

Planar Graphs Have Bounded Queue-Number

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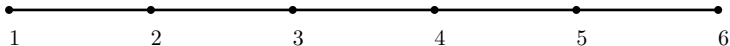
⁴Carleton University

⁵Karlsruhe Institute of Technology

⁶Monash University

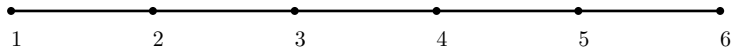
Seminar Talk by Shengzhe Wang
April 19, 2023

Queue Layout

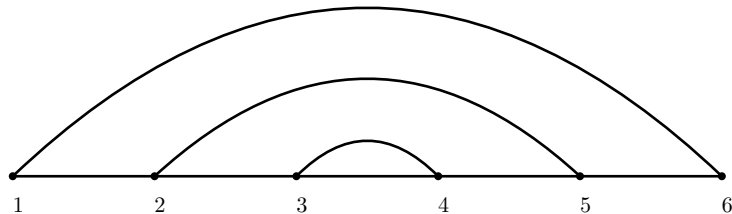


A path graph with 6 vertices

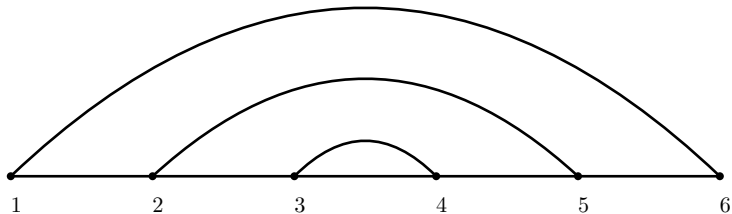
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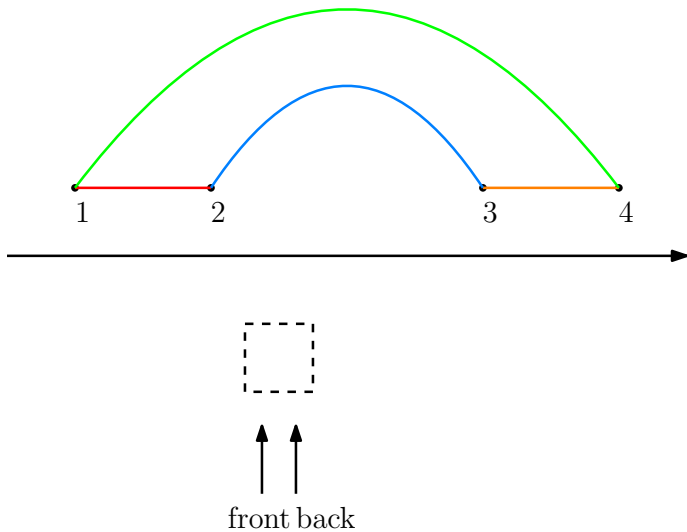


Queue Layout

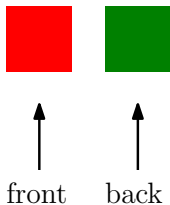
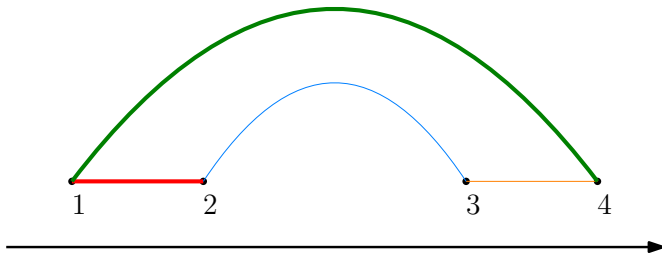


Not Valid Queue-Layout

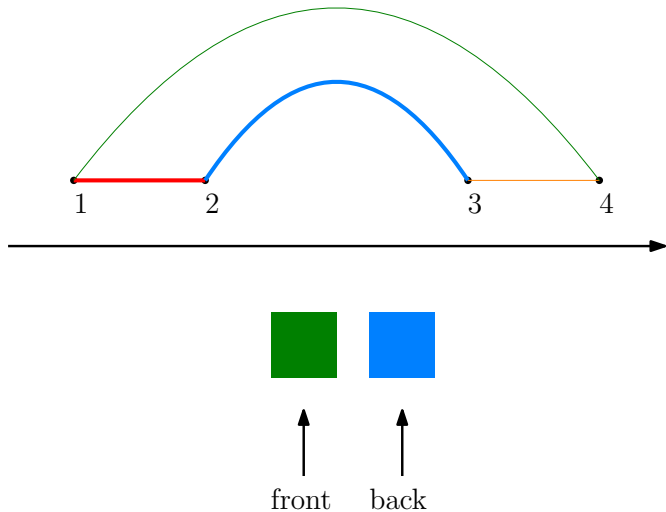
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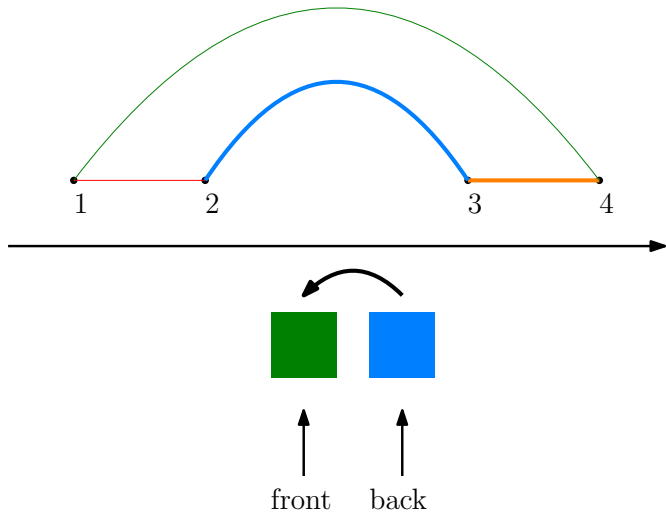
Queue Layout



