Planar Graphs Have Bounded Queue-Number

VIDA DUJMOVIĆ [1] GWENAËL JORET [2] PIOTR MICEK [3]
PAT MORIN [4] TORSTEN UECKERDT [5] DAVID R. WOOD [6]

[1]University of Ottawa

[2]Université Libre de Bruxelles

[3]Jagiellonian University

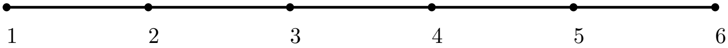
[4]Carleton University

[5]Karlsruhe Institute of Technology

[6]Monash University

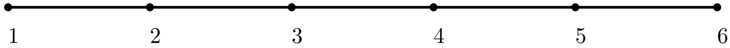
Seminar Talk by Shengzhe Wang
April 19, 2023

Queue Layout

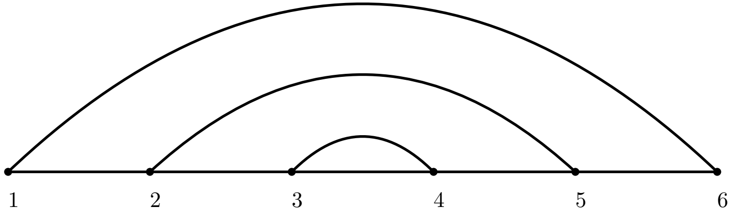


A path graph with 6 vertices

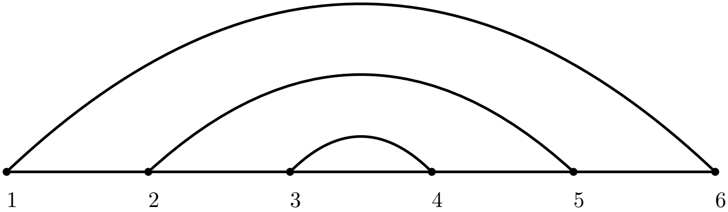
Queue Layout



A path graph with 6 vertices

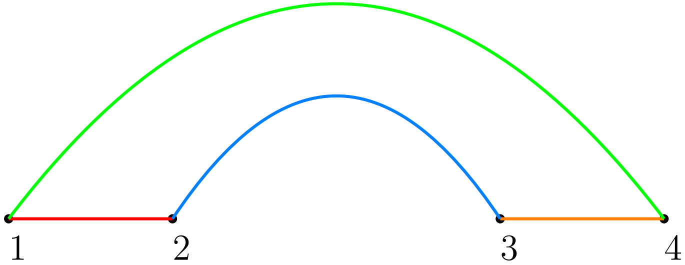


Queue Layout

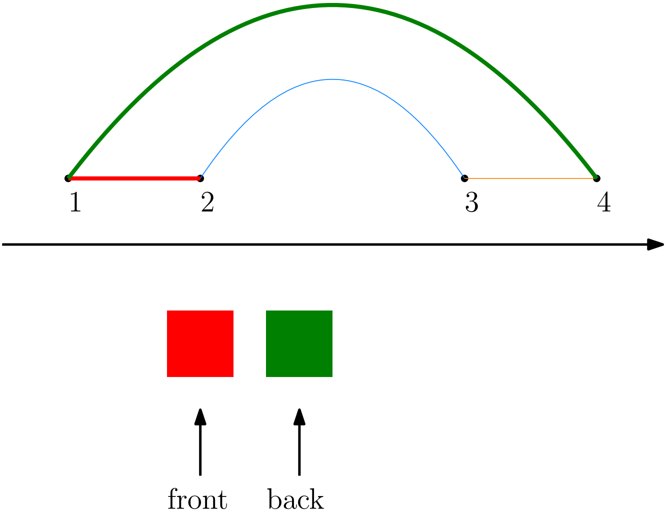


Not Valid Queue-Layout

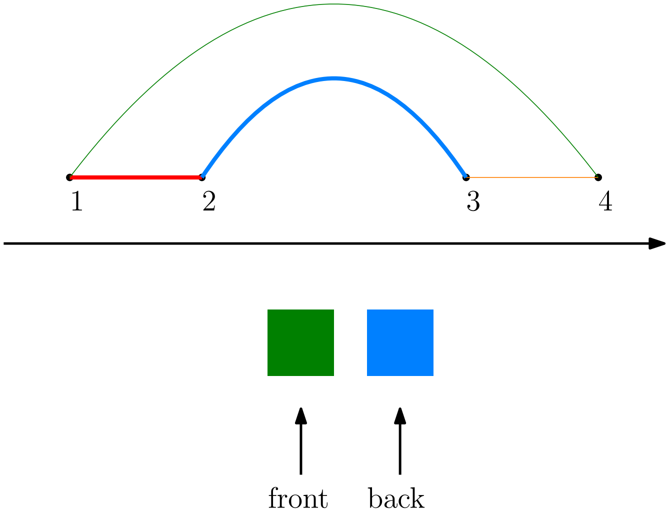
Queue Layout



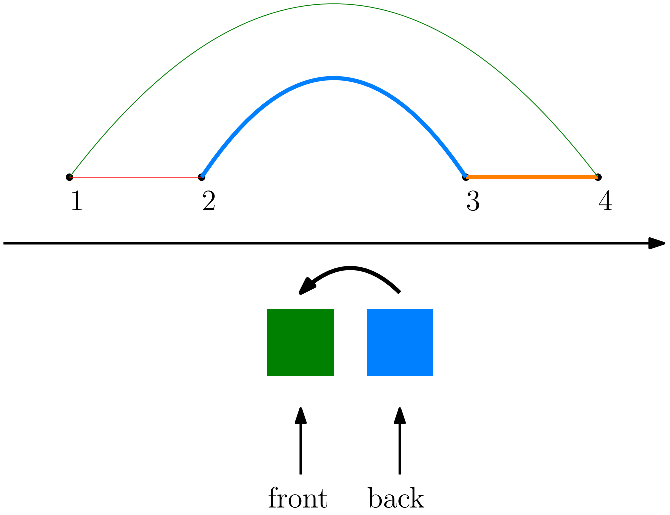
Queue Layout



Queue Layout



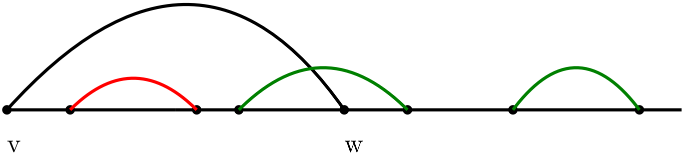
Queue Layout



Queue Layout

Definition: Queue

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, a queue of G is a set of edges $E' \subseteq E$ such that any disjoint edges $vw, xy \in E'$, w.l.o.g, $v \prec w, x \prec y$ and $v \prec x$, we have $w \prec y$.



