choice of b is

$$b = a^{-1}h = gc^{-1},$$

which is unique by uniqueness of inverses.

Conversely, suppose that (1) is a pullback. We can set $g, h = e$ and $c = a$ so that $ae = ea = a$. Instantiating the pullback property on these elements gives b such that $ab = e$ and $ba = e$, that is, $b = a^{-1}$. ◀

Proposition 2.1 holds generally for a monoid object in a cartesian monoidal category, where the elementwise proof still applies thanks to the following standard observation.

► **Remark 2.2.** Given an object M in a cartesian monoidal category \mathcal{D} , there is a bijection between internal monoid structures on M and monoid structures on every hom-set $\mathcal{D}(X, M)$ such that precomposition with any $f : X \rightarrow Y$ defines a monoid homomorphism

$$\mathcal{D}(Y, M) \longrightarrow \mathcal{D}(X, M).$$

The proof is straightforward by the Yoneda lemma. It follows that Proposition 2.1 holds for internal monoids in cartesian monoidal categories in general.

For the consideration of categorical probability, we now recall the simplest version of a commutative monad of measures. This works with measures taking values in any semiring instead of $[0, \infty)$ (see e.g. [7, Section 5.1]), but we restrict to the concrete case of $[0, \infty)$ for simplicity.

► **Definition 2.1.** Let X be a set. Denote by MX the set of finitely supported measures on X , i.e. functions $m : X \rightarrow [0, \infty)$ which are zero for all but a finite number of $x \in X$. Given a function $f : X \rightarrow Y$, denote by $Mf : MX \rightarrow MY$ the function sending $m \in MX$ to the assignment

$$(Mf)(m) : y \longmapsto \sum_{x \in f^{-1}(y)} p(x).$$

This makes M into a functor, and even a monad with the unit and multiplication maps

$$\begin{array}{ccc} X & \xrightarrow{\delta} & MX \\ x & \longmapsto & \delta_x, \end{array} \quad \begin{array}{ccc} MMX & \xrightarrow{E} & MX \\ \xi & \longmapsto & E\xi, \end{array}$$

where

$$\delta_x(x') = \begin{cases} 1 & x = x', \\ 0 & x \neq x', \end{cases} \quad (E\xi)(x) = \sum_{m \in MX} \xi(m) m(x).$$

Call M the measure monad on **Set**.

Denote also by $DX \subseteq MX$ the subset of probability measures, i.e. those finitely supported $p : X \rightarrow [0, \infty)$ such that

$$\sum_{x \in X} p(x) = 1.$$

D forms a submonad of M called the distribution monad.

It is well-known that M is even a commutative monad [7]. The corresponding lax monoidal structure

$$MX \times MY \xrightarrow{c} M(X \times Y)$$

is exactly the formation of product measures given by $c(m, m')(x, y) = m(x)m'(y)$. Also D is a commutative monad with the same lax monoidal structure, since the product of probability measures is again a probability measure.

2.1 GS-monoidal and Markov categories

The notion of *gs-monoidal category* has been originally introduced in the context of algebraic approaches to term graph rewriting [3], and then developed in a series of papers [4, 6, 5]. We recall here the basic definitions adopting the graphical formalism of string diagrams, referring to [16] for background on various notions of monoidal categories and their associated diagrammatic calculus.

► **Definition 2.2.** A *gs-monoidal category* is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ with a commutative comonoid structure on each object X consisting of a comultiplication and a counit,

$$\text{copy}_X = \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array} \quad \text{del}_X = \begin{array}{c} \bullet \\ | \\ X \end{array}$$

which satisfy the commutative comonoid equations:

$$\begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array} = \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ \backslash \\ \text{---} \\ \bullet \\ / \\ \text{---} \\ | \\ X \end{array} = \begin{array}{c} | \\ X \end{array} \quad \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array} = \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array}$$

These comonoid structures must be multiplicative with respect to the monoidal structure:

$$\begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \otimes Y \end{array} = \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ | \\ X \otimes Y \end{array} = \begin{array}{c} \bullet \\ | \\ X \end{array} \quad \begin{array}{c} \bullet \\ | \\ Y \end{array}$$

$$\begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ I \end{array} = \begin{array}{c} \text{---} \backslash \text{---} \\ \bullet \\ \text{---} / \text{---} \\ | \\ I \end{array} \quad \begin{array}{c} \bullet \\ | \\ I \end{array} = \begin{array}{c} \bullet \\ | \\ I \end{array}$$

► **Definition 2.3.** A morphism $f : X \rightarrow Y$ in a gs-monoidal category is called *copyable* or *functional* if and only if

$$\begin{array}{c} Y \quad Y \\ \backslash \quad / \\ \bullet \\ | \\ \boxed{f} \\ | \\ X \end{array} = \begin{array}{c} Y \quad Y \\ \boxed{f} \quad \boxed{f} \\ \backslash \quad / \\ \bullet \\ | \\ X \end{array}$$

