Planar Graphs Have Bounded Queue-Number

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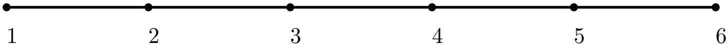
Geometry: Combinatorics and Algorithms Seminar
Shengzhe Wang
April 21, 2023

Queue Layout



A path graph with 6 vertices

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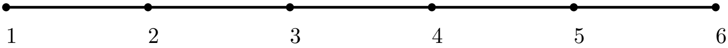

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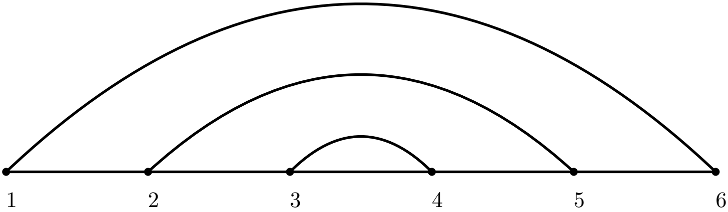

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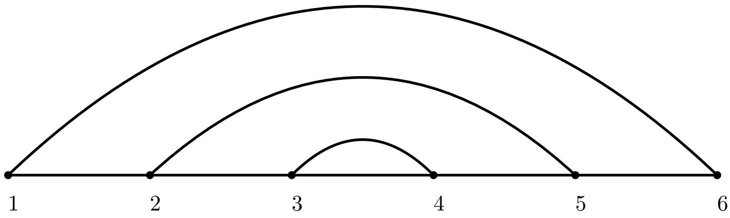
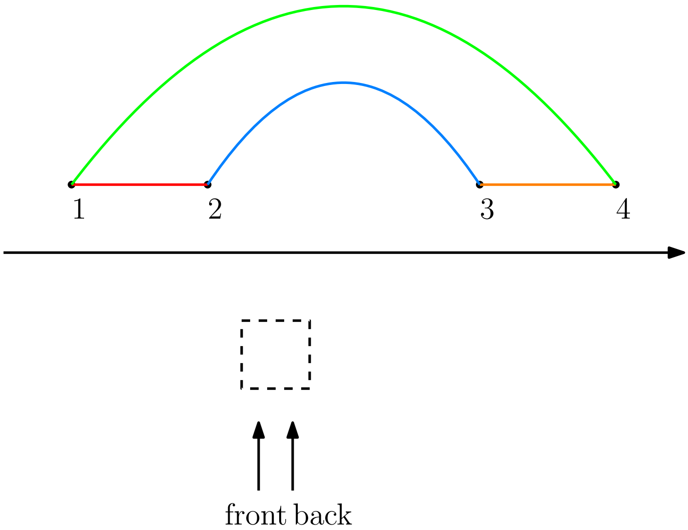


Not Valid Queue-Layout

Queue Layout



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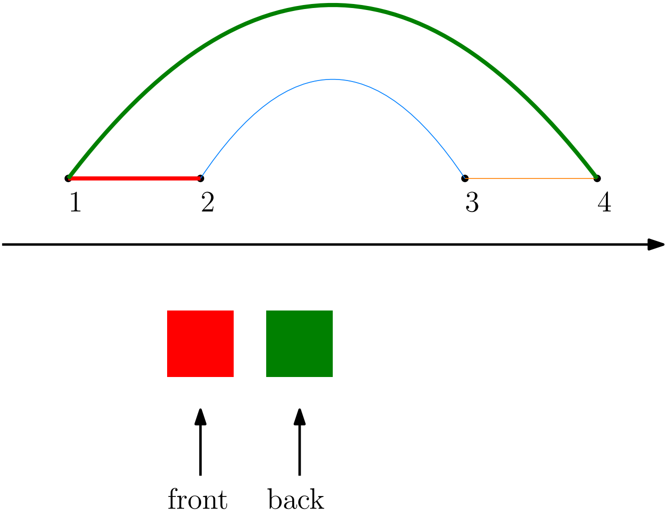

