Magnetic Field of a Current Loop in the Plane of the Loop

PHY 460 - Chapter 5

November 16, 2022

Introduction

The goal of this exercise is to compute the magnetic field at distance x from the center of a thin wire loop of radius R carrying counter-clockwise current I, as sketched below. Assume

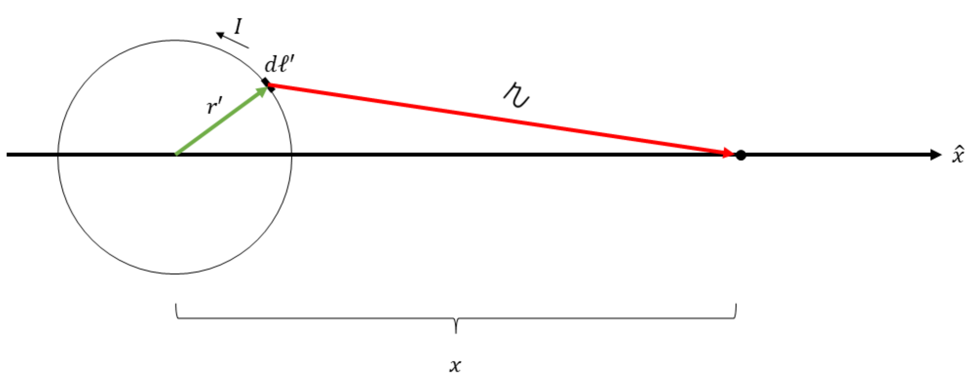


Figure 1: Diagram of a current loop with point of interest in the same plane and outside the loop along the x-axis.

the loop lies in the xy-plane and z is out of the page. Figure 5.21 in the Griffiths textbook shows the diagram for the magnetic field along the z-axis.

Setup

Let's start with the Biot-Savart Laws in Eqn. (1) and determine what we need to solve the problem.

$$\vec{B}\left(\vec{r}\right) = \frac{\mu_o I}{4\pi} \int \frac{d\vec{\ell} \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \tag{1}$$

The general form of the magnitude of the cross product is

$$\left| \vec{a} \times \vec{b} \right| = ab \sin\alpha \tag{2}$$

where α is the angle between $\vec{a}$ and $\vec{b}$.

1. Rewrite the Biot-Savart Law using Eqn. (2) as a magnitude by replacing the cross product. Use the right hand rule to determine the direction of the magnetic field at the point of interest.

2. Reproduce the diagram below and label the angle α between $d\vec{\ell}$ and $\vec{\boldsymbol{\mathscr{r}}}$.

3. In your drawing, add the magnetic field direction at the point of interest.

4. What is the value of the angle between $d\vec{\ell}$ and $\vec{r'}$? Label this appropriately above.

5. Label the angle between $\vec{r'}$ and $\vec{\boldsymbol{\mathscr{r}}}$ as γ.

6. Find an angle relationship using α, γ, and the angle between $d\vec{\ell}$ and $\vec{r'}$.

7. Use the diagrams in Fig. 2 to find the relationship between $\sin\alpha$ and $\cos\gamma$

8. Label the angle between $\vec{r'}$ and the $\hat{x}$ as θ.Notice that the path to integrate ($d\ell$) is the circumference of the current loop. The current loop is in the xy-plane, centered at the origin. Rewrite $d\ell$ in terms of R and θ such that integration $\int d\ell$ gives the circumference of the loop. Rewrite the Biot-Savart Law with integration dependent on θ.

We now have an integral that appears simple because there are no apparent dependencies on θ. However, what happens to α as θ increases from zero counterclockwise around the loop?

9. Use the law of cosines to write an expression using γ below.

10. Use the law of cosines to write an expression for $\mathscr{r}^2$ in terms of θ below.

11. Use steps 9 and 10 above to obtain Eqn. 3.

$$\sin\alpha = \frac{x^2 - R^2 - \mathscr{r}^2}{2R\mathscr{r}} \tag{3}$$

Solution

12. Rewrite the Biot-Savart Law using steps 10 and 11.

Making Sense of It

at $x = 0$ and at $x = \infty$ Plot $B(x)$

Solutions

1. Rewrite the Biot-Savart Law using Eqn. (2). Use the right hand rule to determine the direction of the magnetic field at the point of interest.