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98 It is called *discardable* or *full* if

$$99 \quad \begin{array}{c} \bullet \\ \downarrow \\ \boxed{f} \\ \downarrow \\ X \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ X \end{array}$$

100 ► **Example 2.4.** The category **Rel** of sets and relations with the monoidal operation
 101 $\otimes : \mathbf{Rel} \times \mathbf{Rel} \rightarrow \mathbf{Rel}$ given by the direct product of sets is a gs-monoidal category [6]. In
 102 this gs-monoidal category, the copyable arrows are precisely the partial functions, and the
 103 discardable arrows are the total relations.

104 ► **Remark 2.5.** It is well-known that if every morphism is copyable and discardable, or
 105 equivalently if the copy and discard maps are natural, then the monoidal product is the
 106 categorical product, and thus the category is cartesian monoidal [8]. More concisely, the
 107 following conditions are equivalent for a gs-monoidal category \mathcal{C} :

- 108 ■ \mathcal{C} is cartesian monoidal;
- 109 ■ every morphism is copyable and discardable;
- 110 ■ the copy and discard maps are natural.

add ref

111 In recent works [?] it has been shown that gs-monoidal categories naturally arise in several
 112 ways, such as Kleisli categories of commutative monads or span categories. In the following
 113 proposition, we recall the result regarding Kleisli categories:

114 ► **Proposition 2.6.** *Let T be a commutative monad on a cartesian monoidal category \mathcal{D} .
 115 Then its Kleisli category \mathbf{Kl}_T is canonically a gs-monoidal category with copy and discard
 116 structure induced by that of \mathcal{D} .*

117 Nowadays, *Markov categories* [9] represent one of the more interesting specializations of the
 118 notion of gs-monoidal category. Based on the interpretation of their arrows as generalised
 119 Markov kernels, Markov categories are considered the foundation for a categorical approach
 120 to probability theory.

121 ► **Definition 2.7.** *A gs-monoidal category is said to be a **Markov category** if any (hence
 122 all) of the following equivalent conditions are satisfied:*

- 123 ■ the monoidal unit is terminal;
- 124 ■ the discard maps are natural;
- 125 ■ every morphism is discardable.

126 We recall from [14, 12] the notion of *affine monad*:

127 ► **Definition 2.8.** *A monad T on a cartesian monoidal category is called **affine** if $T1 \cong 1$.*

128 It was observed in [9, Corollary 3.2] that if the monad preserves the terminal object, then
 129 every arrow of the Kleisli category is discardable, and this makes the Kleisli category into a
 130 Markov category. In other words, we have the following specialization of Proposition 2.6:

131 ► **Proposition 2.9.** *Let T be a symmetric monoidal (equivalently, commutative) monad on a
 132 cartesian monoidal category \mathcal{D} . Then \mathbf{Kl}_T is Markov if and only if T is affine.*

3 Weakly Markov categories and weakly affine monads

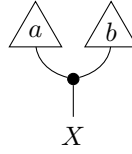
134 In this section, we introduce an intermediate level between gs-monoidal and Markov called
 135 *weakly Markov*, and its corresponding notion for monads, which we call *weakly affine*.

3.1 The monoid of effects

In a gs-monoidal category \mathcal{C} we call a *state* a morphism from the monoidal unit $p : I \rightarrow X$, and *effect* a morphism to the monoidal unit $a : X \rightarrow I$. As is standard convention, we represent such morphisms as triangles as follows.



Effects, i.e. elements of the set $\mathcal{C}(X, I)$, form canonically a commutative monoid as follows: the monoidal unit is the discard map $X \rightarrow I$, and given $a, b : X \rightarrow I$, their product ab is given by copying:¹



If a morphism $f : X \rightarrow Y$ is copyable and discardable, precomposition with f induces a morphism of monoids $\mathcal{C}(Y, I) \rightarrow \mathcal{C}(X, I)$.

Let's now consider the case where the gs-monoidal structure comes from a commutative monad on a cartesian monoidal category \mathcal{D} . For this case, we now describe how the monoid structure on Kleisli morphisms $X \rightarrow 1$ comes from a canonical internal monoid structure on $T1$ in \mathcal{D} . This monoid structure on $T1$ is given by the following unit and multiplication [15, Section 10],

$$1 \xrightarrow{\eta} T1, \quad T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

For example, for the monad of measures M , we obtain $M1 = [0, \infty)$ with its usual multiplication. The resulting monoid structure on Kleisli morphisms $X \rightarrow 1$ is now given as follows. The unit is given by

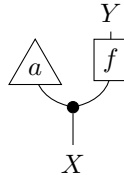
$$X \xrightarrow{!} 1 \xrightarrow{\eta} T1,$$

and the multiplication of Kleisli morphisms $f, g : X \rightarrow 1$ represented by $f^\sharp, g^\sharp : X \rightarrow T1$ is the Kleisli morphism represented by

$$X \xrightarrow{\text{copy}_X} X \times X \xrightarrow{f^\sharp \times g^\sharp} T1 \times T1 \xrightarrow{c} T(1 \times 1) \xrightarrow{\cong} T1.$$

For the monad of measures M , Kleisli morphisms $X \rightarrow 1$ are functions $X \rightarrow [0, \infty)$, and this description shows that their product is the pointwise product.

