Topics Covered

Introduction

Let $\{(x_i, y_i)\}_{i=0,1,\ldots,n}$ a sequence of $n+1$ points such that $x_0 < x_1 < \ldots, x_n$. A spline function of degree $k \geq 0$ having knots $x_0, x_1, \ldots, x_n$ is a function S such that:

- on each interval $[x_i, x_{i+1}]$, S is a polynomial of degree $\leq k$;
- S has continuous $(k-1)^{st}$ derivative on $[x_0, x_n]$.

Therefore, S is a piece-wise polynomial of degree at most k having continuous derivatives of all orders up to $k-1$.

Linear Splines

A spline of degree 1 is a piece-wise linear function that has the following form:

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0 x, \quad x \in [x_0, x_1) \\ S_1(x) = a_1 + b_1 x, \quad x \in [x_1, x_2) \\ \vdots \\ S_{n-1}(x) = a_{n-1} + b_{n-1} x, \quad x \in [x_{n-1}, x_n] \end{cases} \tag{1}$$

For a data point $\{(x_i, f(x_i))\}_{i=0,1,\ldots,n}$, an interpolating spline of degree 1 is given by

$$S(x) = \begin{cases} S_0(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0}, \quad x \in [x_0, x_1] \\ S_1(x) = f(x_1)\frac{x-x_2}{x_1-x_2} + f(x_2)\frac{x-x_1}{x_2-x_1}, \quad x \in [x_1, x_2] \\ \vdots \\ S_{n-1}(x) = f(x_{n-1})\frac{x-x_n}{x_{n-1}-x_n} + f(x_n)\frac{x-x_{n-1}}{x_n-x_{n-1}}, \quad x \in [x_{n-1}, x_n] \end{cases} \tag{2}$$

Notice that $S_{1,n}(x_i) = f(x_i)$ for all $i = 0, 1, 2\ldots, n$.

Example 1

Find an interpolating linear spline such that

i	x_i	$f(x_i)$
0	-1	0
1	0	1
2	1	3

#1

Quadratic Splines

A quadratic spline $S_{2,n}(x)$ is a $\mathcal{C}^1([a,b])$ piecewise quadratic polynomial given by:

$$S_{2,n}(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2, & x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2, & x \in [x_1, x_2] \\ \vdots \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2, & x \in [x_{n-1}, x_n] \end{cases} \tag{3}$$

- $S_{2,n}(x)$ interpolates the data set $\{(x_i, f(x_i))\}_{i=0,1,\ldots,n}$. Therefore:

$$S_{2,n}(x_i) = f(x_i), \quad \forall i = 0, 1, \ldots, n$$

This leads to two conditions:

$$S_i(x_i) = f(x_i) \quad \text{for} \quad i = 0, 1, \ldots, n \tag{4}$$

and,

$$S_{i+1}(x_{i+1}) = S_i(x_{i+1}) \quad \text{for} \quad i = 0, 1, \ldots, n-2 \tag{5}$$

- $S_{2,n}(x)$ has to be smooth at the interior the data point $\{(x_i, f(x_i))\}_{i=1,\ldots,n-1}$. This leads to:

$$S'_{i+1}(x_{i+1}) = S'_i(x_{i+1}) \quad \text{for} \quad i = 0, 1, \ldots, n-2 \tag{6}$$

Equations (4), (5) and (6) combines for $3n - 1$ constrains. To completely determine $S_{2,n}(x)$ given in (3), we need $3n$ coefficients.

Example 2

Find an interpolating quadratic spline such that

i	x_i	$f(x_i)$
0	-1	0
1	0	1
2	1	3

#2

Cubic Splines

A cubic spline $S_{3,n}(x)$ is a $\mathcal{C}^2([a,b])$ piecewise quadratic polynomial given by:

$$S_{3,n}(x) = \begin{cases} S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3, & x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_0)^3, & x \in [x_1, x_2] \\ \vdots \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x-x_{n-1}) + c_{n-1}(x-x_{n-1})^2 + d_{n-1}(x-x_{n-1})^3, & x \in [x_{n-1}, x_n] \end{cases} \tag{7}$$