Queue Layout



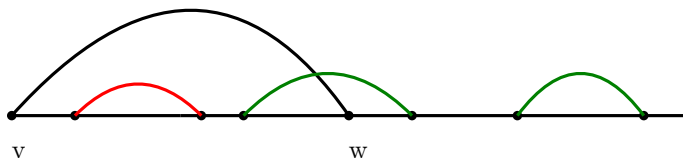
Queue Layout



Queue Layout

Definition: Queue

Let $G = (V, E)$, consider a linear ordering \preceq of V , a queue of G is a set of edges $E' \subseteq E$ such that any disjoint edges $vw, xy \in E'$, w.l.o.g, $v \prec w, x \prec y$ and $v \prec x$, we have $w \prec y$.



Definition: K-Queue Layout

Let $G = (V, E)$, consider a linear ordering \preceq of V , for an integer $k \geq 0$ a k -queue layout of G is a partition of E into E_1, E_2, \dots, E_k such that each E_i is a queue of G .

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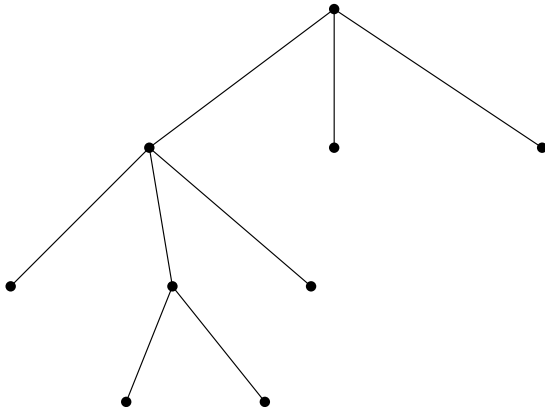
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Definition: Queue-Number

The queue-number of G , denoted by $qn(G)$, is the minimum integer k such that G has a k -queue layout.

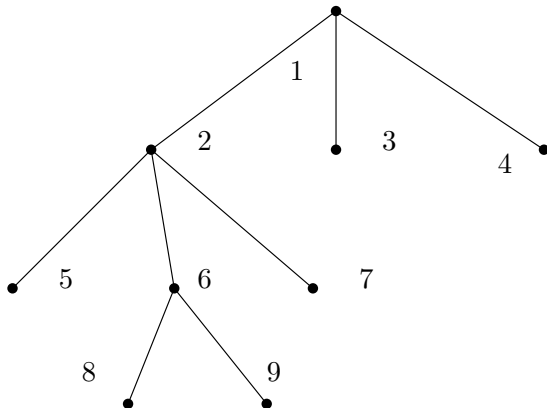
Queue-Number: Tree

What is the queue-number of tree?



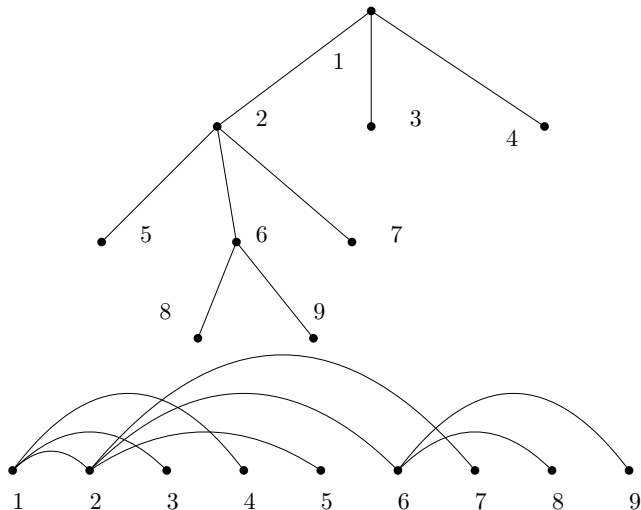
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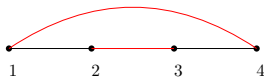
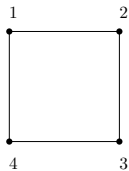


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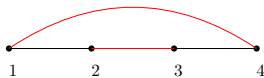
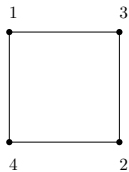
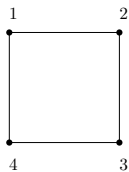
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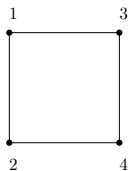
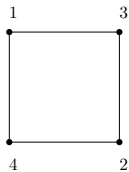
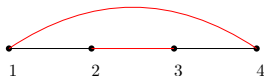
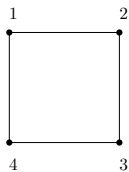
Queue Number: Cycle



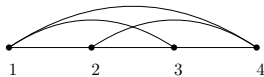
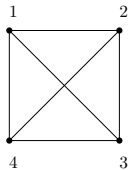
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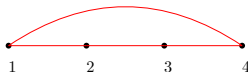
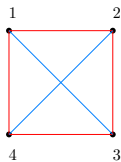
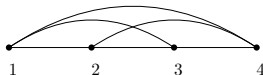
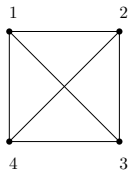
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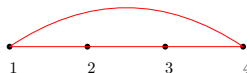
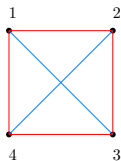
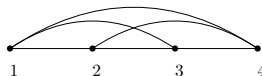
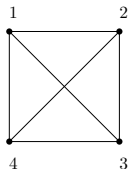
Queue Number: K_4



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Theorem (Heath, Rosenberg, 1992)

The complete graph K_n has queue number $\lfloor \frac{n}{2} \rfloor$.

