0.1 Control Volume Form

Differential conservation laws, equipped with adequate constitutive, boundary, and initial data, define all the requirements for functions that describe how the conserved quantities evolve at every point in space-time, as long as the derivatives are defined. However, solving for these underlying functions for an arbitrary set of differential equations and data is a seemingly intractable, if not impossible, task. As such, numerical methods are often used to approximately satisfy, in some sense, the set of conservation laws on a finite set of domains in space-time as opposed to the infinite set satisfied by the underlying continuous functions.

0.1.1 Generic Bulk Flow

The set of conservation equations for a fluid in three dimensional form using index notation is

$$\partial_t \rho + \partial_j(\rho u_j) = S_\rho \tag{1a}$$

$$\partial_t \rho u_i + \partial_j(\rho u_i u_j - \sigma_{ij}) + \partial_i P = \rho\, g_i + S_{\rho u} \tag{1b}$$

$$\partial_t \rho e + \partial_j[u_j(\rho e + P)] - \partial_j(u_i \sigma_{ij}) = S_{\rho e}, \tag{1c}$$

where ρ is density, u is velocity, g is the gravity vector, e is the total energy of flow, P is the thermodynamic pressure, σ_{ij} is the viscous stress tensor, and S_* are arbitrary sinks or sources. Integration of equation 1 over some arbitrary, time-independent volume Ω with boundary Γ yields

$$\partial_t \int_\Omega \rho\, \partial\Omega + \int_\Omega \partial_j(\rho u_j)\, \partial\Omega = \int_\Omega S_\rho\, \partial\Omega \tag{2a}$$

$$\partial_t \int_\Omega \rho u_i\, \partial\Omega + \int_\Omega \partial_j(\rho u_i u_j - \sigma_{ij}) + \partial_i P\, \partial\Omega = \int_\Omega \rho\, g_i + S_{\rho u}\, \partial\Omega \tag{2b}$$

$$\partial_t \int_\Omega \rho e\, \partial\Omega + \int_\Omega \partial_j[u_j(\rho e + P)] - \partial_j(u_i \sigma_{ij})\, \partial\Omega = \int_\Omega S_{\rho e}\, \partial\Omega. \tag{2c}$$

The unweighted volume average of a quantity α is denoted by $\overline{\alpha}$ and defined as

$$\overline{\alpha} = \frac{1}{\Omega} \int_{\Omega} \alpha \, \partial\Omega. \tag{3}$$

Using this defintion, division of equation 2 by Ω and transformation of the volume integrals of spatial derivatives into surface integrals via the Divergence Theorem gives

$$\partial_t \overline{\rho} + \frac{1}{\Omega} \int_{\Gamma} \rho u_j n_j \, \partial\Gamma = \overline{S}_{\rho} \tag{4a}$$

$$\partial_t \overline{\rho u}_i + \frac{1}{\Omega} \int_{\Gamma} (\rho u_i u_j - \sigma_{ij}) n_j \, \partial\Gamma + \frac{1}{\Omega} \int_{\Gamma} P n_i \, \partial\Gamma = \overline{\rho} g_i + \overline{S}_{\rho u} \tag{4b}$$

$$\partial_t \overline{\rho e} + \frac{1}{\Omega} \int_{\Gamma} [u_j(\rho e + P) - u_i \sigma_{ij}] n_j \, \partial\Gamma = \overline{S}_{\rho e}, \tag{4c}$$

where n denotes the outward unit normal of the surface Γ. Equation 4 is the general, three-dimensional control volume form of the considered conservation equations.

0.1.2 One Dimension

Since the mass and energy equations are inherently scalar, this section will discuss the