MATH 505a Fall 2018 Qual Solution Attempts

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Problem 1

Let X be exponentially distributed random variable with $\mathbb{P}(X > t) = e^{-rt}$ for $t > 0$. Write X as the sum of its integer and fractional parts: $X = Y + Z$ with $Y = \lfloor X \rfloor \in \mathbb{Z}$ and $Z \in [0, 1)$.

(a)

Find $\mathbb{E}(X)$

Solution. Since X only takes non-negative value,

$$\mathbb{E}(X) = \int_0^\infty e^{rt} dt = \frac{1}{r}$$

(b)

Find $\mathbb{P}(Y = n)$, $n = 0, 1, 2, ...$

Solution.

$$\mathbb{P}(Y = n) = \mathbb{P}(n \leq X < n + 1) = e^{-rn} - e^{-r(n+1)}$$

(c)

Find $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.

Solution.

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{n=1}^{\infty} \mathbb{P}(Y \geq n) \\ &= \sum_{n=1}^{\infty} e^{rn} \\ &= \frac{e^{-r}}{1 - e^{-r}}\end{aligned}$$

$$\mathbb{E}(Z) = \mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) = \frac{1}{r} - \frac{e^{-r}}{1 - e^{-r}}$$

(d)

Show that Y and Z are independent.

Proof. It suffices to show that $\mathbb{P}(Y = n|Z = a) = \mathbb{P}(Y = n), \ \forall n$.

$$\begin{aligned}
\mathbb{P}(Y = n|Z = a) &= \frac{\mathbb{P}(X = n + a)}{\sum_{i=0}^{\infty} \mathbb{P}(X = i + a)} \\
&= \frac{re^{-r(n+a)}}{\sum_{i=0}^{\infty} re^{-r(n+i)}} \\
&= e^{-rn} \cdot (1 - e^{-r}) \\
&= e^{-rn} - e^{-r(n+1)} \\
&= \mathbb{P}(Y = n)
\end{aligned}$$

□

Problem 2

Let f and g be bounded nondecreasing functions on $\mathbb{R}$, and let X, Y be independent and identically distributed random variables.

(a)

Show that

$$\mathbb{E}[(f(X) - f(Y))\,(g(X) - g(Y))] \geq 0$$

Proof. By the nondecreasing monotonicity,

$$\mathbb{P}(f(X) - f(Y) \geq 0|X > Y) = \mathbb{P}(g(X) - g(Y) \geq 0|X > Y) = 1$$

$$\mathbb{P}(f(X) - f(Y) \leq 0|X \leq Y) = \mathbb{P}(g(X) - g(Y) \leq 0|X \leq Y) = 1$$

So we can argue that,

$$\begin{aligned}
\mathbb{P}((f(X) - f(Y))\,(g(X) - g(Y)) \geq 0) &= \mathbb{P}((f(X) - f(Y))\,(g(X) - g(Y)) \geq 0|X > Y)\mathbb{P}(X > Y) \\
&\quad + \mathbb{P}((f(X) - f(Y))\,(g(X) - g(Y)) \geq 0|X \leq Y)\mathbb{P}(X \leq Y) \\
&= 1
\end{aligned}$$

Therefore, it follows that

$$\mathbb{E}[(f(X) - f(Y))\,(g(X) - g(Y))] \geq 0$$

□

(b)

Show that $f(X)$ and $g(X)$ are positively correlated, that is,

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)] \cdot \mathbb{E}[g(X)].$$

Proof.

$$\begin{aligned}
\mathbb{E}[(f(X)-f(Y))(g(X)-g(Y))] &= \mathbb{E}(f(X)g(X) - f(X)g(Y) - f(Y)g(X) + g(Y)f(Y)) \\
&\overset{(*)}{=} \mathbb{E}(f(X)g(X)) - \mathbb{E}(f(X))\mathbb{E}(g(Y)) - \mathbb{E}(f(Y))\mathbb{E}(g(X)) + \mathbb{E}(g(Y)f(Y)) \\
&\overset{(**)}{=} 2\mathbb{E}(f(X)g(X)) - 2\mathbb{E}(f(X))\mathbb{E}(g(X)) \\
&= 2\text{Cov}(f(X), g(X)) \\
&\overset{(***)}{\geq} 0
\end{aligned}$$

$(*)$ X, Y independent.
$(**)$ X, Y identically distributed.
$(***)$ by the result from (a) □

Problem 3

Suppose that X and Y have joint density $f(x,y)$ given by $f(x,y) = ce^{-x}$ for $x > 0$ and $-x < y < x$ and $f(x,y) = 0$ otherwise.

(a)

Show that $c = 1/2$.

Solution.

$$\begin{aligned}
\int_0^\infty \int_{-x}^x f(x,y)dydx &= 1 \\
\int_0^\infty \int_{-x}^x ce^{-x}dydx &= 1 \\
2c\int_0^\infty xe^{-x}dx &= 1 \\
2c &= 1 \\
c &= \frac{1}{2}
\end{aligned}$$

(b)

Find the marginal densities of X and Y, and the conditional density of Y given X.

Solution.

$$\begin{aligned}
f_X(x) &= \int_{-x}^{x} \frac{1}{2}e^{-x}dy \\
&= xe^{-x},\ x > 0 \\
f_Y(y) &= \int \frac{1}{2}e^{-x}\mathbf{1}_{(-x,x)}(y)dx \\
&= \int_{|y|}^{\infty} \frac{1}{2}e^{-x}dx \\
&= \frac{1}{2}e^{-|y|} \\
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{1}{2x},\ x > 0,\ -x < y < x.
\end{aligned}$$

(c)

Find $\mathbb{P}(X > 2Y)$

Solution.

$$\begin{aligned}
\mathbb{P}(X \geq 2Y) &= \int_0^{\infty} \mathbb{P}\left(Y \leq \frac{X}{2}|X = x\right) f_X(x)dx \\
&= \int_0^{\infty} \int_{-\infty}^{x/2} \frac{1}{2x}\mathbf{1}_{(-x,x)}(y) \cdot xe^{-x}dydx \\
&= \frac{3}{4}\int_0^{\infty} xe^{-x}dx \\
&= \frac{3}{4}
\end{aligned}$$