Do we have some tools to help bound the queue-number?

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Theorem (Wiechert,2017)

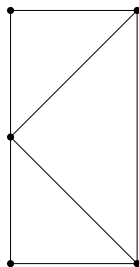
Every graph with treewidth k has queue-number at most $2^k - 1$.

Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T) . T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non-empty connected subtree of T .



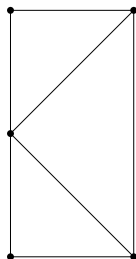
G

Tree-decomposition

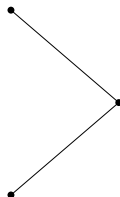
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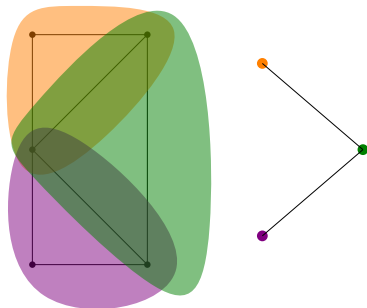
T

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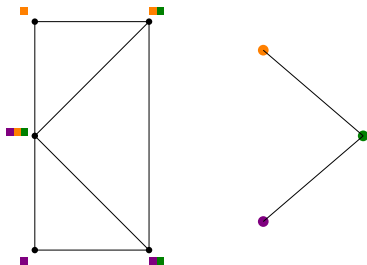


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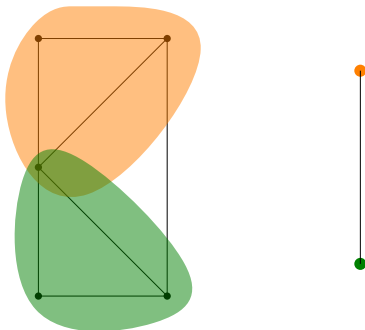


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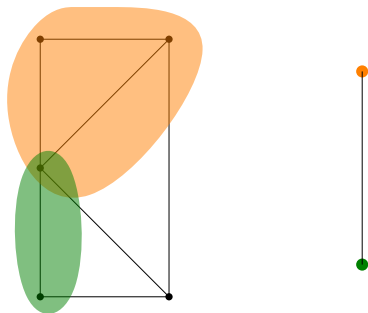


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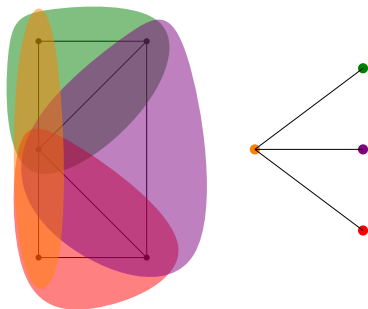


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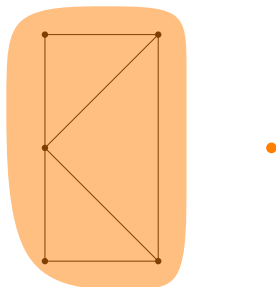


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Definition: Width of Tree-decomposition

The width of a tree-decomposition of G is $\max_{x \in V(T)} |B_x| - 1$

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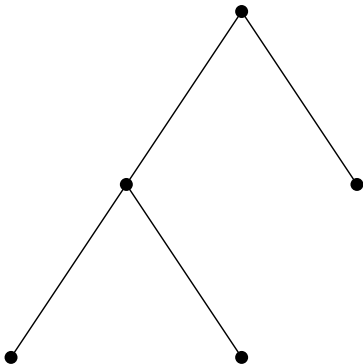
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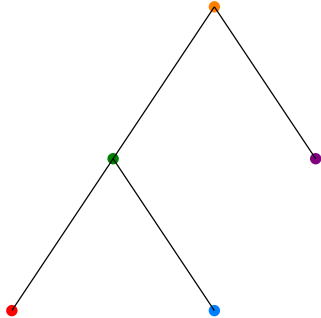
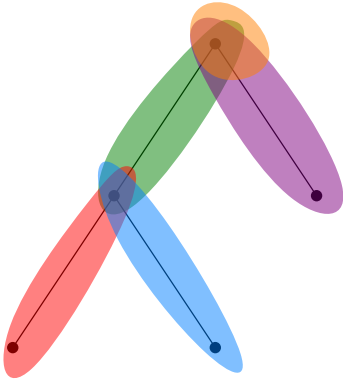
Definition: Treewidth

The treewidth of a graph G is the minimum width of all tree-decompositions of G .