Queue-Number

Definition: K-Queue Layout

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, for an integer $k \geq 0$ a k-queue layout of G is a partition of E into $E_1, E_2, ..., E_k$ such that each E_i is a queue of G.

Queue-Number

Definition: K-Queue Layout

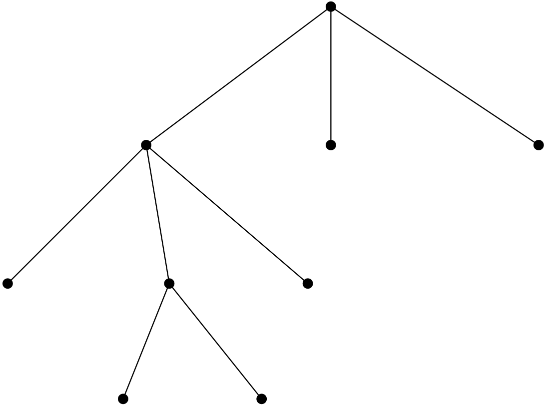
Let $G = (V, E)$, consider a linear odering $\preceq$ of V, for an integer $k \geq 0$ a k-queue layout of G is a partition of E into $E_1, E_2, ..., E_k$ such that each E_i is a queue of G.

Definition: Queue-Number

The queue-number of G, denoted by $\mathrm{qn}(G)$, is the minimum integer k such that G has a k-queue layout.

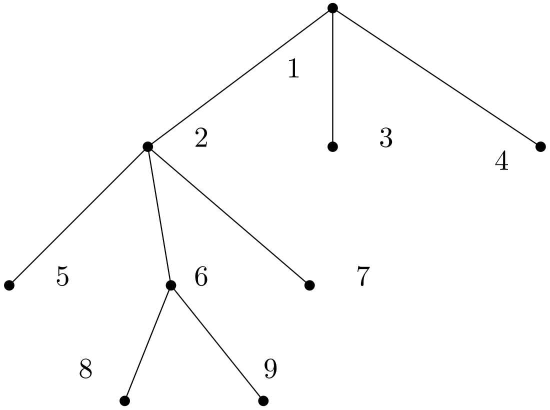
Queue-Number: Tree

What is the queue-number of tree?



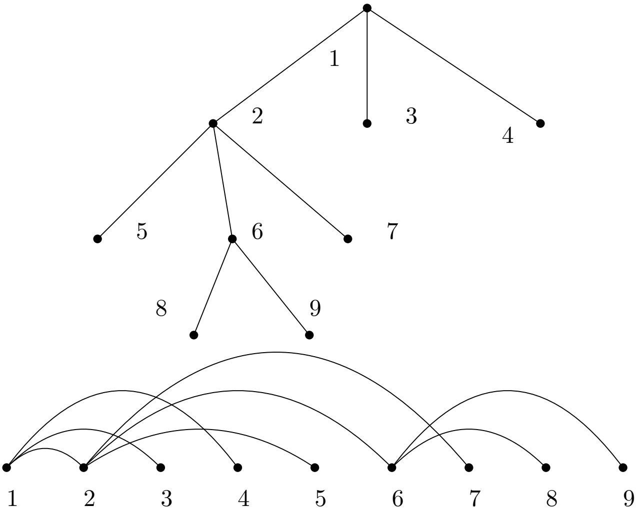
Queue-Number: Tree

What is the queue-number of tree?

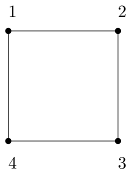


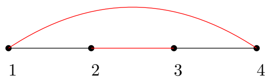
Queue-Number: Tree

What is the queue-number of tree?