2. Reproduce the diagram below and label the angle α between $d\vec{\ell}$ and $\vec{\mathscr{r}}$.

3. In your drawing, add the magnetic field direction at the point of interest.

4. What is the angle between $d\vec{\ell}$ and $\vec{r'}$? Label this appropriately above.right angle

5. Label the angle between $\vec{r'}$ and $\vec{\mathscr{r}}$ as γ.

$$\left|\vec{B}\left(\vec{r}\right)\right| = \frac{\mu_o I}{4\pi}\int \frac{d\ell \sin\alpha}{\mathscr{r}^2}$$

In this case, $d\ell$ is the circumference of the current loop and can be rewritten as $d\ell = Rd\theta$.

$$\begin{aligned} \left|\vec{B}\left(\vec{r}\right)\right| &= \frac{\mu_o I}{4\pi}\int \frac{Rd\theta \sin\alpha}{\mathscr{r}^2} \\ &= \frac{\mu_o I R}{4\pi}\int \frac{\sin\alpha}{\mathscr{r}^2} d\theta \end{aligned}$$

6. Find an angle relationship using α, γ, and the angle between $d\vec{\ell}$ and $\vec{r'}$.

7. Use the diagrams in Fig. 2 to find the relationship between $\sin\alpha$ and $\cos\gamma$

$$\begin{aligned} \alpha + \gamma + 90^\circ &= 360^\circ \\ \alpha + \gamma &= 270^\circ \\ \sin\alpha &= -\cos\gamma \end{aligned}$$

8. Label the angle between $\vec{r'}$ and the $\hat{x}$ as θ.

9. Use the law of cosines to write an expression using γ below.

10. Use the law of cosines to write an expression for $\mathscr{r}^2$ in terms of θ below.

$$
\begin{aligned}
x^2 &= \mathscr{r}^2 + r'^2 - 2\mathscr{r}r'\cos\gamma \\
x^2 &= \mathscr{r}^2 + R^2 - 2\mathscr{r}R\cos\gamma \\
x^2 &= \mathscr{r}^2 + R^2 - 2\mathscr{r}R\left(-\sin\alpha\right) \\
2\mathscr{r}R\sin\alpha &= x^2 - \mathscr{r}^2 - R^2 \\
\sin\alpha &= \frac{x^2 - \mathscr{r}^2 - R^2}{2R\mathscr{r}}
\end{aligned}
$$

The denominator...

$$
\mathscr{r}^2 = x^2 + R^2 - 2xR\cos\theta
$$

First, let's look at the integrand only.

$$
\begin{aligned}
\frac{\sin\alpha}{\mathscr{r}^2} &= \frac{x^2 - R^2 - \mathscr{r}^2}{2R\mathscr{r}\left(R^2 + x^2 - 2Rx\cos\theta\right)} \\
&= \frac{x^2 - R^2 - \left(x^2 + R^2 - 2Rx\cos\theta\right)}{2R\left(x^2 + R^2 - 2Rx\cos\theta\right)^{1/2}\left(R^2 + x^2 - 2Rx\cos\theta\right)} \\
&= \frac{2Rx\cos\theta - 2R^2}{2R\left(x^2 + R^2 - 2Rx\cos\theta\right)^{3/2}} \\
&= R\cdot\frac{\frac{x}{R}\cos\theta - 1}{\left(x^2 + R^2 - 2Rx\cos\theta\right)^{3/2}}
\end{aligned}
$$

Now, we can go back to the full Biot-Savart Law and continue to simplify.

$$
\begin{aligned}
\left|\vec{B}\left(\vec{r}\right)\right| &= \frac{\mu_o IR}{4\pi}\int R\cdot\frac{\frac{x}{R}\cos\theta - 1}{\left(x^2 + R^2 - 2Rx\cos\theta\right)^{3/2}}d\theta \\
&= \frac{\mu_o IR^2}{4\pi}\int \frac{\frac{x}{R}\cos\theta - 1}{\left(x^2 + R^2 - 2Rx\cos\theta\right)^{3/2}}d\theta \qquad (4)
\end{aligned}
$$

This integral is not analytically solvable. First, let's look at the limit $x = 0$.