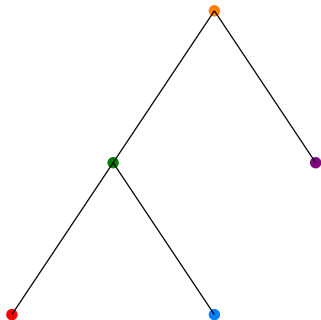
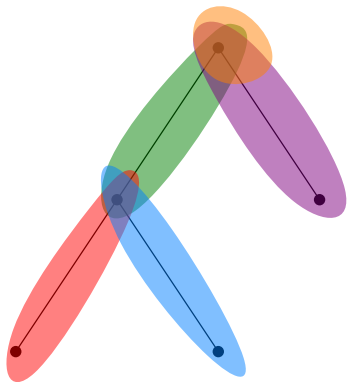
Treewidth: Tree



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Treewidth: Tree



Tree has treewidth 1

Treewidth: Planar graph

Theorem (Wiechert,2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

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Theorem

Planar graph on n vertices has treewidth $O(\sqrt{n})$ and the bound is tight.

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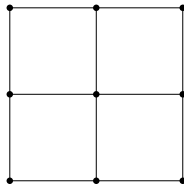
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Partitions

Maybe we need more structure

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Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G , if G has an H -partition of layered width ℓ and H has treewidth k , then

$$\text{qn}(G) \leq 3\ell(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, \dots, P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of \mathcal{P} .

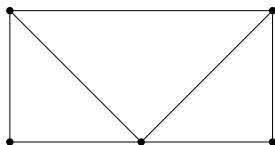
The quotient of \mathcal{P} is a graph, denoted by G/\mathcal{P} , with each vertex v_i corresponds P_i . For any two vertices v_i, v_j in G/\mathcal{P} , they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G .

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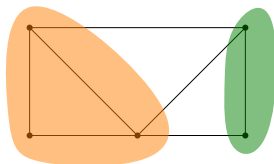


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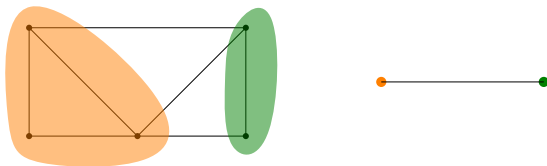


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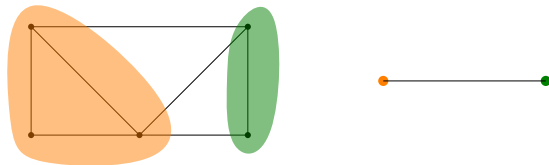
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Partitions

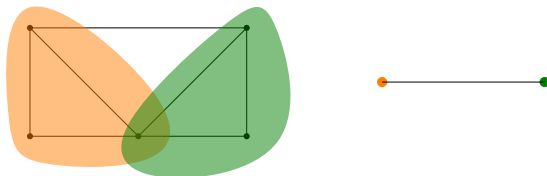
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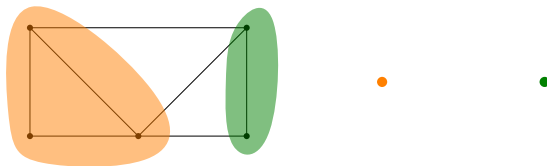
Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, \dots, P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of \mathcal{P} .

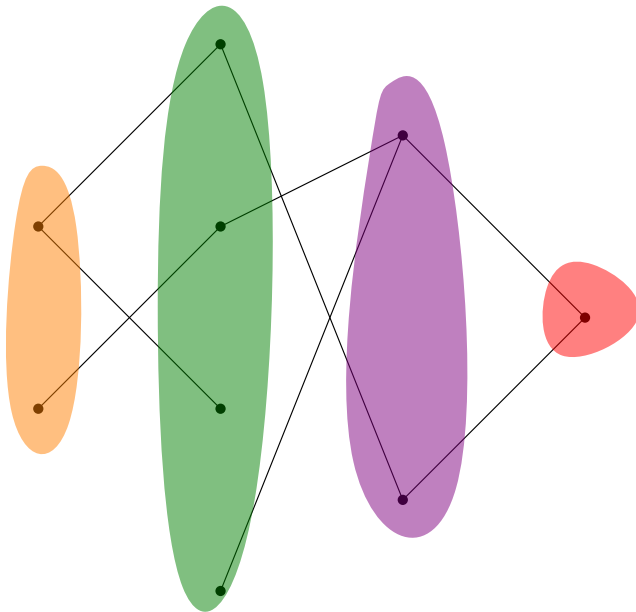
The quotient of \mathcal{P} is a graph, denoted by G/\mathcal{P} , with each vertex v_i corresponds P_i . For any two vertices v_i, v_j in G/\mathcal{P} , they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G .

Definition: H-partition

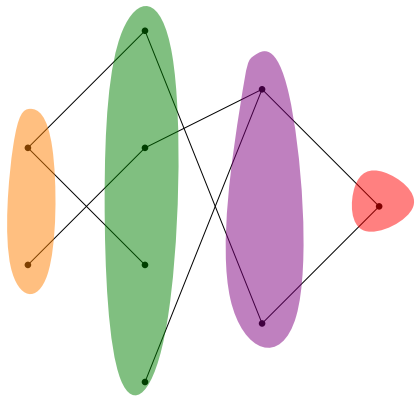
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Layering



Layering



Layered width

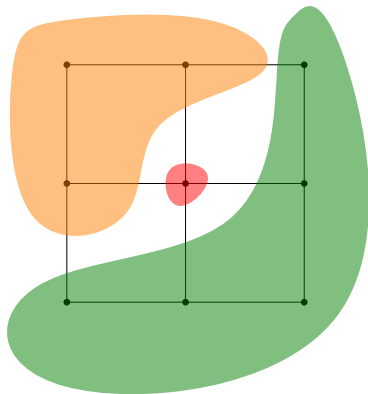
Definition: Layered width

The layered width of a partition \mathcal{P} of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G , each element in \mathcal{P} has at most ℓ vertices in each element of path-partition.