For general \mathcal{C} , note that the commutative monoid $\mathcal{C}(X, I)$ acts on the set $\mathcal{C}(X, Y)$: given $a : X \rightarrow I$ and $f : X \rightarrow Y$, the resulting $a \cdot f$ is given as follows,



¹ See also e.g. the \odot product in [2, Proposition 3.10].

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It is straightforward to see that this indeed amounts to an action of the monoid $\mathcal{C}(X, I)$ on the set $\mathcal{C}(X, Y)$. For the monad of measures M , this action is given by pointwise rescaling. Moreover, for general \mathcal{C} the operation

$$\begin{aligned} \mathcal{C}(X, Y) \times \mathcal{C}(X, Z) &\longrightarrow \mathcal{C}(X, Y \otimes Z) \\ (f, g) &\longmapsto f \cdot g := (f \otimes g) \circ \text{copy}_X \end{aligned}$$

commutes with this action in each variable (separately).

3.2 Main definitions

► **Definition 3.1.** A gs-monoidal category \mathcal{C} is called **weakly Markov** if for every object X , the monoid $\mathcal{C}(X, I)$ is a group.

Every Markov category is weakly Markov: for each X , the monoid $\mathcal{C}(X, I)$ is the trivial group.

► **Definition 3.2.** Given two parallel morphisms $f, g : X \rightarrow Y$ in a weakly Markov category \mathcal{C} , we say that f and g are called **equivalent**, denoted $f \sim g$, if they lie in the same orbit for the action of $\mathcal{C}(X, I)$, i.e. if there is $a \in \mathcal{C}(X, I)$ such that $a \cdot f = g$.

Note that if $a \cdot f = g$ for some a , then a is unique. This can be seen by marginalizing over Y the following diagram.

$$\begin{array}{c} \triangle a \quad \square f \\ \quad \quad \quad \bullet \\ \quad \quad \quad | \\ X \end{array} = \begin{array}{c} Y \\ | \\ \square g \\ | \\ X \end{array}$$

In other words, the action of $\mathcal{C}(X, I)$ on $\mathcal{C}(X, Y)$ is free, i.e. it has trivial stabilizers.

T: Is it worth noting that the equivalence classes form a Markov cat, which is isomorphic to the Markov cat of discardable morphisms?

P: It's not entirely clear to me how composition is well defined for generic (non-copyable) morphisms.

Let's now look at the Kleisli case.

► **Definition 3.3.** A commutative monad T on a cartesian monoidal category is called **weakly affine** if $T1$ with its canonical internal commutative monoid structure is a group.

This choice of terminology is motivated by the following proposition, which can be seen as a “weakly” version of Proposition 2.9.

► **Proposition 3.4.** Let \mathcal{D} be a cartesian monoidal category, and let T be a commutative monad on \mathcal{D} . Then the Kleisli category of T is weakly Markov if and only if T is weakly affine.

Proof. First, suppose that $T1$ is an internal group, and denote by $\iota : T1 \rightarrow T1$ its inversion map. The inverse of a Kleisli morphism $a : X \rightarrow 1$ in $\text{Kl}_T(X, 1)$ represented by $a^\# : X \rightarrow T1$

196 is represented by $\iota \circ a^\sharp$: indeed, the following diagram in \mathcal{D} commutes,

$$\begin{array}{ccccc}
 X & \xrightarrow{\text{copy}_X} & X \times X & & \\
 \downarrow a^\sharp & & \downarrow a^\sharp \times a^\sharp & \searrow a^\sharp \times (\iota \circ a^\sharp) & \\
 T1 & \xrightarrow{\text{copy}_X} & T1 \times T1 & \xrightarrow{\text{id} \times \iota} & T1 \times T1 \xrightarrow{c} T(1 \times 1) \\
 \downarrow ! & & \downarrow \eta & & \downarrow \cong \\
 1 & \xrightarrow{\eta} & T1 & &
 \end{array}$$

198 where the bottom rectangle commutes since ι is the inversion map for $T1$. The analogous
 199 diagram with $\iota \times \text{id}$ in place of $\text{id} \times \iota$ commutes analogously.