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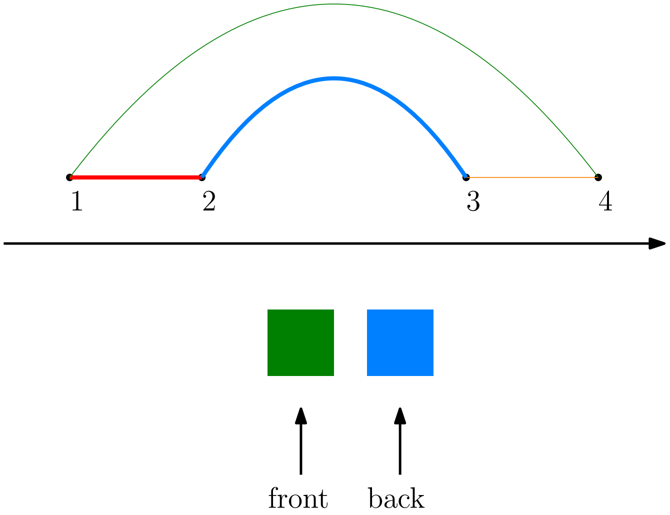

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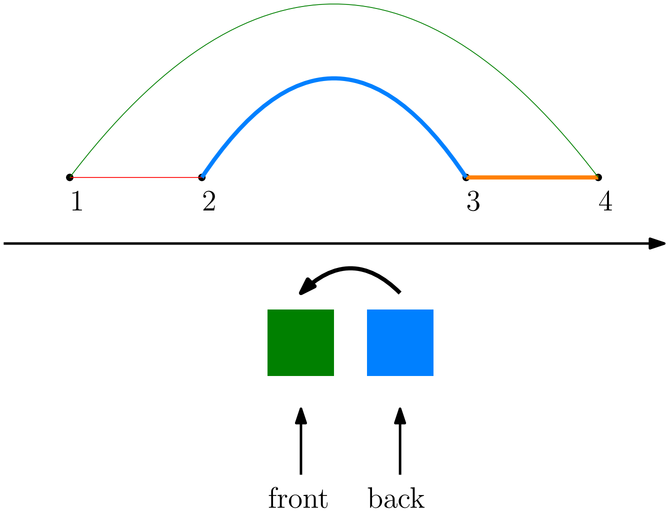
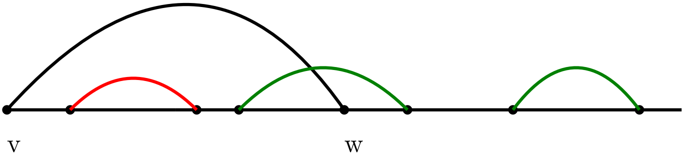


Queue Layout

Definition: Queue

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, a queue of G is a set of edges $E' \subseteq E$ such that any disjoint edges $vw, xy \in E'$, w.l.o.g, $v \prec w, x \prec y$ and $v \prec x$, we have $w \prec y$.



Queue-Number

Definition: K-Queue Layout

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, for an integer $k \geq 0$ a k-queue layout of G is a partition of E into $E_1, E_2, ..., E_k$ such that each E_i is a queue of G with respect to the ordering $\preceq$.

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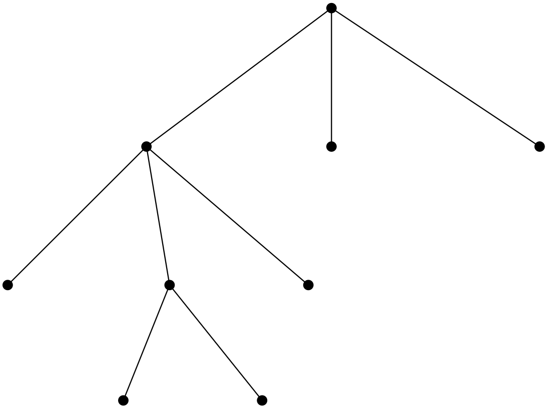
Definition: Queue-Number

The queue-number of G, denoted by $\mathrm{qn}(G)$, is the minimum integer k such that G has a k-queue layout for some ordering $\preceq$.

Queue-Number: Tree

What is the queue-number of a tree?