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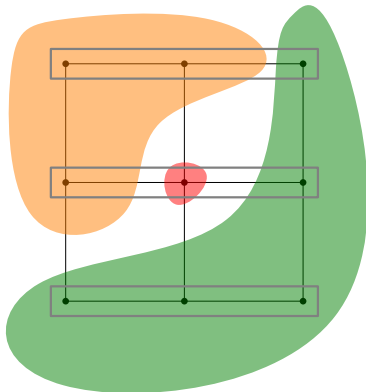
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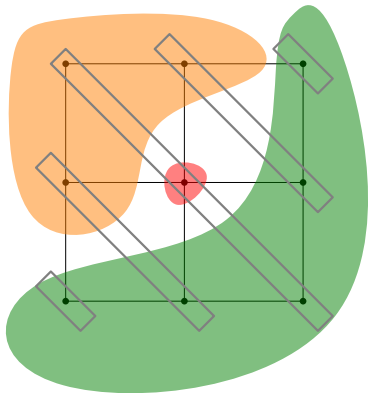
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Layered width: Tree

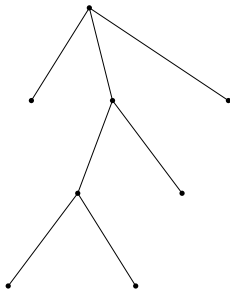
Definition: Vertical Path

Let T be a tree rooted at a vertex r , a non-empty path (x_1, \dots, x_p) in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.

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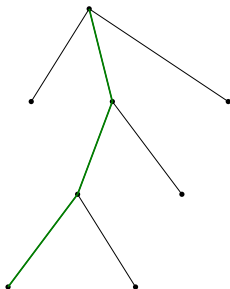
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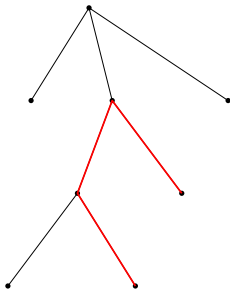
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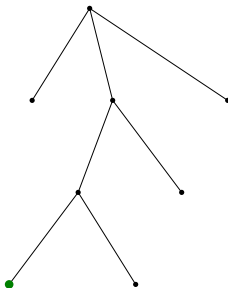
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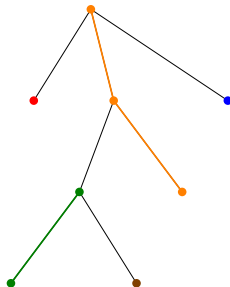
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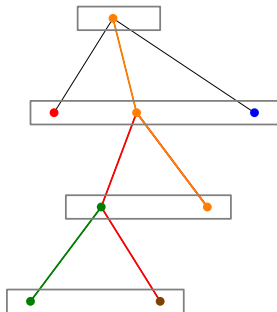
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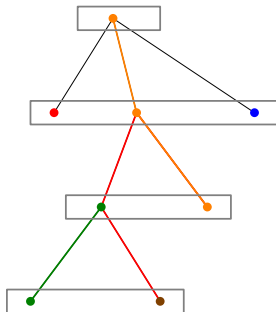
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A partition of the tree where each part is a vertical path has layered width 1.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G , if G has an H -partition of layered width ℓ and H has treewidth k , then

$$\text{qn}(G) \leq 3l(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$$

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Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex r on outerface of G^+ . For any cycle F in G^+ , which can be partitioned into at most 6 pairwise disjoint vertical paths of T , with $F = [P_1, \dots, P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F , then G has a partition \mathcal{P} into vertical paths of T , and $P_1, \dots, P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

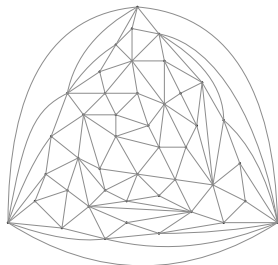
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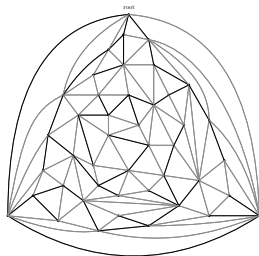


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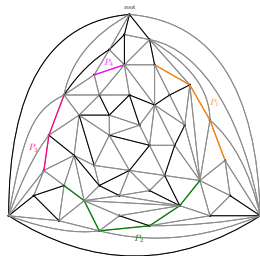


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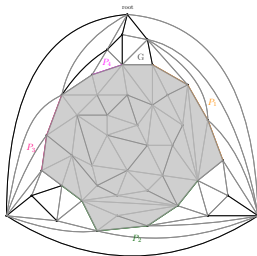


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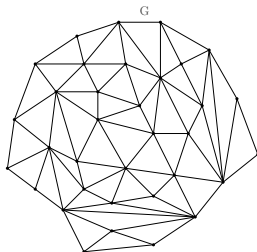


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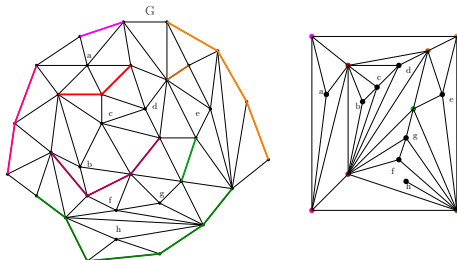