200 Conversely, suppose that for every X , the monoid structure on $\text{Kl}_T(X, 1)$ has inverses.
 201 Then in particular we can take $X = T1$, and the inverse of the Kleisli morphism $\text{id} : T1 \rightarrow T1$
 202 is an inversion map for $T1$. \blacktriangleleft

203 This result can also be thought of in terms of the Yoneda embedding, see the details in
 204 ??.

205 rewrite what we are trying to say here

206 3.3 Examples of weakly affine monads

207 Every affine monad is a weakly affine monad. Here are less trivial examples.

208 ► **Example 3.5.** Let $M^* : \mathbf{Set} \rightarrow \mathbf{Set}$ be the monad assigning to every set the set of finitely
 209 supported discrete *nonzero* measures on M^* , or equivalently let $M^*(X)$ for any set X be
 210 the set of nonzero finitely supported functions $X \rightarrow [0, \infty)$. It is a submonad $M^* \subseteq M$,
 211 meaning that the monad structure is defined in terms of the same formulas as for the monad
 212 of measures M (Definition 2.1). Similarly, the lax structure components $c_{X,Y}$ are also
 213 given by the formation of product measures, or equivalently pointwise products of functions
 214 $X \rightarrow [0, \infty)$.

215 Since $M^*1 \cong (0, \infty) \not\cong 1$, this monad is not affine. However the monoid structure of
 216 $(0, \infty)$ induced by M^* is the usual multiplication of positive real numbers, which form a
 217 group. Therefore M^* is weakly affine, and its Kleisli category is weakly Markov.

T: More generally, we could consider nonzero measures with values in any positive semi-field, see the corresponding monads considered in arXiv:2108.10718. Not sure though if it's interesting enough to mention?

218 On the other hand, if the zero measure is included, we have $M1 \cong [0, \infty)$ which is not a
 219 group under multiplication, so M is not weakly affine.

221 ► **Example 3.6.** Let A be a commutative monoid. Then the functor $T_A := A \times -$ on \mathbf{Set}
 222 has a canonical structure of commutative monad, where the lax structure components $c_{X,Y}$
 223 are given by multiplying elements in A while carrying the elements of X and Y along.

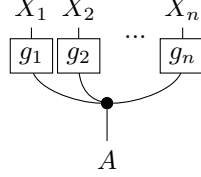
224 Since $T_A(1) \cong A$, the monad T_A is weakly affine if and only if A is a group, and affine if
 225 and only if $A \cong 1$.

226 ► **Example 3.7.** Here is a negative example. Consider the free abelian group monad F on
 227 \mathbf{Set} . Its functor takes a set X and forms the set FX of finite multisets (with repetition,
 228 where order does not matter) of elements of X and their formal inverses. We have that
 229 $F1 \cong \mathbb{Z}$, which is an abelian group under addition. However, the monoid structure on $F1$
 230 induced by the monoidal structure of the monad corresponds to the *multiplication* on \mathbb{Z} ,
 231 which does not have inverses. Therefore F is not weakly affine.

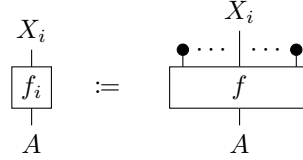
4 Conditional independence in weakly Markov categories

Markov categories have a rich theory of conditional independence in the sense of probability theory [11]. Some of those ideas can be translated and generalized to the setting of weakly Markov categories.

► **Definition 4.1.** A morphism $f : A \rightarrow X_1 \otimes \cdots \otimes X_n$ in a gs-monoidal category \mathcal{C} is said to exhibit **conditional independence of the X_i given A** if and only if it can be expressed as a product of the following form.



Note that this formulation is a bit different from the earlier definitions given in [1, Definition 6.6] and [9, Definition 12.12], which were formulated for morphisms in Markov categories and state that f exhibits conditional independence if the above holds with the g_i being the *marginals* of f , which are



Indeed in a Markov category, conditional independence in our sense holds if and only if it holds with $g_i = f_i$ [9, Lemma 12.11]. We also say that f is the *product of its marginals*.

► **Example 4.2.** In the Kleisli category of the distribution monad D , which is Markov, a morphism $f : A \rightarrow X \otimes Y$ exhibits conditional independence if and only if its value at every $a \in A$ is the product of its marginals [9, Section 12].

Here is what conditional independence looks like in the Kleisli case.

