The improvement and verification for the logistic population growth equation

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Abstract. The classical logistic equation is generalized to a new model with power exponent to the rate of growth of the population. The solutions are presented and a comparing is made with the population census of WU–Gong County, China. The results indicated that the population is affected strongly by the power exponent and the associated characteristics of growth of the population are discussed.

Keywords: logistic equation, growth of the population, differential equation, power exponent.

1. Introduction

At the very end of 18 century, the problem of population was brought into sharper focus by the Rev. Thommas Robert Malthus in An essay on the principle of population[1]. Malthus generalize the inherent growth of the population in the following equation[2,4-6]:

$$\frac{dp}{dt} = kp \tag{1}$$

where $p(t)$ is the population number at the time t and $p_0 = p(t_0)$ at $t = t_0$. k is the unrestricted rate of increase per individual, birth rate minus death rate, in the kind of ideal population that we are considering, is often called Malthusian parameter. The solution of equation (1) is $p(t) = p_0 e^{k(t-t_0)}$. Equation (1) is correct in theory , but which here must be rejected since there is no upper bound.

2. The logistic equation of Pierre-Francois Verhulst

Pierre-Francois Verhulst considered the problem in essentially modern constructing the simplest mathematical model of a continuously growing population with an upper limit M, it is written

$$\frac{dp}{dt} = r(M - p)p\,, \quad r > 0 \tag{2}$$

Integrating gives

$$p(t) = \frac{Mp_0}{[p_0 + (M - p_0)e^{-rM(t-t_0)}]} \tag{3}$$

which express, in the simplest possible way, the growth of a population increasing in a biological manner to an maximum of population M.

Actually, the logistic had been used and had become fairly widely known, though not under the name. For example, the growth of a population of bacteria in a limited medium, the growth of individual organisms and the human population growth.

In reality, the growth of human population was influenced by many factors, such like early marriages or postponement of marriage, environment, immigrations, homosexuality, morality, and diseases.

It is evident that the logistic does not provide a sufficiently predictive model for human demography. We should use some more complex models to illustrate those complicated problems.

3. The improvement of logistic equation

In general, the Malthusian parameter or unrestricted rate of increase, k, should be a function of $M - p$, i.e., $k = f(M - p)$. The power exponent models are widely used in scientific research and engineering. We shall now rewrite the unrestricted rate of increase $k = r(M - p)^{\alpha}$. (where $r, M, \alpha > 0$ are constants), becomes

$$\frac{dp}{dt} = r(M - p)^{\alpha} p, r > 0 \tag{4}$$

The case $\alpha = 1$ is the classical logistic equation of Pierre-Francois Verhulst and the $\alpha > 0, \alpha \neq 1$ can be used to indicate some more complex circumstance.

In terms of equation (4), yields the following second-order conditions

$$\begin{aligned} p'' &= rp'(M - p)^{\alpha} + rp\alpha(M - p)^{\alpha-1}(-p') \\ &= rp'(M - p)^{\alpha-1}[M - (1+\alpha)p] = 0 \end{aligned}$$

We get,

$$p = \frac{M}{1+\alpha} \tag{5}$$

It is indicates that the rate of increase per individual of increasing gets its maximum value as the population $p(t)$ reaches $\frac{M}{1+\alpha}$ and then decreases to zero.

For some special cases, such like $\alpha = 2$, 3, we may obtain the analytical solution by integrating equation (4)

When $\alpha = 2$, integration gives,

$$\frac{1}{M}\frac{1}{M-p} + \frac{1}{M^2}\ln\frac{p}{M-p} = rt + c_1 \tag{6}$$

writing $t = t_0$, $p = p_0$, becomes,

$$\frac{M}{M-p} + \ln\frac{p}{M-p} = rM^2(t-t_0) + \frac{M}{M-p_0} + \ln\frac{p_0}{M-p_0} \tag{7}$$