$$\begin{aligned}
\left|\vec{B}\left(\vec{r}\right)\right| &= \frac{\mu_o I R^2}{4\pi}\int \frac{0-1}{\left(0+R^2-0\right)^{3/2}}d\theta \\
&= \frac{\mu_o I R^2}{4\pi}\int \frac{-1}{\left(R^2\right)^{3/2}}d\theta \\
&= \frac{\mu_o I}{4\pi R}\int_0^{2\pi} d\theta \\
&= \frac{\mu_o I}{4\pi R}2\pi \\
&= \frac{\mu_o I}{2R}
\end{aligned}$$

where we dropped the negative sign because we only care about magnitude. We know from the right hand rule that it points out of the page. The solution is what we found previously for the center of a ring of current. Next, let's look in the limit $x \gg R$. To do this it is simpler if we factor x^3 out of the denominator.

$$\begin{aligned}
\left|\vec{B}\left(\vec{r}\right)\right| &= \frac{\mu_o I R^2}{4\pi}\int \frac{\frac{x}{R}\cos\theta - 1}{\left[\left(x^2\right)\left(1+\frac{R^2}{x^2}-\frac{2R}{x}\cos\theta\right)\right]^{3/2}}d\theta \\
&= \frac{\mu_o I R^2}{4\pi}\int \frac{\frac{x}{R}\cos\theta - 1}{\left(x^3\right)\left(1+\frac{R^2}{x^2}-\frac{2R}{x}\cos\theta\right)^{3/2}}d\theta \\
&= \frac{\mu_o I R^2}{4\pi x^3}\int \frac{\frac{x}{R}\cos\theta - 1}{\left(1+\frac{R^2}{x^2}-\frac{2R}{x}\cos\theta\right)^{3/2}}d\theta
\end{aligned}$$

Now, we can eliminate $(R/x)^2$ because it is much smaller than the other terms. Then, use a binomial expansion on the remaining denominator.

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3}\int \frac{\frac{x}{R}\cos\theta - 1}{\left(1-\frac{2R}{x}\cos\theta\right)^{3/2}}d\theta$$

$$\left(1-\frac{2R}{x}\cos\theta\right)^{-3/2} \approx \left(1+\frac{3R}{x}\cos\theta\right)$$

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3}\int \left(\frac{x}{R}\cos\theta - 1\right)\left(1-\frac{3R}{x}\cos\theta\right)d\theta$$

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3}\int \left(\frac{x}{R}\cos\theta - 1 - \frac{3R}{x}\cos\theta + 3\cos^2\theta\right)d\theta$$

The first and third terms integrate to $\sin\theta$ from 0 to 2π. These are zero.

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3} \int_0^{2\pi} \left(-1 + 3\cos^2\theta\right) d\theta$$

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3} \int_0^{2\pi} \left(-1 + 3\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right)\right) d\theta$$

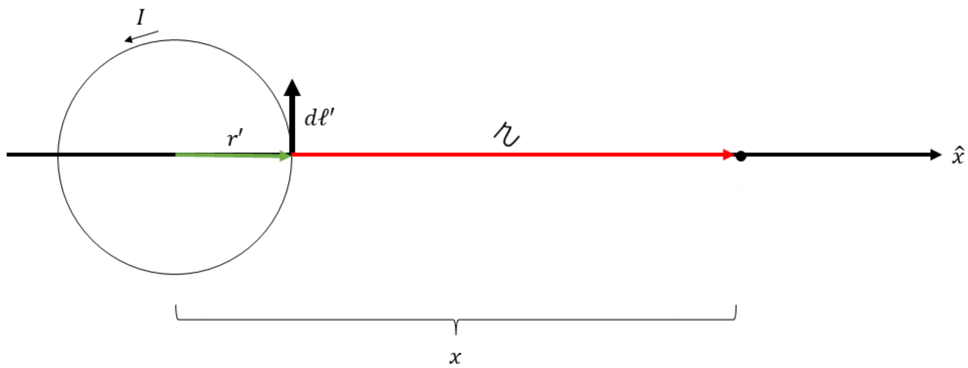
$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3} \left(-\theta + \frac{3}{4}\sin 2\theta + \frac{3\theta}{2}\right)\Bigg|_0^{2\pi}$$

$$\left|\vec{B}\left(\vec{r}\right)\right| \approx \frac{\mu_o I R^2}{4\pi x^3} \left(\pi\right)$$

$$\vec{B}\left(\vec{r}\right)| \approx \frac{\mu_o \vec{m}}{4\pi x^3} \cdot \frac{1}{2}$$

To evaluate intermediate values, we will need to use a computer to numerically integrate equation (4) above. See the Jupyter notebook to continue.