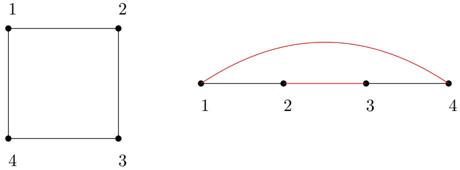


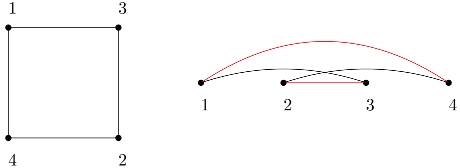
Queue Number: Cycle



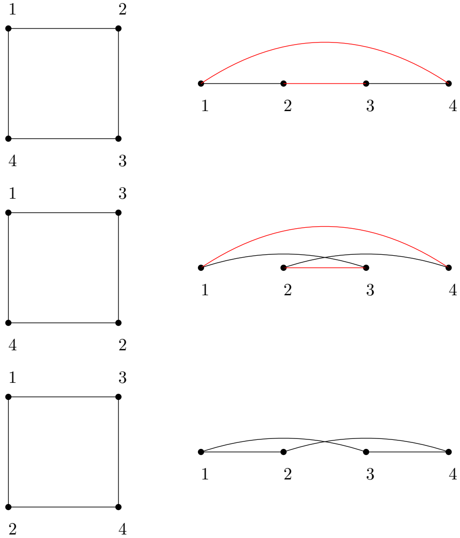


Queue Number: Cycle

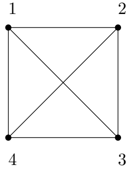


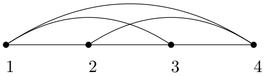


Queue Number: Cycle

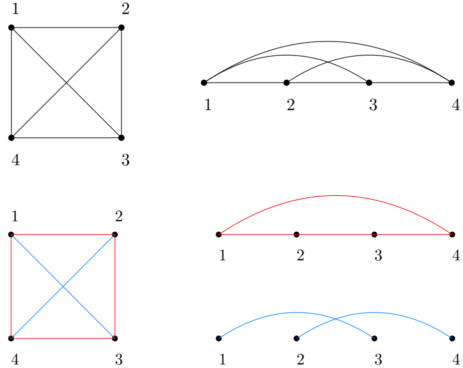


Queue Number: K_4

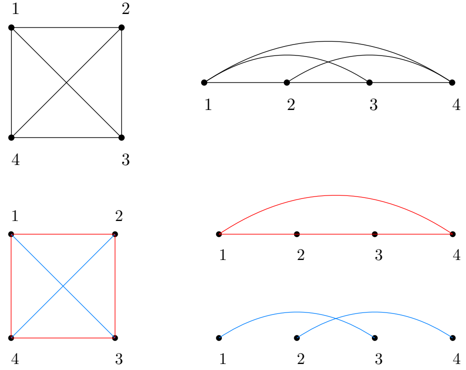




Queue Number: K_4



Queue Number: K_4



Theorem (Heath, Rosenberg, 1992)

The complete graph K_n has queue number $\lfloor \frac{n}{2} \rfloor$.

Treewidth

Do we have some tools to help bound the queue-number?

Treewidth

Do we have some tools to help bound the queue-number?

Theorem (Wiechert,2017)

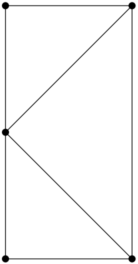
Every graph with treewidth k has queue-number at most $2^k - 1$.

Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

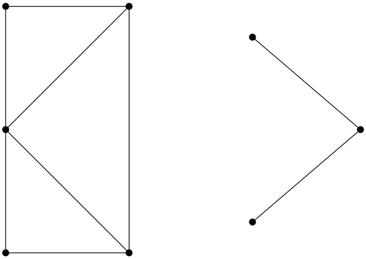


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

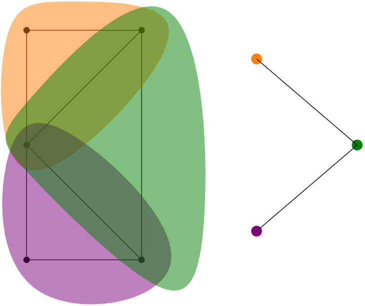


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

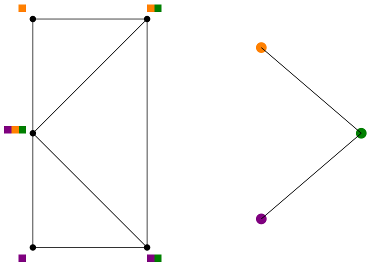


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

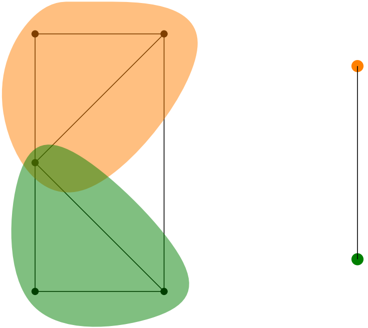


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

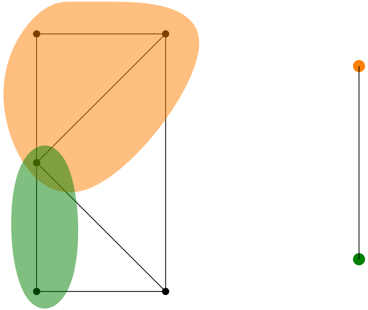


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

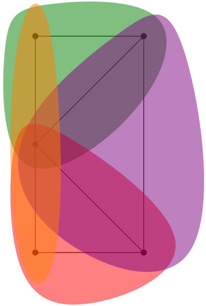


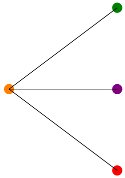
Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.





Treewidth

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

Definition: Width of Tree-decomposition

The width of a tree-decomposition of G is $\max_{x \in V(T)} |B_x| - 1$

Treewidth

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall\{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

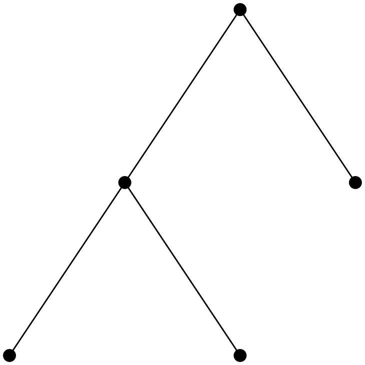
Definition: Width of Tree-decomposition

The width of a tree-decomposition of G is $\max_{x \in V(T)} |B_x| - 1$

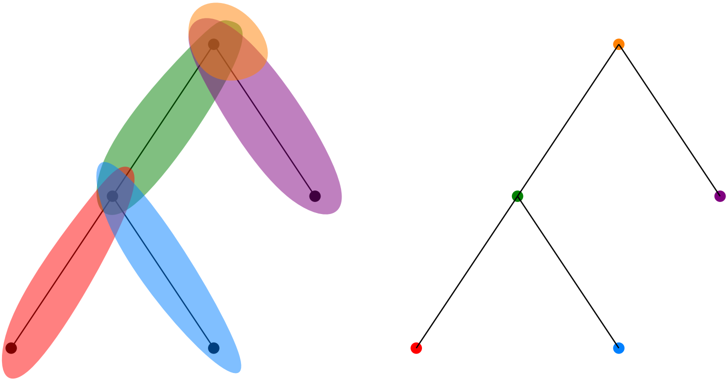
Definition: Treewidth

The treewidth of a graph G is the minimum width of all tree-decomposition of G.

