

MATH 505a Fall 2019 Qual Solution Attempts

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Problem 1

Suppose A, B, C are pairwise independent, $A \cap B \cap C = \emptyset$, and $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = p$.

(a)

What is the largest possible value of p ?

Solution. First, notice that $A \cap B \cap C = \emptyset \implies (A \cap B) \cap (B \cap C) = \emptyset$. Then

$$p = \mathbb{P}(B) \geq \mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) = \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(B)\mathbb{P}(C) = 2p^2$$

We have

$$p \leq \frac{1}{2}$$

Then, we let $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$. ($\mathbb{P}(A \cap B) = \frac{1}{4}$ and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{3}{4}$). And let $C = (A \cup B) \setminus (A \cap B)$. Now we want to show that such A, B, C satisfies the assumptions.

$$\mathbb{P}(C) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = \frac{1}{2}$$

$$\mathbb{P}(C \cap A) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \frac{1}{4}$$

$$\mathbb{P}(C \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{1}{4}$$

$$A \cap B \cap C = (A \cap B) \cap ((A \cup B) \setminus (A \cap B)) = \emptyset$$

Therefore $\frac{1}{2}$ is the largest possible value for p .

(b)

Is it possible that $\mathbb{P}(A \cup B \cup C) = 1$? Prove or disprove.

Solution. Impossible.

Proof. By inclusion-exclusion theorem,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - (\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)) + \mathbb{P}(A \cap B \cap C) \\ &= 3p - 3p^2\end{aligned}$$

which reaches maximum at $p = 0.5$ and $\mathbb{P}(A \cup B \cup C) = 0.75$. \square

Problem 2

Consider two coins: coin 1 shows heads with probability p_1 and coin 2 shows heads with probability p_2 . Each coin is tossed repeatedly. Let T_i be the time of first heads for coin i , and define the event $A = \{T_1 < T_2\}$.

(a)

Find $\mathbb{P}(A)$. HINT: One possible method is to condition on one of the variables.

Solution. Notice that $T_i \sim \text{Geometric}(p_i)$.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{t=2}^{\infty} \mathbb{P}(T_1 \leq t-1) \mathbb{P}(T_2 = t) \\ &= \sum_{t=2}^{\infty} (1 - (1 - p_1)^{t-1}) (p_2 (1 - p_2)^{t-1}) \\ &= p_2 \left(\sum_{t=1}^{\infty} (1 - p_2)^t - \sum_{t=1}^{\infty} [(1 - p_2)(1 - p_1)]^t \right) \\ &= \frac{p_2(1 - p_2)}{1 - (1 - p_1)(1 - p_2)}\end{aligned}$$

(b)

Find $\mathbb{P}(T_1 = k|A)$ for all $k \geq 1$.

Solution.

$$\begin{aligned}\mathbb{P}(T_1 = k|A) &= \frac{\mathbb{P}(T_1 = k) \mathbb{P}(T_2 > k)}{\mathbb{P}(A)} \\ &= \frac{[1 - (1 - p_1)(1 - p_2)](1 - p_1)^{k-1} p_1 (1 - p_2)^k}{p_2(1 - p_2)} \\ &= [1 - (1 - p_1)(1 - p_2)](1 - p_1)^{k-1} (1 - p_2)^{k-1}\end{aligned}$$

Problem 3

Player A and B are having a table tennis match; the first player to win 3 games wins the match. One of the players is better than the other; this better player wins each game with probability 0.7. Carl comes to watch the match. He does not know who is the better player so (based on Carl's information) A, B each initially have probability 0.5 to be the better player. Then Carl sees A win 2 of the first 3 games.

(a)

What is now the probability (after the 3 games, based on Carl's information) that A is the better player? Simplify your answer to a single fraction or decimal.