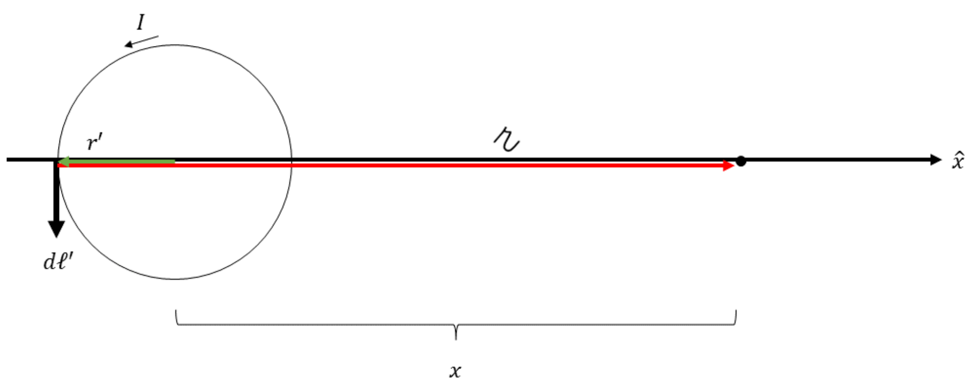
I have exaggerated $d\ell'$ magnitude to show the angle α.



It's important to think about crossing $d\ell' \times \mathscr{r}$
This points into the page, and the rotation α to align the two vectors is

$$\alpha = 270^o$$
$$\gamma = 0^o$$
$$\sin\alpha = -1$$
$$\cos\gamma = 1$$

I have exaggerated $d\ell'$ magnitude to show the angle α.
I have shifted   downward to show its beginning.



$$\alpha = 90^o$$
$$\gamma = 180^o$$
$$\sin\alpha = 1$$
$$\cos\gamma = -1$$

Figure 2: Example diagrams to assist finding a relationship between $\sin\alpha$ and $\cos\gamma$.

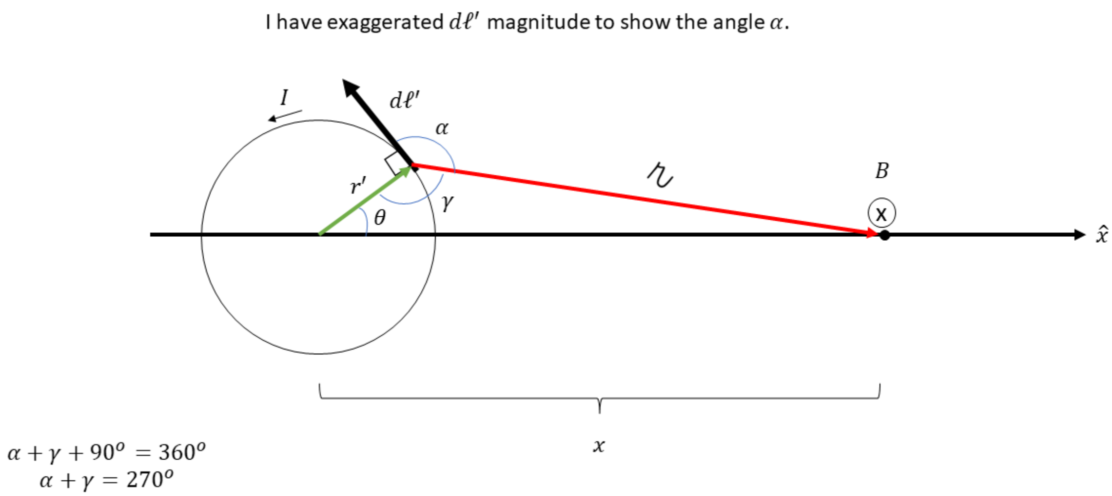


Figure 3: Solution to 1-5.