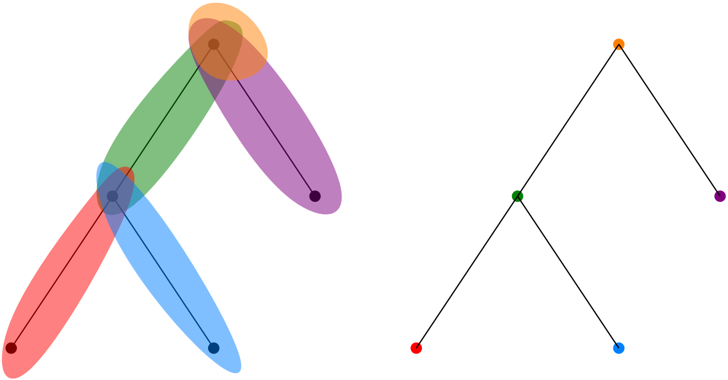
Treewidth: Tree



Treewidth: Tree



Treewidth: Tree



Tree has treewidth 1

Treewidth: Planar graph

Theorem (Wiechert,2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Treewidth: Planar graph

Theorem (Wiechert,2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Theorem

Planar graph on n vertices has treewidth $O(\sqrt{n})$ and the bound is tight.

Treewidth: Planar graph

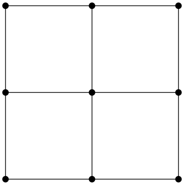
Theorem (Wiechert,2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Theorem

Planar graph on n vertices has treewidth $O(\sqrt{n})$ and the bound is tight.



Partitions

Maybe we need more structure

Partitions

Maybe we need more structure

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Partitions

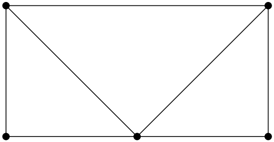
Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Partitions

Definition: Partition and Quotient

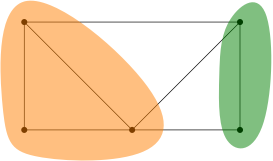
A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.



Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

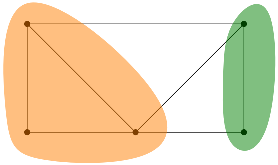


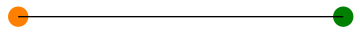
Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.

The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.





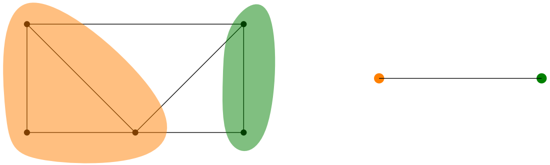
Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

A H-partition of a graph G is a pair (A, H). $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.



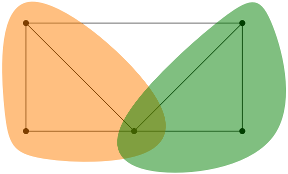
Partitions

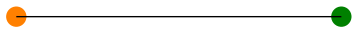
Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

A H-partition of a graph G is a pair (A, H). $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.





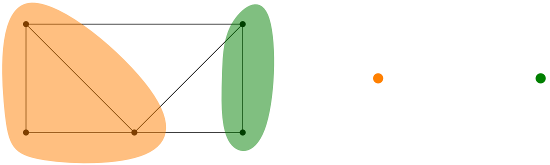
Partitions

Definition: Partition and Quotient

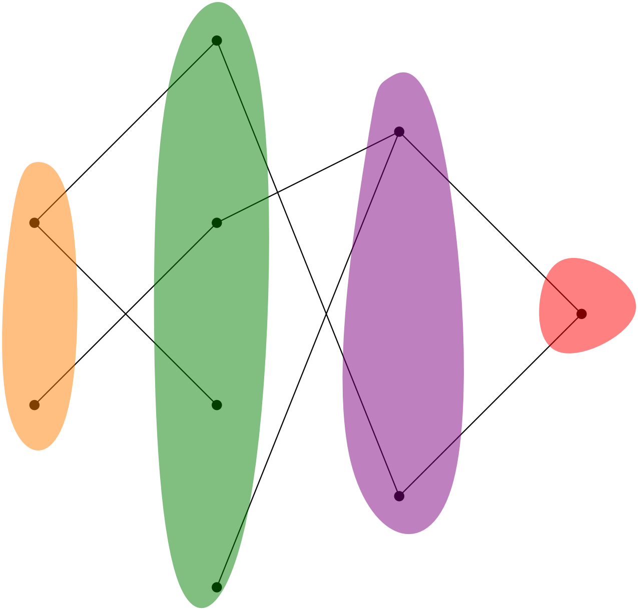
A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, with each vertex v_i corresponds P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

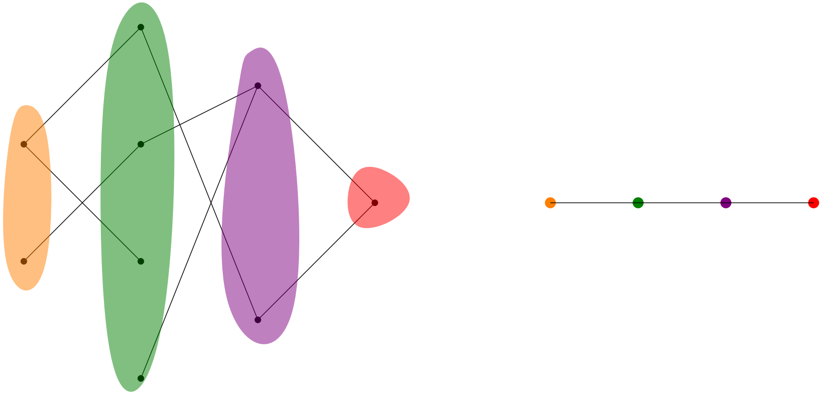
A H-partition of a graph G is a pair (A, H). $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.



