

## Chapter 1

# Jacobian-Free Newton-Krylov Solver

A Jacobian-Free Newton-Krylov (JFNK) solver is a generic name for a method that solves a system of equations, which are typically nonlinear. As implied by the name, these solvers are defined by two main characteristics: lack of a need to form the exact Jacobian and Newton-like updates solved via a Krylov method. Since the ability to avoid creating the exact Jacobian comes from the usage of a Krylov method for the linear solves, a brief overview of those methods will be given and followed by an in-depth discussion of the particular method leveraged in this work's JFNK solver: Generalized Minimal Residual (GMRES). Krylov methods come in many different varieties and only differ in

## 1.1 Krylov Methods

Krylov methods are a class of techniques used to solve linear systems of the form

$$Ax = b \tag{1.1}$$

where  $x$  and  $b$  are  $N \times 1$  vectors and  $A$  is an  $N \times N$  nonsingular matrix. Different methods place other limitations on  $A$ , but the most general condition (and the only condition required by GMRES) is that the coefficient matrix be invertible. The exact solution to equation 1.1 is, trivially,

$$x = A^{-1}b. \tag{1.2}$$

However, if  $N$  is large, the exact inversion of  $A$  can be computationally and memory intensive. The computational cost of inverting  $A$  is only exacerbated when solving nonlinear problems with a Newton-like scheme since several, if not many, linear solves are required to make one nonlinear advancement in the solution.

Krylov methods take an iterative approach to solving the linear problem. Given an initial guess of the solution to equation 1.1  $x_0$ , an initial residual vector  $r_0$  can be defined as  $b - Ax_0$ , and a shift from  $x_0$  in the  $\delta x$  direction will result in a different residual vector  $\tilde{r}$ :

$$\tilde{r} = r_0 - A\delta x. \quad (1.3)$$

Thus, the main goal of a Krylov method is to find a shift  $\delta x$ , called the update, such that the norm of  $\tilde{r}$  is eventually minimized to some small, prescribed value. This goal is realized by expressing the exact solution  $x$  in terms of the initial guess and the exact solution to the update:

$$x = x_0 + A^{-1}r_0. \quad (1.4)$$

This formulation is similar to equation 1.2 but relies on the residual and not  $b$ . We now define the eponymous Krylov subspace of dimension  $k$  to be

$$\mathcal{K}_m(A, r_0) = \text{Span}\{r_0, Ar_0, A^2r_0, \dots, A^{m-1}r_0\}. \quad (1.5)$$

The importance of the Krylov subspace is, by the CayleyHamilton theorem, that the inverse of a matrix is spanned by and can be expressed as a linear combination of the its Krylov subspace.

## **1.2 GMRES**

### **1.2.1 Alogorthim**

## **1.3 Jacobian-Free Augmentation**