Solution. Let $C = \{A \text{ won 2 of the first 3 games}\}$, $A = \{A \text{ is better}\}$, and $B = \{B \text{ is better}\}$.

$$\begin{aligned}\mathbb{P}(A|C) &= \mathbb{P}(C|A) \frac{\mathbb{P}(A)}{\mathbb{P}(C)} \\ &= \binom{3}{2} (0.7)^2 (0.3) \frac{0.5}{\mathbb{P}(C \cap A) + \mathbb{P}(C \cap B)} \\ &= \binom{3}{2} (0.7)^2 (0.3) \frac{0.5}{\binom{3}{2} (0.7)^2 (0.3) 0.5 + \binom{3}{2} (0.3)^2 (0.7) 0.5} \\ &= \frac{7}{10}\end{aligned}$$

(b)

What is now the probability (after 3 games, based on Carl's information) that A will go on to win the match? NOTE: Express your answer for (b) in terms of numbers; you do not need to simplify to a single number. An answer in a form like $\frac{5}{4} + 7(2 - \frac{9}{5})$ is OK.

Solution. Let $W_A = \mathbb{P}(A \text{ win a game}|C) = 0.7 \cdot 0.7 + 0.3 \cdot 0.3 = 0.58$, so $W_B = \mathbb{P}(B \text{ win a game}|C) = 0.42$. So,

$$\begin{aligned}\mathbb{P}(A \text{ win the match}|C) &= W_A + W_B W_A \\ &= 0.58 + 0.42 \cdot 0.58 \\ &= 0.8236\end{aligned}$$

Problem 4

Suppose X_n is binomial with parameters (n, p) with $0 \leq p \leq 1$, and X is Poisson(λ).

(a)

Find the moment generating function of X_n .

Solution.

$$\begin{aligned}\mathbb{E}(e^{X_n t}) &= \sum_{k=0}^n e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} p (pe^t)^k (1-p)^{n-k} \\ &\stackrel{(*)}{=} (1-p+pe^t)^n\end{aligned}$$

(*) Binomial formula.

(b)

Suppose $n \rightarrow \infty$ and $p = p_n \rightarrow 0$ with $np \rightarrow \lambda \in (0, \infty)$. Show that $\mathbb{P}(X_n = k) \rightarrow \mathbb{P}(X = k)$ as $n \rightarrow \infty$, for all $k \geq 0$. HINT: $(1 - \frac{c_n}{n})^n \rightarrow e^{-c}$ if $c_n \rightarrow c$.

Proof.

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{P}(X_n = k) &= \lim_{n \rightarrow \infty} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \lim_{n \rightarrow \infty} \frac{1}{k!} \frac{n!}{(n-k)!} p^k \left(1 - \frac{np}{n}\right)^n \left(1 - \frac{np}{n}\right)^{-k} \\ &= \frac{1}{k!} \lim_{n \rightarrow \infty} \frac{n-k+1}{n} \frac{n-k+2}{n} \cdots \frac{n-1}{n} \frac{n}{n} (np)^k \left(1 - \frac{np}{n}\right)^n \left(1 - \frac{np}{n}\right)^{-k} \\ &= \frac{1}{k!} \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot \lambda^k \cdot e^{-\lambda} \cdot 1 \\ &= e^{-\lambda} \frac{\lambda^k}{k!}\end{aligned}$$

□

(c)

For n, p as in part(b), show that $\mathbb{P}(X_n > k) \rightarrow \mathbb{P}(X > k)$ as $n \rightarrow \infty$, for all $k \geq 0$.

Proof.

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{P}(X_n > k) &= \lim_{n \rightarrow \infty} (1 - \mathbb{P}(X_n \leq k)) \\ &= 1 - \lim_{n \rightarrow \infty} \sum_{x=0}^k \mathbb{P}(X_n = x) \\ &= 1 - \sum_{x=0}^k \lim_{n \rightarrow \infty} \mathbb{P}(X_n = x) \\ &\stackrel{(**)}{=} 1 - \sum_{x=0}^k \mathbb{P}(X = x) \\ &= \mathbb{P}(X > k)\end{aligned}$$

(**) Conclusion of (b).

□