Chapter 1

Theory

The theory encompassed by this work can be divided into four basic parts: conservation laws, numerical methods, thermohydraulics, and stability. Conservation laws, such as those for momentum and energy, form a fundamental basis of analysis in all branches of science. Therefore, a precise definition for a generic conserved quantity and an associated conservation law will be given. Following this generic treatment, the specific conservation laws for thermohydraulics will be presented. Numerical methods for solving the thermohydraulic conservation laws will then be derived. Finally, the stability theory for the thermohydraulic system will be discussed.

1.1 Conservation Laws

A conserved quantity q refers to any physical property whose time evolution within an arbitrary, closed volume exactly balances with its surface fluxes and volume sources. For the purpose of mathematical discussion and analysis, the conserved quantity q will simultaneously represent a function of space and time that conforms to the requirements of the physical property. These definitions define the following scalar conservation law over a volume Ω with a closed surface Γ:

$$\partial_t \int_\Omega q(x,t)\,\partial\Omega = \int_\Gamma -F_i(q;x,t)n_i\,\partial\Gamma + \int_\Omega S(q,x,t)\,\partial\Omega, \tag{1.1}$$

where F_i is the surface flux of q, n_i is the outward unit normal of the surface Γ, and S is the scalar volume source of q. Such that the units in the equation agree, both q and S are taken on a per volume basis and F_i on a per area basis. The negative sign in the surface integral ensures that outward fluxes act as sinks and inward fluxes as sources to the time evolution of q's volume integral. Equation 1.1 is the most general scalar conservation law that will be presented and is always physically valid regardless of the functions' behavior.

A general conservation laws can also be presented as a differential equation. To achieve this, the Divergence Theorem allows the surface integral in equation 1.1 to be equated to a volume integral:

$$\int_{\Gamma} -F_i(q;x,t)n_i\,\partial\Gamma = \int_{\Omega} -\partial_i F_i(q;x,t)\,\partial\Omega. \tag{1.2}$$

Substituting equation 1.2 into equation 1.1 and moving all terms to the left-hand side gives:

$$\int_{\Omega} \partial_t q(x,t) + \partial_i F_i(q;x,t) - S(q,x,t)\,\partial\Omega = 0, \tag{1.3}$$

where the time derivative operator could be moved into the integral since the volume is assume time-independent. Now, since the integration volume in equation 1.3 is arbitrary, we take limit of the equality as the volume shrinks to zero. In this limit, the enforcement of the equality in equation 1.3 changes from one over a finite volume into one that is enforced at every point in a particular domain. To ensure this point-wise enforcement, the integrand itself is required to be equally zero at every point in the domain and, at last, yields a differential equation conservation law:

$$\partial_t q(x,t) + \partial_i F_i(q;x,t) - S(q,x,t) = 0 \tag{1.4}$$

Differential conservation laws, equipped with adequate constitutive, boundary, and initial data, define the requirements for all sufficiently smooth functions that describe how the conserved quantities evolve at every point in space-time.

However, solving for these functions for an arbitrary set of laws and data is a formidable, if not impossible, task. Therefore, solutions are often left to numerical methods.

to approximately satisfy, in some sense, the set of conservation laws on a finite set of domains in space-time as opposed to the infinite set satisfied by the underlying continuous functions.

1.1.1 Generic Bulk Flow

The set of conservation equations for a fluid in three dimensional form using index notation is

$$\partial_t \rho + \partial_j(\rho u_j) = S_\rho \tag{1.5a}$$

$$\partial_t \rho u_i + \partial_j(\rho u_i u_j - \sigma_{ij}) + \partial_i P = \rho\, g_i + S_{\rho u} \tag{1.5b}$$

$$\partial_t \rho e + \partial_j[u_j(\rho e + P)] - \partial_j(u_i \sigma_{ij}) = S_{\rho e}, \tag{1.5c}$$

where ρ is density, u is velocity, g is the gravity vector, e is the total energy of flow, P is the thermodynamic pressure, σ_{ij} is the viscous stress tensor, and S_* are arbitrary sinks or sources. Integration of equation 1.5 over some arbitrary, time-independent volume Ω with boundary Γ yields

$$\partial_t \int_\Omega \rho\, \partial\Omega + \int_\Omega \partial_j(\rho u_j)\, \partial\Omega = \int_\Omega S_\rho\, \partial\Omega \tag{1.6a}$$

$$\partial_t \int_\Omega \rho u_i\, \partial\Omega + \int_\Omega \partial_j(\rho u_i u_j - \sigma_{ij}) + \partial_i P\, \partial\Omega = \int_\Omega \rho\, g_i + S_{\rho u}\, \partial\Omega \tag{1.6b}$$

$$\partial_t \int_\Omega \rho e\, \partial\Omega + \int_\Omega \partial_j[u_j(\rho e + P)] - \partial_j(u_i \sigma_{ij})\, \partial\Omega = \int_\Omega S_{\rho e}\, \partial\Omega. \tag{1.6c}$$

The unweighted volume average of a quantity α is denoted by $\overline{\alpha}$ and defined as

$$\overline{\alpha} = \frac{1}{\Omega} \int_{\Omega} \alpha \, \partial\Omega. \tag{1.7}$$

Using this defintion, division of equation 1.6 by Ω and transformation of the volume integrals of spatial derivatives into surface integrals via the Divergence Theorem gives

$$\partial_t \overline{\rho} + \frac{1}{\Omega} \int_{\Gamma} \rho u_j n_j \, \partial\Gamma = \overline{S}_{\rho} \tag{1.8a}$$

$$\partial_t \overline{\rho u}_i + \frac{1}{\Omega} \int_{\Gamma} (\rho u_i u_j - \sigma_{ij}) n_j \, \partial\Gamma + \frac{1}{\Omega} \int_{\Gamma} P n_i \, \partial\Gamma = \overline{\rho} g_i + \overline{S}_{\rho u} \tag{1.8b}$$

$$\partial_t \overline{\rho e} + \frac{1}{\Omega} \int_{\Gamma} [u_j(\rho e + P) - u_i \sigma_{ij}] n_j \, \partial\Gamma = \overline{S}_{\rho e}, \tag{1.8c}$$

where n denotes the outward unit normal of the surface Γ. Equation 1.8 is the general, three-dimensional control volume form of the considered conservation equations.

1.1.2 One Dimension

Since the mass and energy equations are inherently scalar, this section will discuss the