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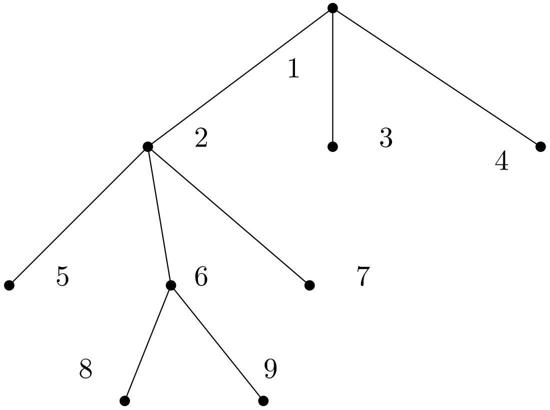

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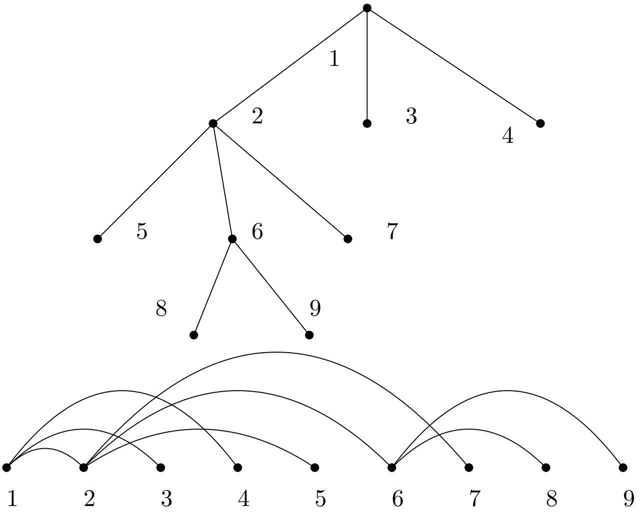
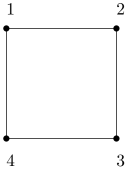
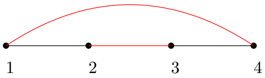


Queue Number: Cycle





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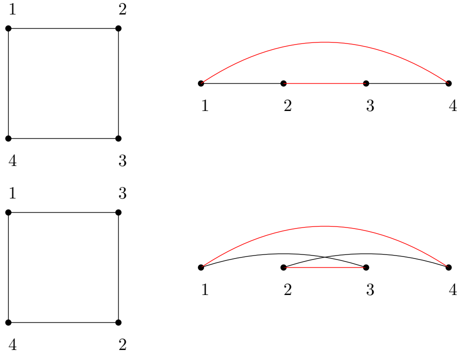

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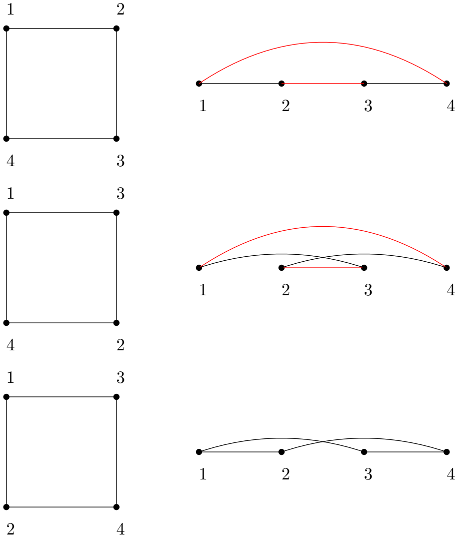
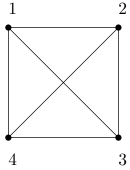
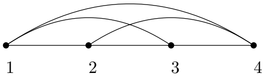


Queue Number: K_4





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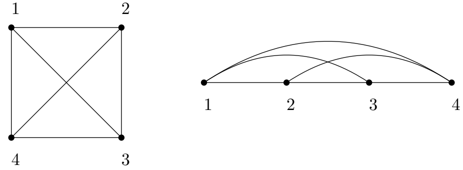

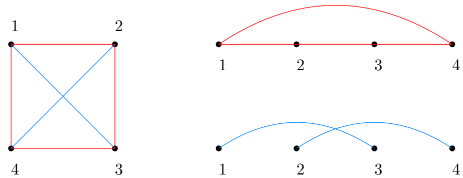

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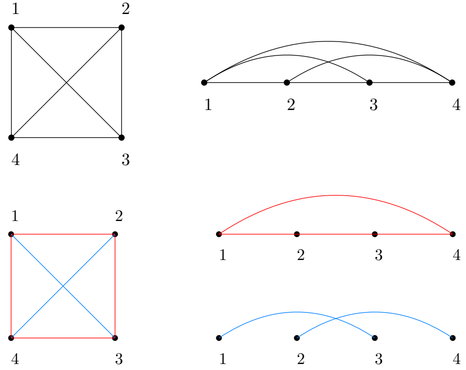

Theorem (Heath, Rosenberg, 1992)

The complete graph K_n has queue number $\lfloor \frac{n}{2} \rfloor$.

