Topics Covered

Introduction

The bisection method is a technique based on the intermediate value theorem that consists of finding a root of a non linear equation $f(x) = 0$ on a interval $[a, b]$, where f is a continuous function.

Example 1

Consider the function $f(x) = x^2 - 1$.

1.) Show that $f(x)$ has a root in $[a, b] = [0, 3]$.

2.) Let $p_1 = \frac{a+b}{2}$. Compute $f(p_1)$ and show that $f(x)$ has a root in $[0, p_1]$ or $[p_1, 3]$.

3.) Let $p_2 = \frac{a+p_1}{2}$. Compute $f(p_2)$ and show that $f(x)$ has a root in $[p_1, p_2]$.

4.) Repeat the process of 2.) and 3.) until $f(p_n) \approx 0$

#1

The idea is to find a succession of smaller intervals than $[a, b]$ that contains a root of $f(x)$. That is, intervals on which the IVT is applicable.

General Approach

Let $a_1 = a$ and $b_1 = b$. Let p_1 the midpoint of $[a, b]$. Hence $p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}$.

☞ If $f(p_1) = 0$, then $p = p_1$ is the root of $f(x) = 0$.

☞ If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as $f(a_1)$ or $f(b_1)$

- ✔ If $f(p_1)f(a_1) > 0 \implies p \in (p_1, b_1)$ and set $a_2 = p_1$ and $b_2 = b_1$.
- ✔ If $f(p_1)f(a_1) < 0 \implies p \in (a_1, p_1)$ and set $a_2 = a_1$ and $b_2 = p_1$.

☞ We restart the process with the new interval $[a_2, b_2]$.

☞ Continue the process until we get $f(p_N) = 0$, or $f(p_N) \approx 0$ to stop the iteration.

Stopping criteria

There are three main stopping criteria:

1.
2.
3.

Matlab Program

Example 2

Write a Matlab program that implements the bisection method on the function $f(x) = x^6 - x - 1$ on $[1, 2]$.

#2

Convergence of the Bisection method

Theorem 0.1. *Suppose that f is continuous on $[a, b]$ and $f(a)f(b) < 0$. The bisection method generates a sequence $(p_n)_{n\geq 1}$ approximation p with*

$$|p_n - p| \leq \frac{b-a}{2^n}$$

Proof.

$\square$

Example 3

Find the number of iterations necessary for convergence with a tolerance ε.

#3

Example 4

Find the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} on $[1, 2]$.

#4