Layering



Layering



Layered width

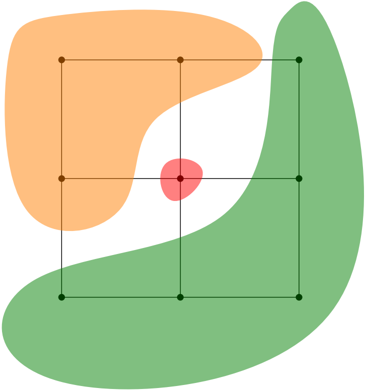
Definition: Layered width

The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, each element in $\mathcal{P}$ has at most ℓ vertices in each element of path-partition.

Layered width

Definition: Layered width

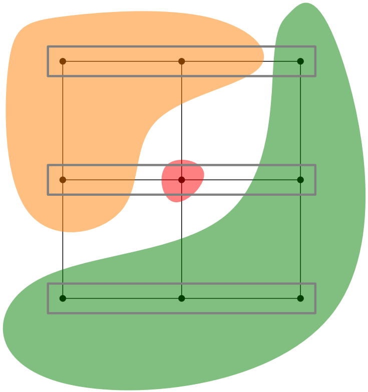
The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, each element in $\mathcal{P}$ has at most ℓ vertices in each element of path-partition.



Layered width

Definition: Layered width

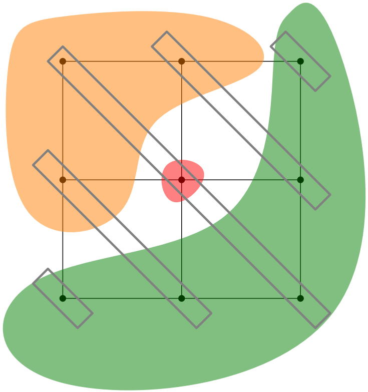
The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, each element in $\mathcal{P}$ has at most ℓ vertices in each element of path-partition.



Layered width

Definition: Layered width

The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, each element in $\mathcal{P}$ has at most ℓ vertices in each element of path-partition.



Layered width: Tree

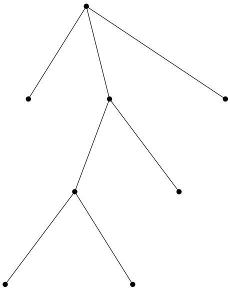
Definition: Vertical Path

Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.

Layered width: Tree

Definition: Vertical Path

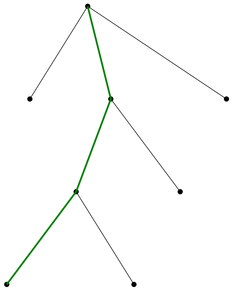
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

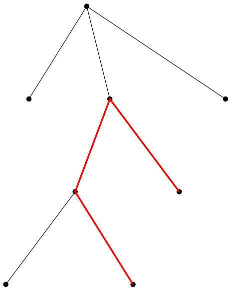
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

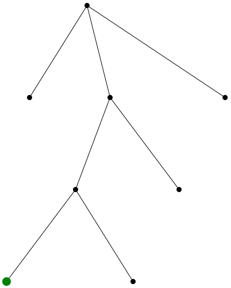
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

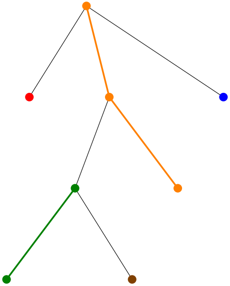
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

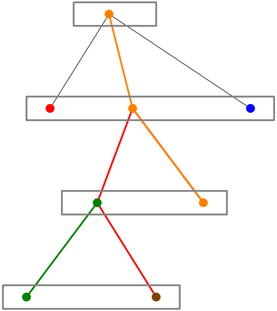
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

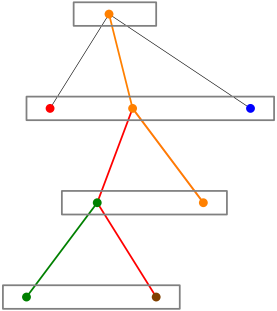
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



A partition of the tree where each part is a vertical path has layered width 1.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has queue-number at most

$$3(2^8 - 1) + \lfloor \frac{3}{2} \rfloor = 766$$

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

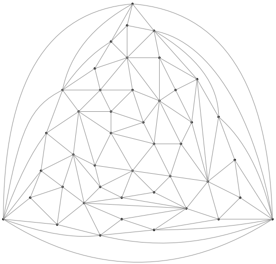
- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

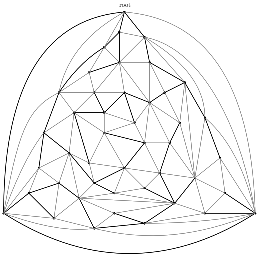


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

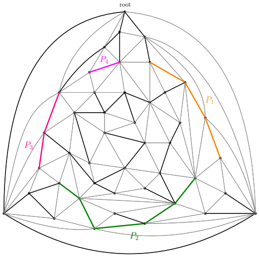


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

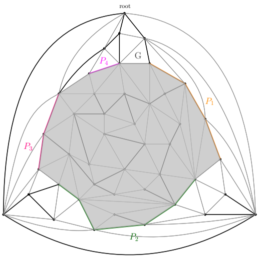


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

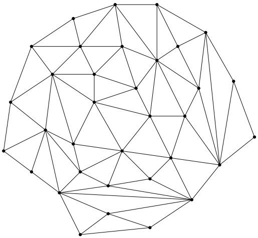


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

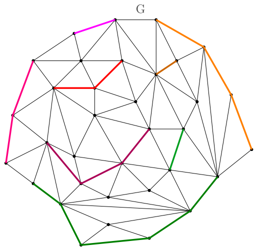


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

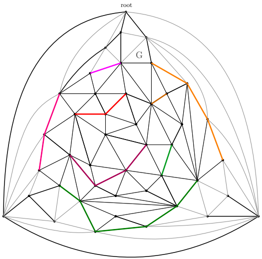


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

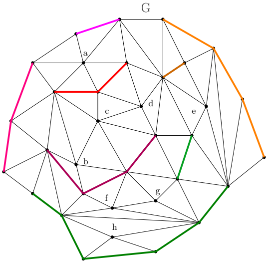


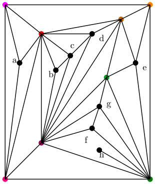
Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x



