

# MATH 505a Fall 2021 Qual Solution Attempts

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## Problem 1

(a)

Let  $X$  be a non-negative random variable with finite expectation. Show that

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \leq E[X] < 1 + \sum_{i=1}^{\infty} \mathbb{P}(X \geq i).$$

*Proof.* Since  $X$  is non-negative,

$$\begin{aligned} \mathbb{E}[X] &= \int_{[0, \infty)} xf(x)dx \\ &= \sum_{i=0}^{\infty} \int_{[i, i+1)} xf(x)dx \end{aligned}$$

Then notice that,

$$i \int_{[i, i+1)} f(x)dx \leq \int_{[i, i+1)} xf(x)dx \leq (i+1) \int_{[i, i+1)} f(x)dx$$

That is,

$$i\mathbb{P}(i \leq X \leq i+1) \leq \int_{[i, i+1)} xf(x)dx \leq (i+1)\mathbb{P}(i \leq X \leq i+1)$$

Plugging into the sum, the lower bound becomes:

$$\begin{aligned} &\sum_{i=0}^{\infty} i\mathbb{P}(i \leq X \leq i+1) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \end{aligned}$$

Similarly, the upper bound:

$$\begin{aligned}
 & \sum_{i=0}^{\infty} (i+1) \mathbb{P}(i \leq X \leq i+1) \\
 &= \sum_{i=0}^{\infty} i \mathbb{P}(i \leq X \leq i+1) + \sum_{i=0}^{\infty} \mathbb{P}(i \leq X \leq i+1) \\
 &= \sum_{i=1}^{\infty} \mathbb{P}(X \geq i) + 1
 \end{aligned}$$

□

**(b)**

Show that if  $X$  takes values only in  $\{0, 1, \dots, n\}$  for some  $n$ , then the first inequality in (a) is an equality:

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) = \mathbb{E}[X].$$

*Proof.* Note that if  $X$  only take natural number values, we have  $\mathbb{P}(X \geq i) = \sum_{k=i}^{\infty} \mathbb{P}(X = k)$ . Plug this into the left hand side:

$$\begin{aligned}
 \sum_{i=1}^{\infty} \mathbb{P}(X \geq i) &= \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \mathbb{P}(X = k) \\
 &= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \dots \\
 &\quad + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \dots \\
 &\quad + \mathbb{P}(X = 3) + \dots \\
 &\quad + \dots \\
 &= \sum_{i=1}^{\infty} i \mathbb{P}(X = i) \\
 &= \mathbb{E}[X]
 \end{aligned}$$

□

**(c)**

Let  $M$  be the minimum value seen in 4 die rolls. Find  $\mathbb{E}[M]$ . You don't need to simplify to one number, just get an expression in terms of numbers only.

*Proof.* Note that  $M$  only takes values in  $\{1, 2, 3, 4, 5, 6\}$ , let  $X_i$  denotes the value of  $i$ -th dice roll we can use the conclusion from (b) that:

$$\begin{aligned}\mathbb{E}[M] &= \sum_{i=1}^6 \mathbb{P}(M \geq i) \\ &= \sum_{i=1}^6 \prod_{j=1}^4 \mathbb{P}(X_j \geq i) \\ &= \sum_{i=1}^6 \left(1 - \frac{i-1}{6}\right)^4\end{aligned}$$

□

## Problem 2

Suppose  $X$  and  $Y$  are independent continuous random variables with uniform distribution on  $[0, 1]$ .

(a)

Find the density function of  $X + 2Y$ .

*Solution.* By conditioning on  $X$ , we have:

*Case 1.*  $z \in [0, 1]$ ,

$$\begin{aligned}\mathbb{P}(X + 2Y \leq z) &= \int_0^z \frac{1}{2}(z - x) \, dx \\ &= \frac{z^2}{4}\end{aligned}$$

*Case 2.*  $z \in [1, 2]$ ,

$$\begin{aligned}\mathbb{P}(X + 2Y \leq z) &= \int_0^1 \frac{1}{2}(z - x) \, dx \\ &= \frac{2z - 1}{4}\end{aligned}$$

*Case 3.*  $z \in [2, 3]$ ,

$$\begin{aligned}\mathbb{P}(X + 2Y \leq z) &= (z - 2) + \int_{z-2}^1 \frac{1}{2}(z - x) \, dx \\ &= z - 2 - \frac{z^2 - 2z - 3}{4}\end{aligned}$$

So compute the pdf by differentiating:

$$f_{X+Y}(z) = \begin{cases} \frac{z}{2} & 0 \leq z < 1 \\ \frac{1}{2} & 1 \leq z < 2 \\ -\frac{z}{2} + \frac{3}{2} & 2 \leq z \leq 3 \end{cases}$$

(b)

Find the joint density function for  $X - Y, X + Y$ .

*Solution.* Let  $U = X + Y, V = X - Y$ . Then  $X = \frac{U+V}{2}, Y = \frac{U-V}{2}$ . We can compute the absolute value of Jacobian of the map  $(u, v) \mapsto (x, y)$ :

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

then We have the following:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot |J| \\ &= \frac{1}{2} \mathbb{1}_{0 \leq \frac{u+v}{2} \leq 1, 0 \leq \frac{u-v}{2} \leq 1}(u, v) \\ &= \frac{1}{2} \mathbb{1}_{0 \leq u+v \leq 2, 0 \leq u-v \leq 2}(u, v) \end{aligned}$$

### Problem 3

Consider Bernoulli trials with success probability  $p \in (0, 1)$ . Let  $p_n$  be the probability of an odd number of successes in  $n$  trials.

(a)

Express  $p_n$  in terms of  $p_{n-1}$ .

$$p_n = p \cdot (1 - p_{n-1}) + (1 - p) \cdot p_{n-1}$$

(b)

Based on (a), for what value  $\lambda$  does  $p_{n-1} = \lambda$  imply  $p_n = \lambda$ ?

$$\begin{aligned} \lambda &= p \cdot (1 - \lambda) + (1 - p) \cdot \lambda \\ \implies \lambda &= \frac{1}{2} \end{aligned}$$

(c)

Show that  $\lim_n p_n = \lambda$ , the value you found in (b). HINT: Write  $p_n$  as  $\lambda + \epsilon_n$ , for the  $\lambda$  you found in (b).

$$\begin{aligned} \lambda + \epsilon_n &= p \cdot (1 - (\lambda + \epsilon_{n-1})) + (1 - p) \cdot (\lambda + \epsilon_{n-1}) \\ \epsilon_n &= (1 - 2p)\epsilon_{n-1} \\ \stackrel{(*)}{\implies} \lim_{n \rightarrow \infty} \epsilon_n &= \lim_{n \rightarrow \infty} (1 - 2p)^{n-1} \epsilon_1 = 0 \end{aligned}$$

(\*) :  $|1 - 2p| < 1$