► **Proposition 4.3.** Let \mathcal{D} be a cartesian monoidal category, and let T be a commutative monad on \mathcal{D} . A Kleisli morphism represented by $f^\sharp : A \rightarrow T(X_1 \times \cdots \times X_n)$ exhibits conditional independence of the X_i given A if and only if it factors as follows

$$\begin{array}{ccc} A & \xrightarrow{f^\sharp} & T(X_1 \times \cdots \times X_n) \\ \downarrow (g_1^\sharp, \dots, g_n^\sharp) & \searrow & \\ TX_1 \times \cdots \times TX_n & \xrightarrow{c} & T(X_1 \times \cdots \times X_n) \end{array}$$

for some Kleisli maps $g_i^\sharp : A \rightarrow TX_i$, where the map c above is the one obtained by iterating the lax monoidal structure (which is unique by associativity).

Proof. In terms of the base category \mathcal{D} , a Kleisli morphism in the form of Definition 4.1 reads as follows.

$$A \xrightarrow{\text{copy}} A \times \cdots \times A \xrightarrow{g_1^\sharp \times \cdots \times g_n^\sharp} TX_1 \times \cdots \times TX_n \xrightarrow{c} T(X_1 \times \cdots \times X_n).$$

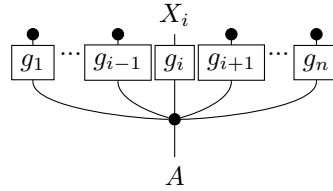
Therefore $f^\sharp : A \rightarrow T(X_1 \times \cdots \times X_n)$ exhibits the conditional independence if and only if it is of the form above. ◀

► **Example 4.4.** In the Kleisli category of the measure monad M , and for any objects, the morphism $A \rightarrow X_1 \otimes \cdots \otimes X_n$ given by the zero measure on every $a \in A$ exhibits conditional independence of its outputs given its input. For example, for $A = 1$, the zero measure on $X \times Y$ is the product of the zero measure on X and the zero (or any other) measure on Y . Notice that both marginals of the zero measure are zero measures—therefore, the factors appearing in the product are not necessarily related to the marginals.

In a weakly Markov category, the situation is similar to the Markov case discussed above, but up to equivalence: an arrow exhibits conditional independence if and only if it is *equivalent* to the product of all its marginals.

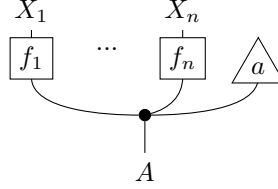
► **Proposition 4.5.** Let $f : A \rightarrow X_1 \otimes \cdots \otimes X_n$ be a morphism in a weakly Markov category \mathcal{C} . Then f exhibits conditional independence of the X_i given A if and only if it is equivalent to the product of all its marginals.

Proof. Denote the marginals of f by f_1, \dots, f_n . Suppose that f is a product as in Definition 4.1. For each $i = 1, \dots, n$, by marginalizing, we get that f_i is equal to the following.



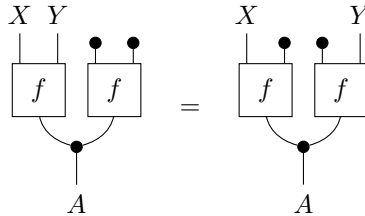
Therefore for each i we have that $f_i \sim g_i$.

Conversely, suppose that f is equivalent to the product of its marginals, i.e. that there exists $a : X \rightarrow I$ such that f is equal to the following.



One can then choose $g_i = f_i$ for all $i < n$, and $g_n = a \cdot f_n$, so that f is in the form of Definition 4.1. ◀

► **Remark 4.6.** For $n = 2$, a morphism $f : A \rightarrow X \otimes Y$ in a weakly Markov category \mathcal{C} exhibits conditional independence of X and Y given A if and only if the following equation holds.



4.1 Main result

The concept of conditional independence for general weakly Markov categories allows us to give an equivalent characterization of weakly affine monads. The condition is in terms of a pullback condition on the associativity diagram, and can be seen as a generalization of Proposition 2.1.

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► **Theorem 4.7.** *Let \mathcal{D} be a cartesian monoidal category, and let T be a commutative monad on \mathcal{D} . Then the following conditions are equivalent.*

1. *T is weakly affine;*
2. *The Kleisli category \mathbf{Kl}_T is weakly Markov;*
3. *For all objects X, Y , and Z , the following associativity diagram is a pullback.*

$$\begin{array}{ccc}
 T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\
 \downarrow c_{X,Y} \times \text{id} & & \downarrow c_{X,Y \times Z} \\
 T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z)
 \end{array} \quad (2)$$

We prove the theorem by means of the following property of weakly Markov categories.