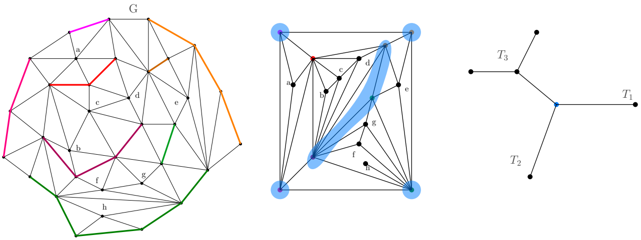


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, \ldots P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, \ldots, P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, \ldots, P_k$ in H is in B_x

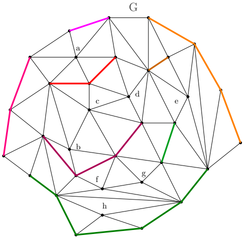


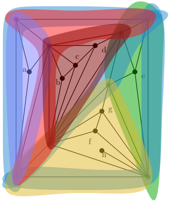
Planar Graph Decomposition

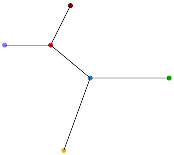
Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

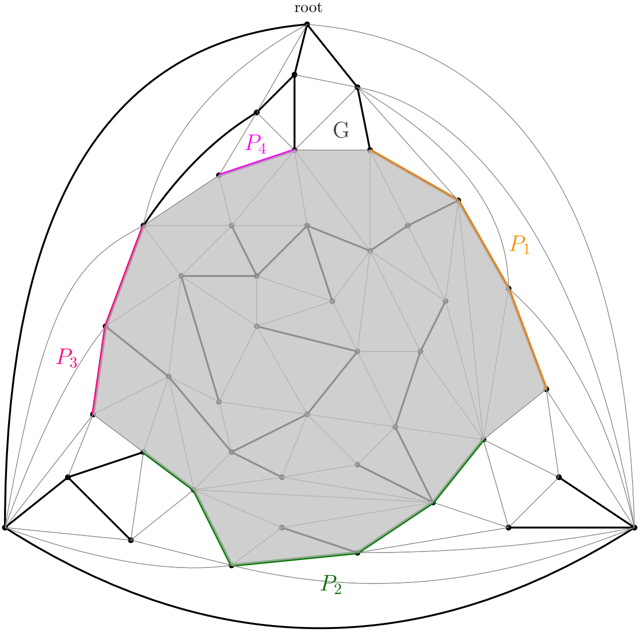
- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x



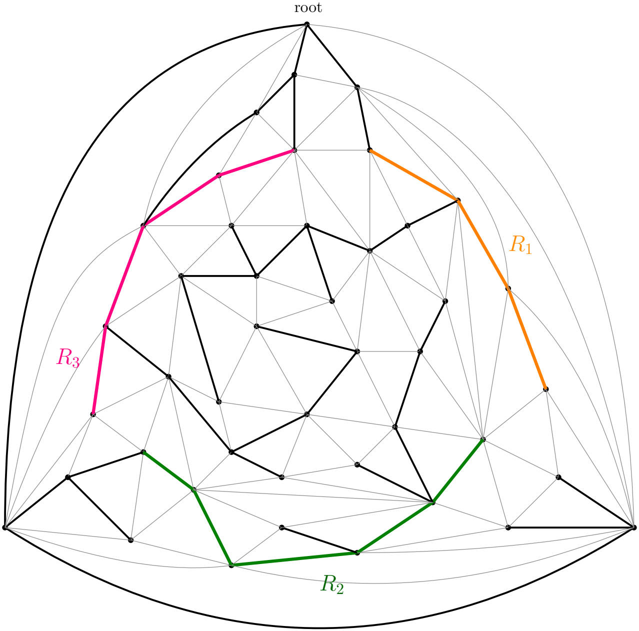




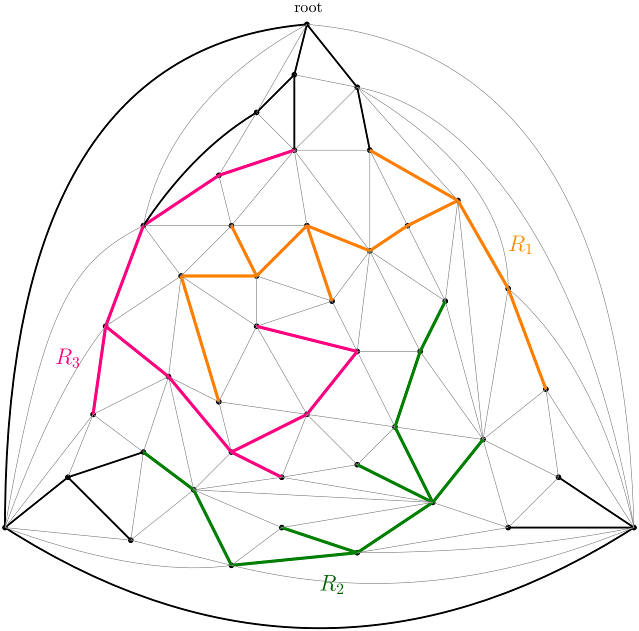
Planar Graph Decomposition



Planar Graph Decomposition



Planar Graph Decomposition



Planar Graph Decomposition

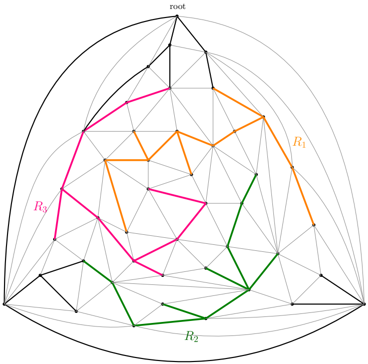
Lemma (Sperner's Lemma)

Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.