To determine $S_{3,n}(x)$, we need to find a_i, b_i, c_i and d_i in (7) for $i = 0, 1, \ldots, n-1$.
☞: We have **4n** unknown.
To satisfy the interpolation conditions and conserving the smoothness properties, we obtain the following conditions:

1. $S_j(x_j) = y_j, \quad i = 0, 1, \ldots, n-1$ and $S_{n-1}(x_n) = y_n$
2. $S_j(x_{j+1}) = S_{j+1}(x_{j+1}), \quad j = 0, 1, \ldots, n-2,$
3. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}), \quad j = 0, 1, \ldots, n-2,$
4. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}), \quad j = 0, 1, \ldots, n-2.$

These conditions lead to $\mathbf{4n-2}$ linear constraints. We need two more conditions to have the same number of unknowns and linear equations.

Natural Cubic Splines

A cubic splines is called **natural** if the remaining 2 conditions are:

$$S''_0(x_0) = S''_{n-1}(x_n) = 0.$$

Clamped Cubic Splines

A cubic splines is called **clamped** if the remaining 2 conditions are:

$$S_0'(x_0) = f'(x_0) \quad \text{and} \quad S_{n-1}'(x_n) = f'(x_n).$$

Example 3

Find an interpolating natural cubic spline with the data:

i	x_i	$f(x_i)$
0	5	5
1	7	2
2	9	4

#3

Derivation of the method

In this section, we construct a general method on how to determine the coefficients of the spline on each interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n$, using the 4 previous conditions.

We have:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad \text{for} \quad j = 0, 1, \ldots, n - 1.$$

Interpolation conditions and continuity at nodes

☞ By setting $h = x_{j+1} - x_j$, show that the conditions 1.) and 2.) imply that, for $j = 0, 1, 2 \ldots, n - 1$,

$$\begin{cases} a_j = f(x_j) \\ a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \end{cases} \tag{8}$$

Hence, from (8), we obtain a_j and n equations.

Continuity of the first derivatives at nodes

☞ Show that condition 3.) implies that, for $j = 0, 1, 2 \ldots, n - 1$,

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \tag{9}$$

Continuity of the second derivatives at nodes

☞ Show that condition 4.) implies that, for $j = 0, 1, 2 \ldots, n-1$,

$$c_{j+1} = c_j + 3d_j h_j \tag{10}$$

Finally, equations (8), (9) and (10) imply that, for $j = 0, 1, 2 \ldots, n-1$, we have:

$$\begin{cases} a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \\ b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \\ c_{j+1} = c_j + 3d_j h_j \end{cases} \tag{11}$$

Note that the quantities a_j and h_j in (11) are known and have $3n$ equations.
☞ Show that (11) is equivalent to:

$$\begin{cases} a_{j+1} = a_j + b_j h_j + \dfrac{h_j^2}{3}(2c_j + c_{j+1}) \\ b_{j+1} = b_j + h_j(c_j + c_{j+1}) \end{cases} \tag{12}$$

(Note that (12) has n less equation that (11)).

By solving the first equation in (12), show that:

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \tag{13}$$

By moving the index backward, we obtain:

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j) \tag{14}$$

and by using the second equation in (12) and moving the index backward, we obtain

$$b_j = b_{j-1} + h_{j-1}(c_{j-1} + c_j) \tag{15}$$

By substituting (13) and (14) into (15), show that

$$h_{j-1}c_{j-1} + 2c_j(h_{j-1} + h_j) + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) \quad j = 1, 2, \ldots, n-1. \tag{16}$$

Note that we have $n-1$ equations for $n+1$ unknown.

For natural splines, we have $S_0''(x_0) = S_{n-1}''(x_n) = 0$. Show that $c_n = c_0 = 0$.

Show that (16) is equivalent to $Ac = y$ where A and y are to be determined.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \dots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \frac{3}{h_2}(a_3-a_2) - \frac{3}{h_1}(a_2-a_1) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 0 \end{pmatrix}$$

By solving $Ac = y$ by using linear algebra tools, to obtain c_i, for $i = 0, 1, 2\dots, n$. Then, we use (13) and the last equation of (11) to obtain b_j and d_j for $i = 0, 1, 2\dots, n$.

☞ Find A and y in the particular case where $h_i = h$ for all i.

Python Code

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