In the same way, when $\alpha = 3$, integration gives,

$$\begin{aligned}&\frac{1}{M^3}\ln\frac{p}{M-p} + \frac{1}{2M}\frac{1}{(M-p)^2} + \frac{1}{M^2}\frac{1}{(M-p)}\\ &= r(t-t_0) + \frac{1}{M^3}\ln\frac{p_0}{M-p_0} + \frac{1}{2M}\frac{1}{(M-p_0)^2} + \frac{1}{M^2}\frac{1}{(M-p_0)}\end{aligned} \tag{8}$$

Equations (7) or (8) indicate that $p(t) \to M$ as $t \to \infty$.

The efficiency of the proposed model can be verified by numerical results for $p \le M$ and a comparison is made with the population census from 1949 to 2004 of Wu-Gong Country, China . By taking $p_0 = 14.16$, M=414300（the census of Wu-Gong Country in 2004） and r=0.0000015887（which is proposed by Pearl and Reed[4]）.

Table 1 and Figure 1 show the profile of population of Wu-Gong in 1949 to 2004 and the prediction of population by solving the equation (4) for power exponent α=0.8. It is seen that, for $\alpha = 0.8$, the new model in this paper can be successfully applied to predict the distribution of population Wu –Gong Country, China.

Table 1

Year	Census	Predict	Error	Year	Census	Predict	Error
1949	14.16	14.1600	0.00%	1977	30.29	30.0323	-0.85%
1950	14.48	14.6619	1.26%	1978	30.62	30.5621	-0.19%
1951	14.81	15.1740	2.46%	1979	30.95	31.0810	0.42%
1952	14.96	15.6958	4.92%	1980	31.26	31.5885	1.05%
1953	15.38	16.2270	5.51%	1981	31.79	32.0840	0.92%
1954	16.09	16.7671	4.21%	1982	32.17	32.5669	1.23%

1955	16.46	17.3155	5.20%	1983	32.53	33.0367	1.56%
1956	17.21	17.8718	3.85%	1984	32.99	33.4929	1.52%
1957	17.62	18.4353	4.63%	1985	33.37	33.9352	1.69%
1958	17.89	19.0055	6.24%	1986	33.89	34.3633	1.40%
1959	18.34	19.5816	6.77%	1987	34.39	34.7768	1.12%
1960	18.91	20.1629	6.63%	1988	34.89	35.1756	0.82%
1961	19.93	20.7488	4.11%	1989	35.64	35.5595	-0.23%
1962	21.21	21.3383	0.60%	1990	37.27	35.9284	-3.60%
1963	21.85	21.9307	0.37%	1991	37.75	36.2823	-3.89%
1964	22.21	22.5251	1.42%	1992	38.23	36.6212	-4.21%
1965	22.47	23.1207	2.90%	1993	38.62	36.9451	-4.34%
1966	23.05	23.7166	2.89%	1994	38.82	37.2541	-4.03%
1967	23.52	24.3119	3.37%	1995	39.11	37.5484	-3.99%
1968	24.15	24.9057	3.13%	1996	39.44	37.8282	-4.09%
1969	24.86	25.4971	2.56%	1997	39.71	38.0937	-4.07%
1970	25.39	26.0851	2.74%	1998	39.91	38.3452	-3.92%
1971	26.45	26.6688	0.83%	1999	40.30	38.5830	-4.26%
1972	27.45	27.2473	-0.74%	2000	40.68	38.8073	-4.60%
1973	28.18	27.8198	-1.28%	2001	40.95	39.0187	-4.72%
1974	28.83	28.3854	-1.54%	2002	41.14	39.2173	-4.67%
1975	29.35	28.9432	-1.39%	2003	41.34	39.4037	-4.68%
1976	29.78	29.4924	-0.97%	2004	41.43	39.5783	-4.47%

Data from the Web of information of Wu-Gong Country Government, http://www.snwugong.gov.cn[4].