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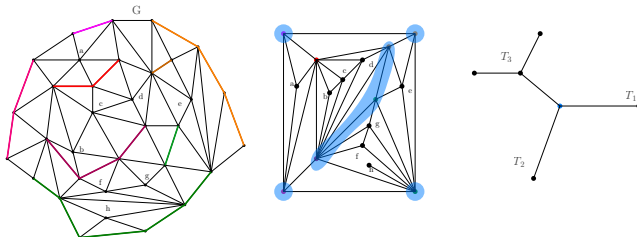


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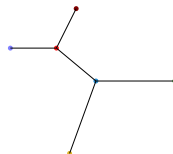
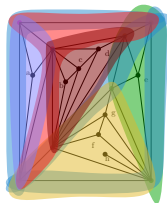
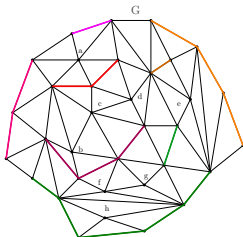


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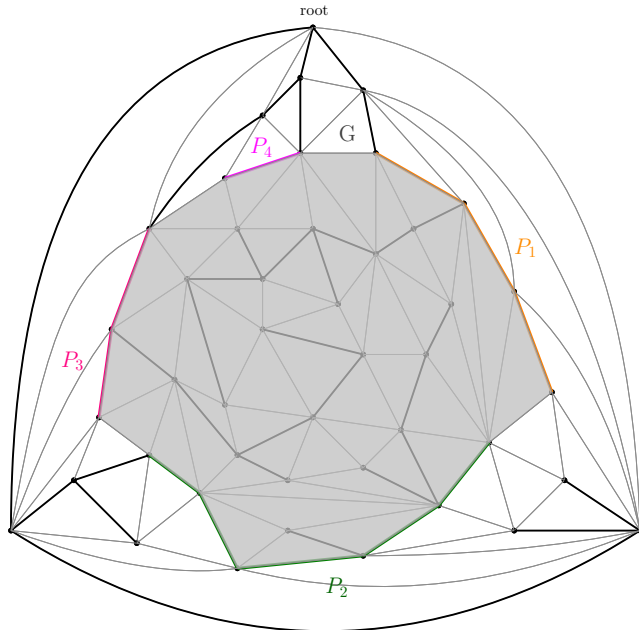
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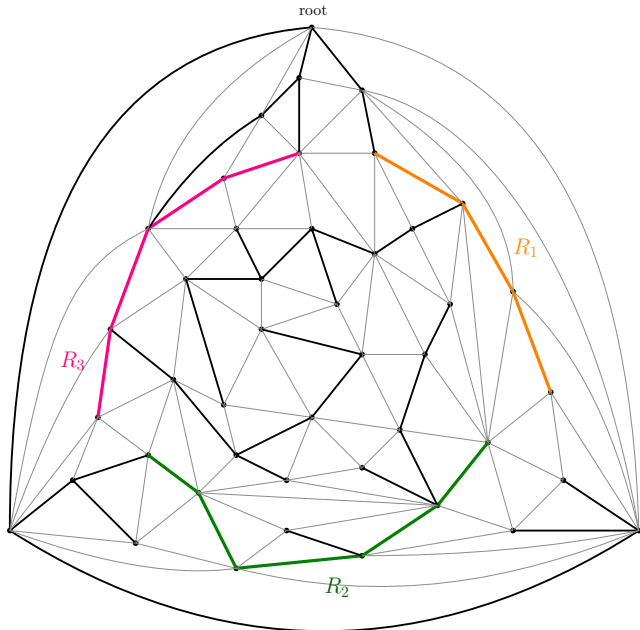
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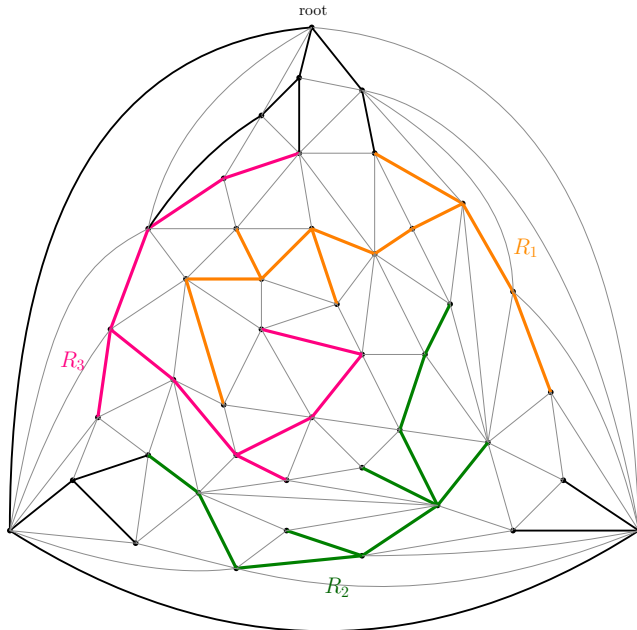
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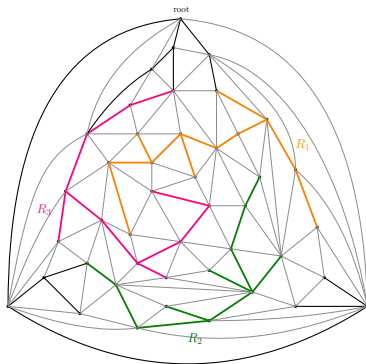
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Planar Graph Decomposition