► **Lemma 4.8** (localized independence property). *Let \mathcal{C} be a weakly Markov category. Whenever a morphism $f : A \rightarrow X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A , as well as conditional independence of X and $Y \otimes Z$ given A , then it exhibits conditional independence of X, Y and Z given A .*

Proof of Lemma 4.8. Suppose $f : A \rightarrow X \otimes Y \otimes Z$ exhibits conditional independence of $X \otimes Y$ (jointly) and Z given A , as well as conditional independence of X and $Y \otimes Z$ given A . By marginalizing out X , we have that f_{YZ} exhibits conditional independence of Y and Z given A . Since by hypothesis f exhibits conditional independence of X and $Y \otimes Z$ given A , by Proposition 4.5 we have that f is equivalent to the product of f_X and f_{YZ} . But, again by Proposition 4.5, f_{YZ} is equivalent to the product of f_Y and f_Z , so we have that f is equivalent to the product of all its marginals. Using Proposition 4.5 in the other direction, this means that f exhibits conditional independence of X, Y and Z given A . ◀

We are now ready to prove the theorem.

Proof of Theorem 4.7. $1 \Leftrightarrow 2$: see Proposition 3.4.

$1 \Rightarrow 3$: By the universal property of products, a cone over the cospan in (2) consists of maps $g_1^\# : A \rightarrow TX$, $g_{23}^\# : A \rightarrow T(Y \times Z)$, $g_{12}^\# : A \rightarrow T(X \times Y)$ and $g_3^\# : A \rightarrow TZ$ such that the following diagram commutes.

$$\begin{array}{ccc}
 A & \xrightarrow{(g_1^\#, g_{23}^\#)} & T(X) \times T(Y \times Z) \\
 \downarrow (g_{12}^\#, g_3^\#) & & \downarrow c_{X,Y \times Z} \\
 T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z)
 \end{array}$$

By Proposition 4.3, this amounts to a Kleisli map $f^\# : A \rightarrow T(X \times Y \times Z)$ exhibiting conditional independence of X and $Y \otimes Z$ given A , as well as of $X \otimes Y$ and Z given A . By the localized independence property (Lemma 4.8), we then have that f exhibits conditional independence of all X, Y and Z given A , and so, again by Proposition 4.3, $f^\#$ factors through the product $TX \times TY \times TZ$. More specifically, by marginalizing over Z , we have that $g_{12}^\#$ factors through $TX \times TY$, i.e. the following diagram on the left commutes for some $h_1^\# : A \rightarrow TX$ and $h_2^\# : A \rightarrow TY$, and similarly, by marginalizing over X , the diagram on the

right commutes for some $\ell_2^\# : A \rightarrow TY$ and $\ell_3^\# : A \rightarrow TZ$.

$$\begin{array}{ccc} A & \xrightarrow{g_{12}^\#} & A \\ (h_1^\#, h_2^\#) \downarrow & \searrow & (\ell_2^\#, \ell_3^\#) \downarrow \\ TX \times TY & \xrightarrow{c} & T(X \times Y) \quad TY \times TZ \xrightarrow{c} T(Y \times Z) \end{array}$$

In other words, the upper and the left curved triangles in the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{(g_1^\#, g_{23}^\#)} & T(X) \times T(Y \times Z) \\ \downarrow (h_1^\#, h_2^\#, g_3^\#) & \searrow (g_1^\#, \ell_2^\#, \ell_3^\#) & \downarrow c_{X,Y \times Z} \\ T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\ \downarrow c_{X,Y} \times \text{id} & & \downarrow c_{X,Y \times Z} \\ T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z) \end{array}$$

By marginalizing over Y and Z , and by weak affinity of T , there exists a unique $a^\# : A \rightarrow T1$ such that $h_1 = a \cdot g_1$. Therefore

$$g_{12} = h_1 \cdot h_2 = (a \cdot g_1) \cdot h_2 = g_1 \cdot (a \cdot h_2),$$

and so in the diagram above we can equivalently replace h_1 and h_2 with g_1 and $a \cdot h_2$. Similarly by marginalizing over X and Y , there exists a unique $c^\# : A \rightarrow T1$ such that $\ell_3 = c \cdot g_3$, so that

$$g_{23} = \ell_2 \cdot \ell_3 = \ell_2 \cdot (c \cdot g_3) = (c \cdot \ell_2) \cdot g_3$$

and in the diagram above we can replace ℓ_2 and ℓ_3 with $c \cdot \ell_2$ and g_3 , as follows.

$$\begin{array}{ccc} A & \xrightarrow{(g_1^\#, g_{23}^\#)} & T(X) \times T(Y \times Z) \\ \downarrow (g_1^\#, (a \cdot h_2)^\#, g_3^\#) & \searrow (g_1^\#, (c \cdot \ell_2)^\#, g_3^\#) & \downarrow c_{X,Y \times Z} \\ T(X) \times T(Y) \times T(Z) & \xrightarrow{\text{id} \times c_{Y,Z}} & T(X) \times T(Y \times Z) \\ \downarrow c_{X,Y} \times \text{id} & & \downarrow c_{X,Y \times Z} \\ T(X \times Y) \times T(Z) & \xrightarrow{c_{X \times Y, Z}} & T(X \times Y \times Z) \end{array}$$

Now, marginalizing over X and Z , we see that necessarily $a \cdot h_2 = c \cdot \ell_2$. Therefore there is a unique map $A \rightarrow TX \times TY \times TZ$ making the whole diagram commute, which means that (2) is a pullback.