Planar Graph Decomposition

Lemma (Sperner's Lemma)

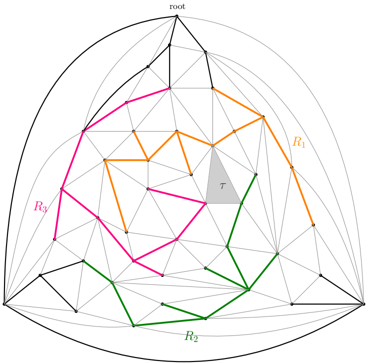
Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.



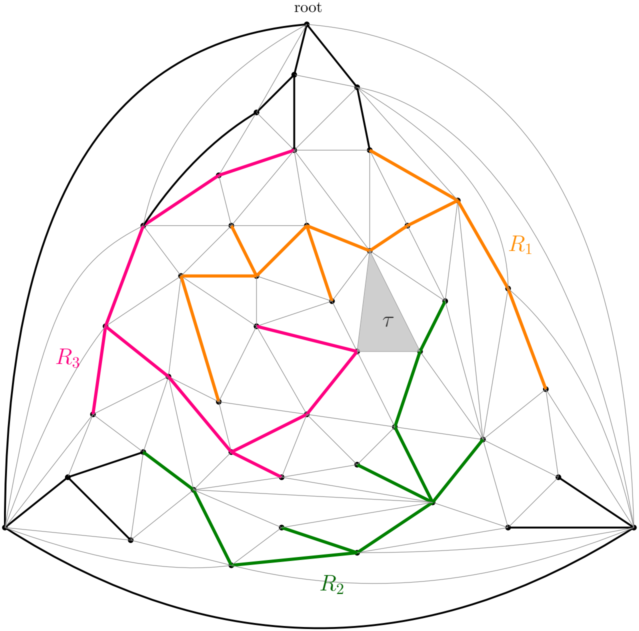
Planar Graph Decomposition

Lemma (Sperner's Lemma)

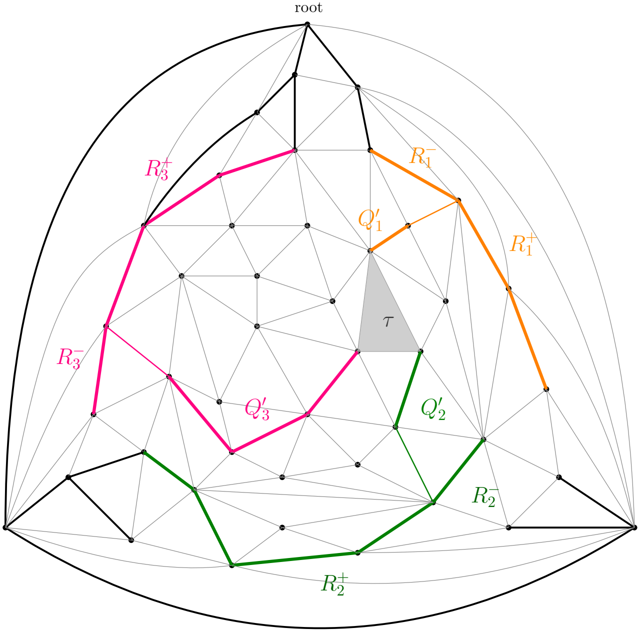
Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.