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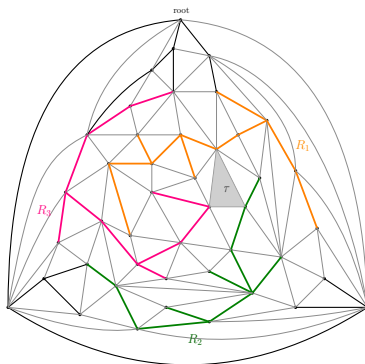
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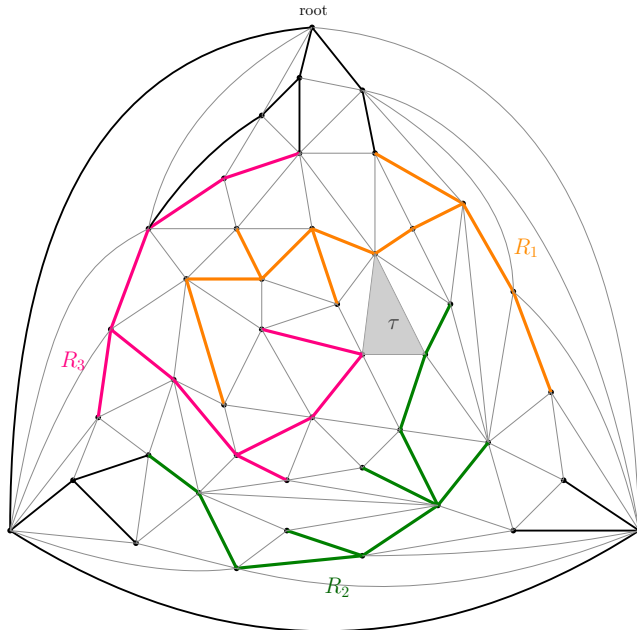
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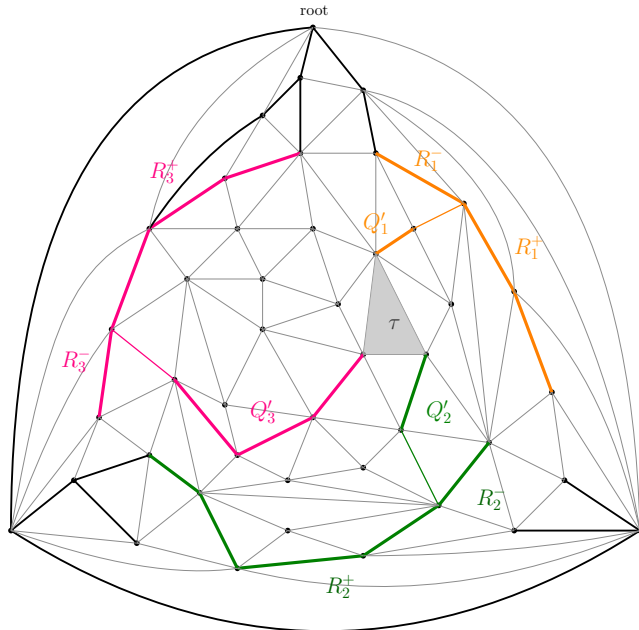
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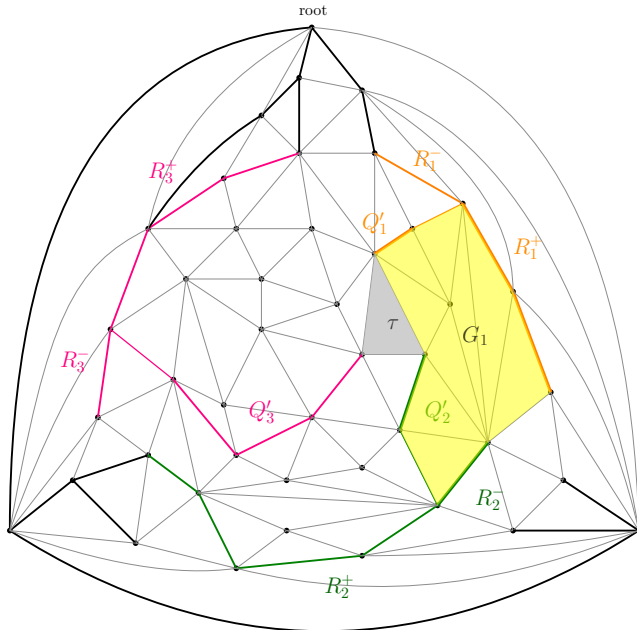
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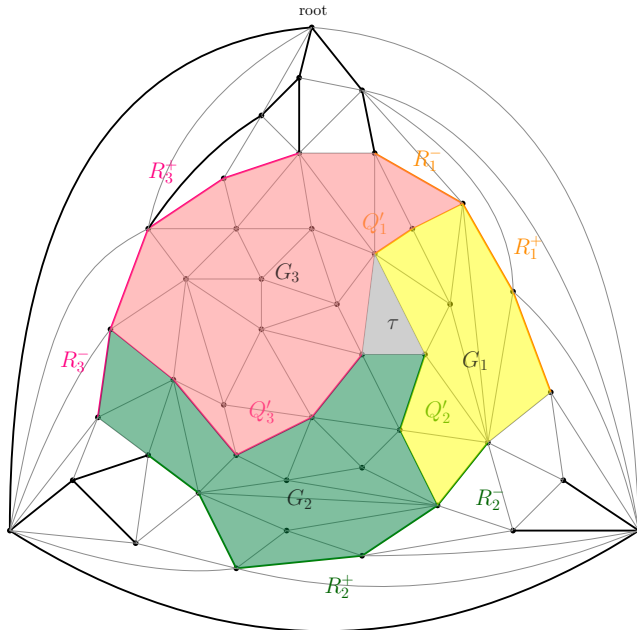
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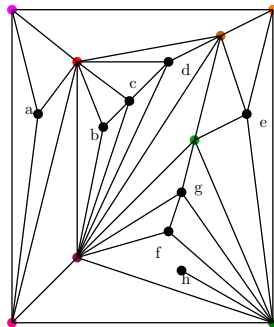
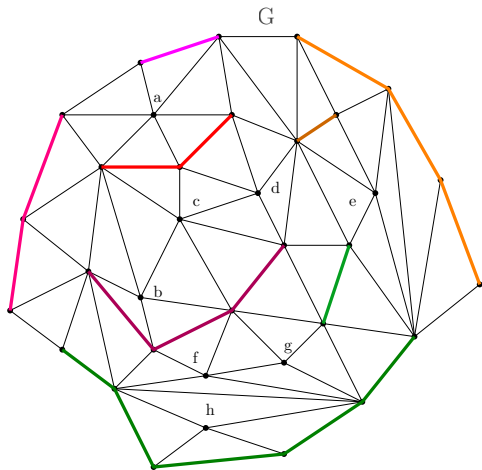
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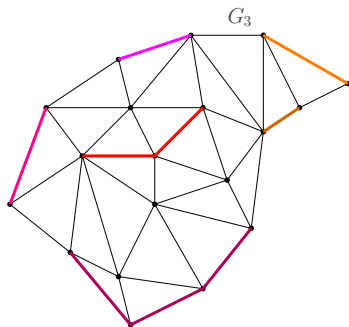
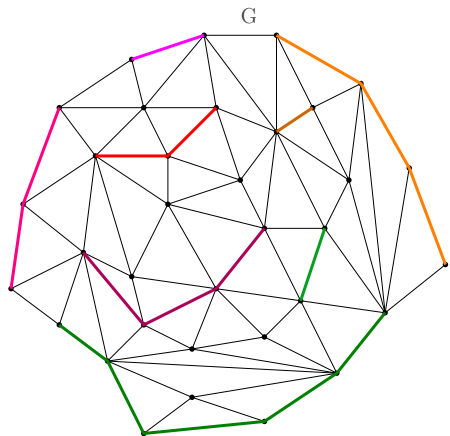
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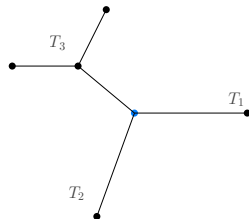
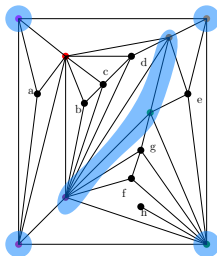
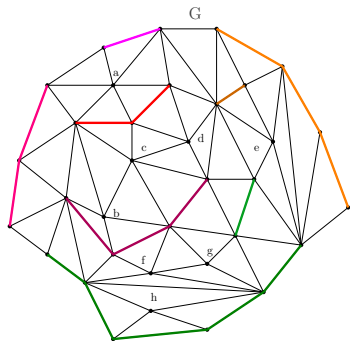
Planar Graph Decomposition