$3 \Rightarrow 1$: If T is weakly affine, then taking $X = Y = Z = 1$ in (2) shows that this monoid must be an abelian group: we obtain a unique arrow $\iota : T(1) \rightarrow T(1)$ making the following diagram commute,

$$\begin{array}{ccccc} T1 & \xrightarrow{(\text{id}, \eta_1!)} & T1 \times T1 \times T1 & \xrightarrow{\text{id} \times c_{1,1}} & T1 \times T(1 \times 1) \xrightarrow{\cong} T1 \times T1 \\ \downarrow (\eta_1!, \text{id}) & \searrow (\text{id}, \iota, \text{id}) & \downarrow c_{1,1} \times \text{id} & \downarrow c_{1,1} \times 1 & \downarrow c_{1,1} \\ T(1 \times 1) \times T1 & \xrightarrow{c_{1 \times 1, 1}} & T(1 \times 1 \times 1) & \xrightarrow{\cong} & T(1 \times 1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ T1 \times T1 & \xrightarrow{c_{1,1}} & T(1 \times 1) & \xrightarrow{\cong} & T1 \end{array}$$

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and the commutativity shows that ι satisfies the equations making it the inversion map for a group structure. ◀

► **Example 4.9.** In the Kleisli category of the measure monad Kl_M (which is not weakly affine) consider the following diagram.

$$\begin{array}{ccc} MX \times MY \times MZ & \xrightarrow{\text{id} \times c_{Y,Z}} & MX \times M(Y \times Z) \\ \downarrow c_{X,Y} \times \text{id} & & \downarrow c_{X,Y \times Z} \\ M(X \times Y) \times MZ & \xrightarrow{c_{X \times Y, Z}} & M(X \times Y \times Z) \end{array}$$

In the top-right corner $MX \times M(Y \times Z)$, take the pair $(0, p)$ where p is any nonzero measure on $Y \times Z$, and similarly, in the bottom-left corner take the pair $(q, 0)$ where q is any nonzero measure on $X \times Y$. Following the diagram, both pairs are mapped to the zero measure in the bottom-right corner. If the diagram was a pullback, we would be able to express the top-right and bottom-left corners as coming from the same triple in $MX \times MY \times MZ$, that is, there would exist a measure m on Y such that $m \cdot 0 = p$ and $0 \cdot m = q$. Since p and q are nonzero, this is not possible.

5 Further results

► **Proposition 5.1.** *Let T be a weakly affine monad. If the diagram*

$$\begin{array}{ccc} T(1) & \xrightarrow{\text{id}} & T(1) \\ \downarrow \iota & & \downarrow \eta_{T1} \\ T(1) & \xrightarrow{T(\eta_1)} & T^2(1) \end{array}$$

commutes, then:

1. $T^2(1) \cong T(1)$ in \mathcal{D} .
2. the internal group $T(1)$ has exponent 2, namely $\iota = \text{id}_{T1}$;
3. the group $\text{Kl}_T(X, 1)$ has exponent 2.

T: Having a nontrivial example of this statement would help to motivate and illustrate it. Like this, its meaning and significance remains quite unclear

Proof. To prove the first claim, it is enough to show that $T(1) \cong 1$ in the Kleisli category Kl_T . By weak affinity, $T(1)$ is a group in \mathcal{D} , where the arrow $\eta_1: 1 \rightarrow T(1)$ is the unit of the group and $\iota: T(1) \rightarrow T(1)$ is the inversion map. Therefore, we have that the composition $\iota\eta_1: 1 \rightarrow T(1)$ has to be equal to η_1 . Hence we can consider the arrows $1 \rightarrow T(1)$ and $T(1) \rightarrow 1$ in the Kleisli category Kl_T represented by $T(\eta_1)\eta_1$ and ι , respectively. The composition $T(\eta_1)\eta_1$ with ι in Kl_T is given by $\mu_{1,1}T(\iota)T(\eta_1)\eta_1$. Employing the naturality of η_1 and the fact that $\iota\eta_1 = \eta_1$, it is direct to check that $\mu T(\iota)T(\eta_1)\eta_1 = \eta_1$, that is the identity $1 \rightarrow 1$ in Kl_T . Now to show that the other composition gives the identity on $T(1)$ in Kl_T , it is enough to show that $T(\eta_1)\iota = \eta_{T(1)}$, but this follows by hypothesis.

For the second claim, we can compose the diagram with the monad multiplication, obtaining $\iota = \text{id}_{T1}$.

