

MATH 505a Spring 2019 Qual Solution Attempts

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Problem 1

Suppose that each of 5 jobs is assigned at random to one of three servers A,B and C. [For example, one possible outcome would be that job 1 goes to server B, job 2 goes to server C, job 3 goes to server C, job 4 goes to server B and job 5 goes to server A. “At random” here means that there are 3^5 equally likely outcomes]

(a)

Find the probability that server C gets all 5 jobs.

$$\mathbb{P}(\text{C gets 5 jobs}) = \left(\frac{1}{3}\right)^5$$

(b)

Let S be the number of servers that get exactly one job. Find $\mathbb{E}S$.

Solution. Let I_A, I_B, I_C denote the indicator functions of A, B, C get exactly one job, respectively. Then,

$$\begin{aligned}\mathbb{E}(S) &= \mathbb{E}(I_A + I_B + I_C) \\ &= \mathbb{P}(I_A = 1) + \mathbb{P}(I_B = 1) + \mathbb{P}(I_C = 1) \\ &= 3 \binom{5}{1} \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^4 \\ &= \frac{80}{81}\end{aligned}$$

(c)

Find the probability that no server gets more than 2 jobs.

$$\begin{aligned}\mathbb{P}(\text{no server gets more than 2 jobs}) &= \mathbb{P}(\text{2 servers get 2 jobs each, 1 server get 1 job}) \\ &= \binom{3}{1} \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right) \\ &= 3 \left(\frac{1}{3}\right)^5 \\ &= \frac{1}{81}\end{aligned}$$

(d)

Take the same story, but with m in place of 5 for the number of jobs, and n in place of 3 for the number of servers. Find the variance of S , in terms of m and n .

Solution. Let I_i denotes the indicator function that the i -th server gets exactly 1 job.

$$\begin{aligned}\text{Var}(S) &= \mathbb{E}(S^2) - (\mathbb{E}S)^2 \\ &= \mathbb{E}\left(\sum_{i=1}^n I_i^2 + 2 \sum_{i \neq j} I_i \cdot I_j\right) - \left(\sum_{i=1}^n \mathbb{E}(I_i)\right)^2 \\ &= \sum_{i=1}^n \mathbb{P}(I_i = 1) + 2 \sum_{i \neq j} \mathbb{P}(I_i = 1, I_j = 1) - \left(\sum_{i=1}^n \mathbb{P}(I_i = 1)\right)^2 \\ &= m \left(\frac{n-1}{n}\right)^{m-1} - \left(m \left(\frac{n-1}{n}\right)^{m-1}\right)^2 + 2 \cdot n(n-1) \cdot m(m-1) \cdot \left(\frac{1}{n}\right)^2 \left(\frac{n-2}{n}\right)^{m-2}\end{aligned}$$

Problem 2

(a)

Suppose that X is Poisson with parameter λ . Find the characteristic function of X .

$$\begin{aligned}\mathbb{E}(e^{itX}) &= \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda(1-e^{it})} \sum_{k=0}^{\infty} e^{-e^{it}\lambda} \frac{(e^{it}\lambda)^k}{k!} \\ &= e^{-\lambda(1-e^{it})}\end{aligned}$$

(b)

Suppose that X_n is Poisson with parameter λ_n and that $\lambda_n \rightarrow \infty$. Show using characteristic functions that $(X_n - \lambda_n)/\sqrt{\lambda_n}$ converges in distribution, and describe the limiting distribution.

Proof.

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{E}(e^{it(X_n - \lambda_n)/\sqrt{\lambda_n}}) &= \lim_{n \rightarrow \infty} \exp(-it\sqrt{\lambda_n} - \lambda_n + \lambda_n e^{i\frac{t}{\sqrt{\lambda_n}}}) \\ &= \lim_{n \rightarrow \infty} \exp\left(-it\sqrt{\lambda_n} - \lambda_n + \lambda_n \sum_{k=0}^{\infty} \frac{(it/\sqrt{\lambda_n})^k}{k!}\right) \\ &= \lim_{n \rightarrow \infty} \exp\left(-it\sqrt{\lambda_n} + \lambda_n(it/\sqrt{\lambda_n}) + \lambda_n \frac{(it/\sqrt{\lambda_n})^2}{2} + \lambda_n \frac{(it/\sqrt{\lambda_n})^3}{6} + \dots\right) \\ &= \lim_{n \rightarrow \infty} \exp\left(-\frac{t^2}{2} - \frac{it^3}{6\lambda_n^{1/2}} + \dots\right) \\ &= \exp\left(-\frac{t^2}{2}\right)\end{aligned}$$

It converges to the standard normal distribution. □

Problem 3

A stick of length 1 is broken into two pieces at a uniformly distributed random point.

(a)

Find the expected length of the smaller piece.

Solution. Let X_1, X_2, U_1 denote the length of the smaller stick, the length of the larger stick, and the location of the first break point, respectively.

$$\begin{aligned}F_{X_1}(x) &= \mathbb{P}(X_1 \leq x, U_1 \leq 1/2) + \mathbb{P}(X_1 \leq x, U_1 > 1/2) \\ &= x + (1 - (1 - x)) \\ &= 2x\end{aligned}$$

$$f_{X_1}(x) = 2, \quad 0 < x < 1/2$$

Similarly, we have

$$\begin{aligned}F_{X_2}(x) &= 1 - \mathbb{P}(X_1 < 1 - x) = 2x - 1 \\ f_{X_2}(x) &= 2, \quad 1/2 < x < 1\end{aligned}$$

From the pdf of X_1 , we have,

$$\mathbb{E}(X_1) = \frac{1}{4}$$

(b)

Find the expected value of the ratio of the smaller length over the large

Solution.

$$\begin{aligned}\mathbb{P}\left(\frac{X_1}{X_2} \leq t\right) &= \mathbb{P}\left(\frac{X_1}{1-X_1} \leq t\right) \\ &= \mathbb{P}\left(X_1 \leq \frac{t}{t+1}\right) \\ &= \frac{2t}{t+1}, \quad 0 < t < 1\end{aligned}$$

Since the ratio only takes non-negative value, we can use complementary cdf to compute expectation:

$$\begin{aligned}\mathbb{E}\left(\frac{X_1}{X_2}\right) &= \int_0^1 1 - \frac{2t}{1+t} dt \\ &= 1 - 2 + 2 \int_0^1 \frac{1}{1+t} dt \\ &= -1 + 2 \ln(2)\end{aligned}$$

(c)

Suppose the larger piece is then broken at a random point, uniformly distributed over its length, independent of the first break point. There are then three pieces. Find the probability the longest of the three has length more than $1/2$.

Solution. Let X_3, U_2 be the length of the longest among three pieces and the second break point location (start from the left end of the X_2). First notice that the pdf of U_2 :

$$f_{U_2|X_2=x}(t) = \frac{1}{x}$$

Now, for the desired probability,

$$\begin{aligned}\mathbb{P}(X_3 > 1/2) &= \int_{1/2}^1 \mathbb{P}(X_3 > 1/2, U_2 > x/2) + \mathbb{P}(X_3 > 1/2, U_2 \leq x/2) \cdot f_{X_2}(x) dx \\ &= \int_{1/2}^1 [\mathbb{P}(U_2 > 1/2|X_2 = x) + \mathbb{P}(U_2 < X_2 - 1/2|X_2 = x)] \cdot 2 dx \\ &= 2 \int_{1/2}^1 1 - \frac{1}{2x} + \frac{1}{x} \left(x - \frac{1}{2}\right) dx \\ &= 2 + 2 \ln(1/2)\end{aligned}$$