Topics Covered

Introduction

The fixed-point is another technique that consists of finding a root of a non linear equation $f(x) = 0$ on a interval $[a, b]$, where f is a continuous function.

Definition 1. *A fixed-point for a given function is number p such that $f(p) = p$.*

Example 1

Consider the function $g(x) = x^2 - x - 3$.
1.) Find the point(s) where the function $g(x)$ and the line $y = x$ intersect.

2.) Are these points also fixed-points of $g(x)$?

#1

3.) Compare the previous points with the roots of $f(x) = g(x) - x$.

Proposition 1. *A root finding problem is equivalent to a fixed-point problem.*

Proof.

□

The fixed-point method consists of replace the equation

$$f(x) = 0 \quad (*)$$

by

$$g(x) = x \quad (**)$$

that have the same solutions.
☞ Replacing $(*)$ by $(**)$ is alway possible, as for example, by choosing $g(x) = x \pm kf(x)$ where k is a constant. These choices may of course not be the best possible.

Example 2

Consider the function $f(x) = x^2 - x - 2$.
Find the four functions $g_i(x)$, for $i = 1, \ldots, 4$, such that $f(x) = 0 \iff g_i(x) = x$.

#2

☞ Give the relationship $(**)$, how to find the fixed-point?

General Approach

- ☞ Choose an initial guess p_0.
- ☞ Generate the sequence $p_n = g(p_{n-1})$.
- ☞ Continue the process until we get $|g(p_N) - p_N| < \varepsilon$ to stop the iteration.

Proposition 2. *Let $g(x) : [a, b] \to [a, b]$ a continuous function and $p_0 \in [a, b]$. If the sequence $(p_n)_{n \geq 0}$ converge to p. Then*

$$p = g(p).$$

Proof.

□

Matlab Programm

Example 3

Write a Matlab program that implements the fixed-point iteration on all four functions $g_i(x)$ obtained in Example 2.

#3

☞ Are all the sequences $p_n = g_i(p_n)$ converging? If so, are converging toward to same value?

Theorem 1.

i) If $g \in \mathcal{C}([a,b])$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then $g(x)$ has at least one fixed-point $p \in [a,b]$.

ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k \quad \forall x \in (a,b),$$

then there exists exactly one fixed-point in $[a,b]$.

Proof.

□

Example 4

Show that the function $g(x) = \frac{x^2-1}{3}$ has a unique fixed-point in $[-1, 1]$.

#4

Remark 1. *The conditions of Theorem 1 are sufficient but not necessary.*

Example 5

Show that the function $g(x) = 3^{-x}$ does not satisfy the conditions of Theorem 1 but has a unique fixed-point in $[0, 1]$.

#5

Convergence of the fixed-point method

Theorem 2 (Fixed-Point Theorem)**.**
Let $g \in \mathcal{C}([a, b])$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose that $g'(x)$ exists on (a, b) and a positive constant $0 < k < 1$ exists with

$$|g'(x)| \leq k \quad \forall x \in (a, b),$$

Then, for any number $p_0 \in [a, b]$, the sequence defined by $p_n = g(p_{n-1}), n \geq 1$, converges to a unique fixed-point $p \in [a, b]$.

Corollary 1.
If g satisfies the hypotheses of the fixed-point theorem, then the bounds for the error involved in using $(p_n)_{n\geq 1}$ to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

Oral Quizz: Prove Theorem 2 and Corollary 1.