The last claim follows by combining the second one with the explicit construction of inverses in $\text{Kl}_T(X, 1)$ (see the proof of Proposition 3.4). ◀

377 ► **Remark 5.2.** Bart Jacobs calls a strong monad T on a cartesian monoidal category *strongly*
 378 *affine* [13] if for every pair of objects X and Y , the following diagram is a pullback,

$$\begin{array}{ccc} X \times TY & \xrightarrow{s} & T(X \times Y) \\ \downarrow \pi_1 & & \downarrow T\pi_1 \\ X & \xrightarrow{\eta} & TX \end{array}$$

380 where s denotes the strength and η denotes the unit of the monad. Every strongly affine
 381 monad is affine. The corresponding condition on the (Markov) category \mathbf{Kl}_T is called
 382 *positivity* [10, Section 2].

383 Note that for a generic commutative monad, the diagram above may even fail to commute
 384 (take for example the measure monad M , and start with $(x, 0)$ in the top left corner). One can
 385 however consider the following diagram, which reduces to the one above (up to isomorphism)
 386 in the affine case,

$$\begin{array}{ccc} X \times TY & \xrightarrow{s} & T(X \times Y) \\ \downarrow \text{id} \times T! & & \downarrow T(\text{id} \times !) \\ X \times T1 & \xrightarrow{s} & T(X \times 1) \cong TX \end{array}$$

388 and which always commutes by naturality of the strength. One can then call the monad
 389 T *positive* if this second diagram is a pullback (and possibly define *positive gs-monoidal*
 390 *categories* analogously to positive Markov categories).

391 — References —

- 392 1 K. Cho and B. Jacobs. Disintegration and Bayesian inversion via string diagrams. *Mathematical*
 393 *Structures in Computer Science*, 29(7):938–971, 2019.
- 394 2 Bob Coecke, Bill Edwards, and Robert W. Spekkens. Phase groups and the origin of non-
 395 locality for qubits. In *Proceedings of the 6th International Workshop on Quantum Physics and*
 396 *Logic (QPL 2009)*, volume 270 of *ENTCS*, 2011. arXiv:1003.5005.
- 397 3 A. Corradini and F. Gadducci. A 2-categorical presentation of term graph rewriting. In
 398 E. Moggi and G. Rosolini, editors, *CTCS 1997*, volume 1290 of *LNCS*, pages 87–105. Springer,
 399 1997.
- 400 4 A. Corradini and F. Gadducci. An algebraic presentation of term graphs, via gs-monoidal
 401 categories. *Applied Categorical Structures*, 7:299–331, 1999.
- 402 5 A. Corradini and F. Gadducci. Rewriting on cyclic structures: equivalence between the
 403 operational and the categorical description. *RAIRO - Theoretical Informatics and Applications*
 404 *- Informatique Théorique et Applications*, 33(4-5):467–493, 1999.
- 405 6 A. Corradini and F. Gadducci. A functorial semantics for multi-algebras and partial algebras,
 406 with applications to syntax. *Theoretical Computer Science*, 286:293–322, 2002.
- 407 7 Dion Coumans and Bart Jacobs. Scalars, monads, and categories. In *Quantum Physics and*
 408 *Linguistics: A Compositional, Diagrammatic Discourse*, pages 184–216. Oxford University
 409 Press, 2013. arXiv:1003.0585.
- 410 8 T. Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4:665–667, 1976.
- 411 9 T. Fritz. A synthetic approach to Markov kernels, conditional independence and theorems on
 412 sufficient statistics. *Advances in Mathematics*, 370:107239, 2020.
- 413 10 Tobias Fritz, Tomáš Gonda, Nicholas Gauguin Houghton-Larsen, Paolo Perrone, and Dario
 414 Stein. Dilations and information flow axioms in categorical probability. arXiv:2211.02507.
- 415 11 Tobias Fritz and Andreas Klingler. The d -separation criterion in categorical probability. *J.*
 416 *Mach. Learn. Res.*, 24(46):1–49, 2023. arXiv:2207.05740.

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- 417 12 B. Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*,
418 69(1):73–106, 1994.
- 419 13 B. Jacobs. Affine monads and side-effect-freeness. In I. Hasuo, editor, *CMCS 2016*, volume
420 9608 of *LNCS*, pages 53–72. Springer, 2016.
- 421 14 A. Kock. Bilinearity and cartesian closed monads. *Mathematica Scandinavica*, 29(2):161–174,
422 1971.
- 423 15 Anders Kock. Commutative monads as a theory of distributions. *Theory Appl. Categ.*, 26,
424 2012. arXiv:1108.5952.
- 425 16 P. Selinger. A survey of graphical languages for monoidal categories. In B. Coecke, editor,
426 *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 289–355. Springer,
427 2011.