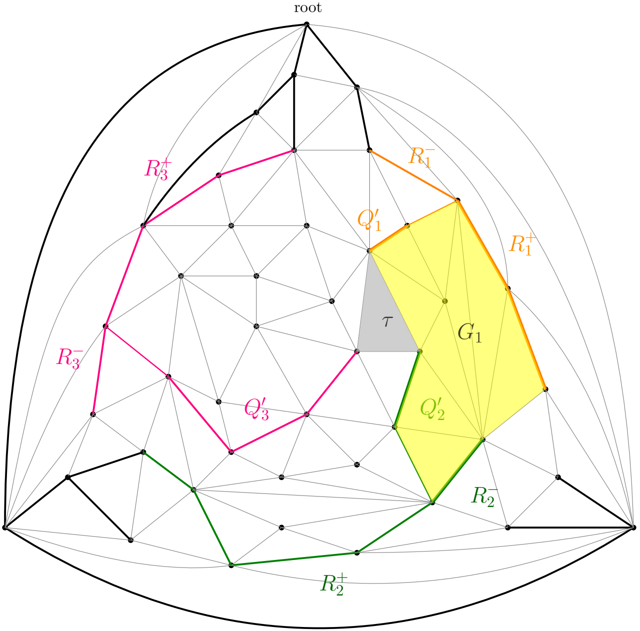
Planar Graph Decomposition



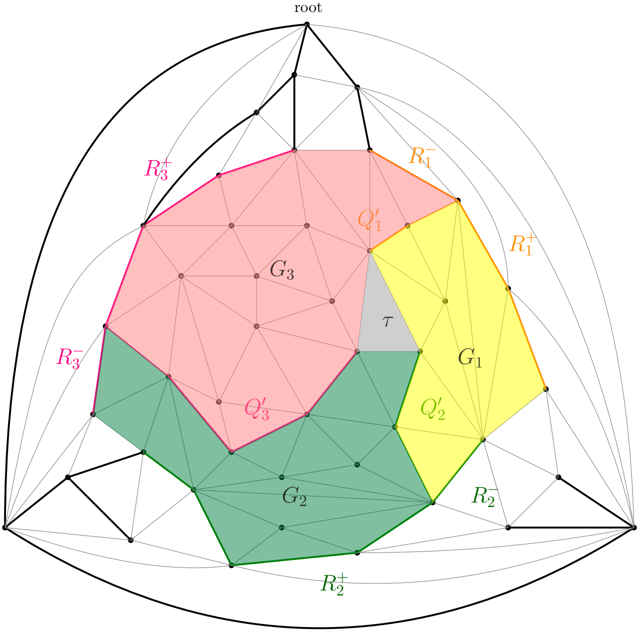
Planar Graph Decomposition



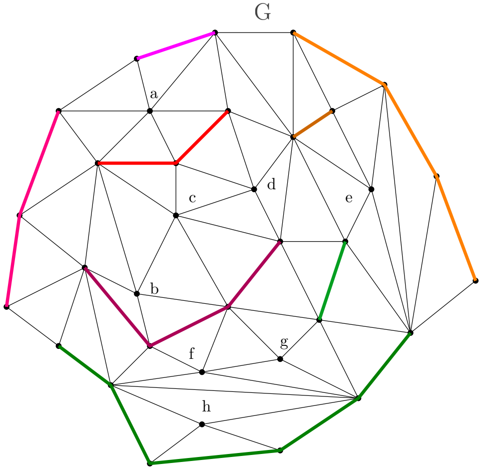
Planar Graph Decomposition

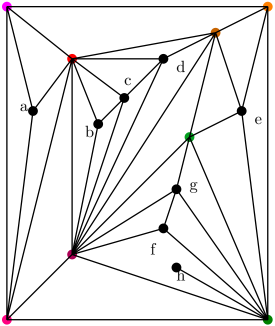


Planar Graph Decomposition

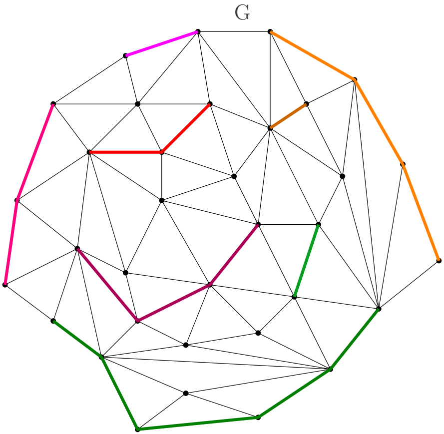


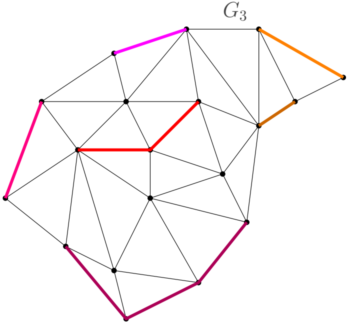
Planar Graph Decomposition



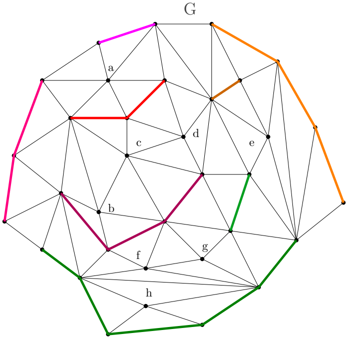


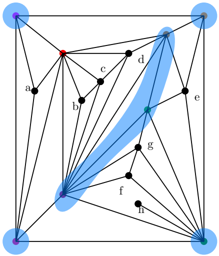
Planar Graph Decomposition

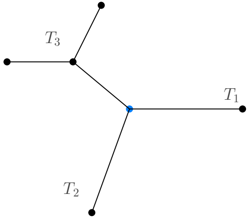




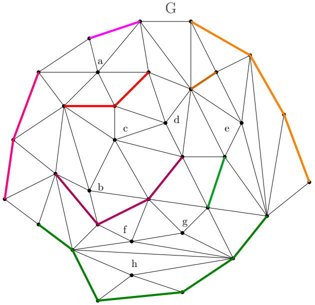
Planar Graph Decomposition

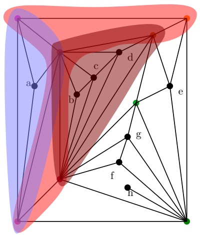


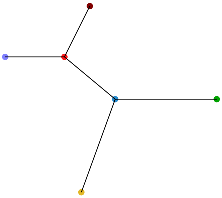


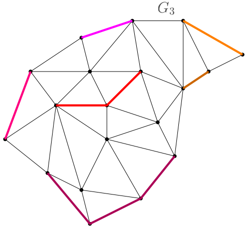


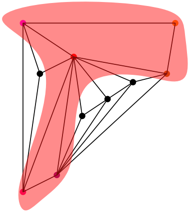
Planar Graph Decomposition

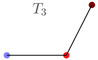




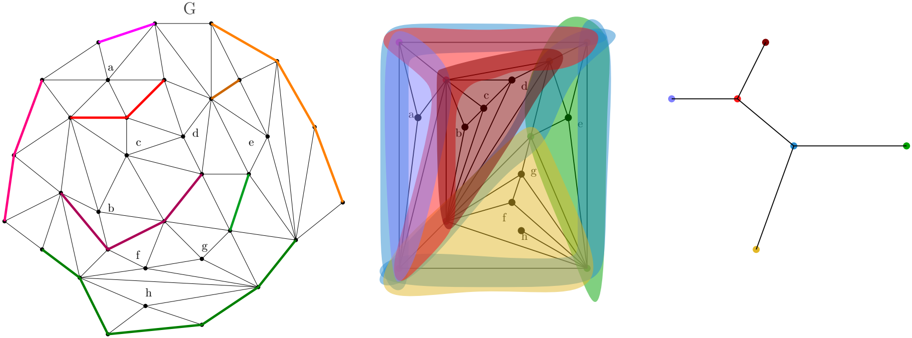








Planar Graph Decomposition



Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has queue-number at most

$$3(2^8 - 1) + \lfloor \frac{3}{2} \rfloor = 766$$

Portal