Figure 2 shows the result for $p_0 = 14.16$, M=414300（Wu –Gong Country .in 2004）and r=0.0000015887 obtained by the classical logistic equation(power exponent α =1.0) and the population census from 1949 to 2004 of Wu-Gong Country, China. It may be seen that the error is much larger than the case of $\alpha = 0.8$ and is not fit to predict the growth of the population in Wu-Gong Country, China.

4. Conclusions

The classical logistic equation of Pierre-Francois Verhulst is generalized to a new model with power law exponent the rate of increasing of the population. The analytical solutions and numerical solutions are both presented. The results indicated that the new model fitted much better to the human population of the census than that of the classical logistic equation by taking proper parameters of maximum population and the unrestricted rate of increase.

References:

1. G.Evelyn Hutchinson, An introduction to population Ecology[M],New Haven and London Yale University Press, 1976.
2. Frank R. Giordano, Maurice D. Weir, and William P. Fox, A first Course in Mathematical Modeling[M], Third Edition, EISBN:0-534-38428-5,Copyright©2003 by Brooks/Cole, a division of Thomson Learning.
3. Yang Qufan Mathematical Modeling[M], Beijing, High Education Publish House, China, 2005, pp.50-90.
4. Web of information of Wu-Gong Country Government, http://www.snwugong.gov.cn.
5. Pearl,R. and Reed, L.J., On the rate of the growth of the population of the United States since 1790, Proceedings of the National Academy of Science,1920,6,pp.275-288.
6. Levins, R. The strategy of model building in population biology[J], American scientist,1966, 54, pp.421-431
7. Frauenthal, James C. Introduction to population modeling[M], Lexington, Ma:Comap,1979.

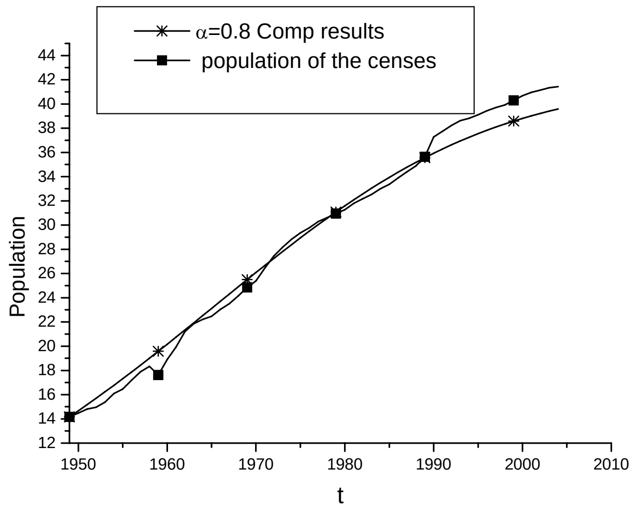

Fig.1 The profiles of population for $\alpha = 0.8$ and the censes results

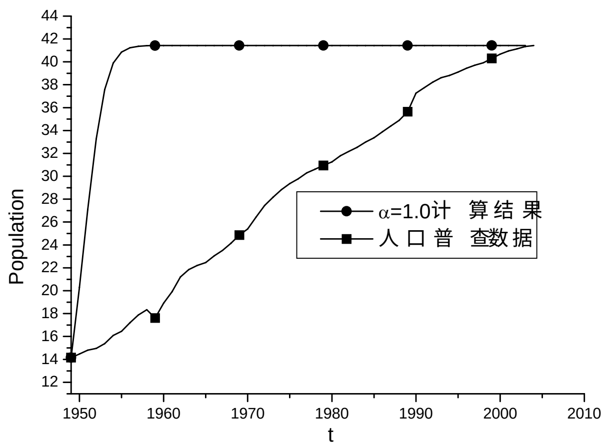

Fig.2 The profiles of population for $\alpha = 1.0$ and the censes results