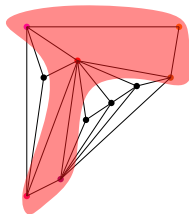
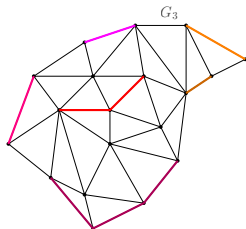
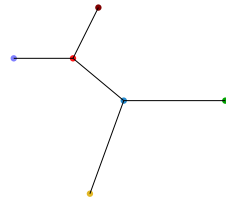
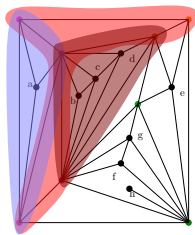
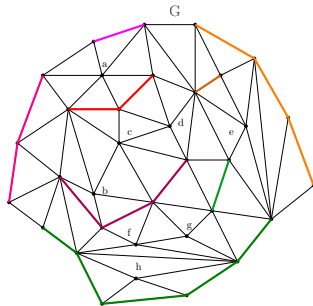
Planar Graph Decomposition



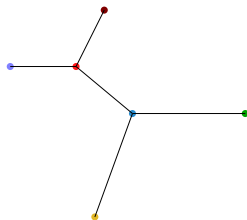
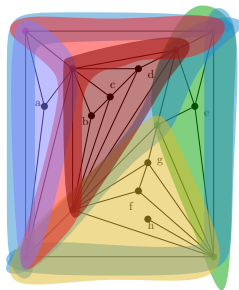
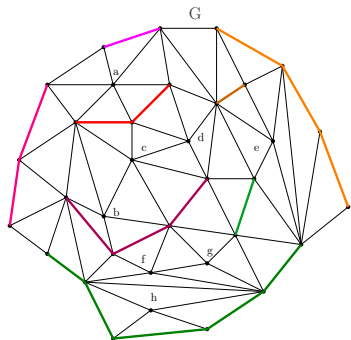
Planar Graph Decomposition



Planar Graph Decomposition



Planar Graph Decomposition



Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition \mathcal{P} with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

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Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has queue-number at most

$$3(2^8 - 1) + \lfloor \frac{3}{2} \rfloor = 766$$

1 Introduction to Queue-Number

- Queue Layout
- Queue-Number

2 Introduction to Treewidth

- Treewidth

3 Introduction to Partitions

- Partitions
- Layering
- Layered Width
- Vertical Path

4 Planar Graph Decomposition

- The Decomposition Lemma
- Induction

5 Summary