Treewidth

Do we have some tools to help bound the queue-number?

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Theorem (Wiechert, 2017)

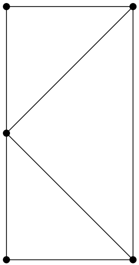
Every graph with treewidth k has queue-number at most $2^k - 1$.

Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.



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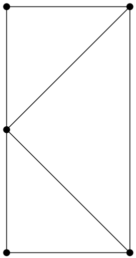

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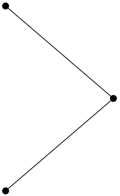

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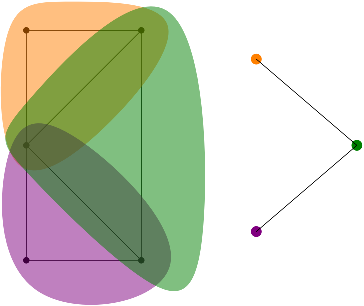

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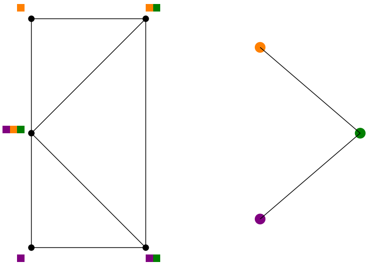

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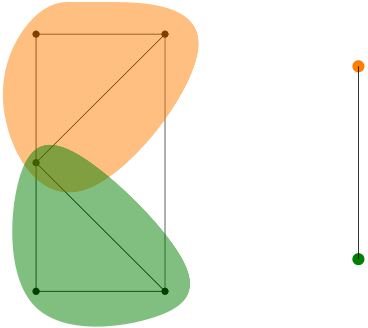

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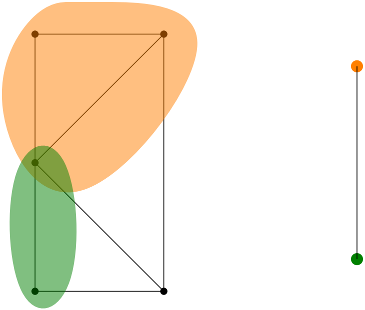

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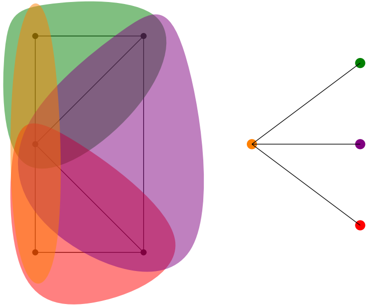

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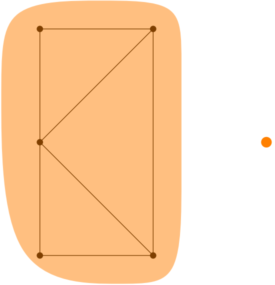

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Definition: Treewidth

The treewidth of a graph G is the minimum width of all tree-decomposition of G.

Treewidth: Fixed-Parameter Tractability

- Computing a maximum independent set in a graph G is NP-hard.

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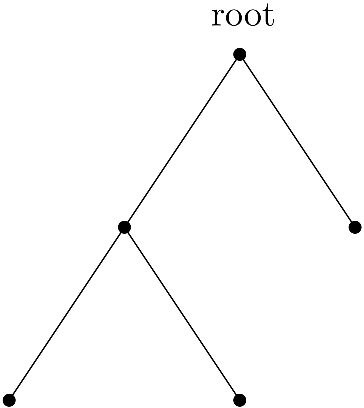
Treewidth: Fixed-Parameter Tractability

- Computing a maximum independent set in a graph G is NP-hard.
- If G has treewidth $\leq k$, then a maximum independent set in G can be computed in time $O(k^2 4^k |V|)$.
- Dynamic Programming on trees is relatively fast.

Treewidth: Tree



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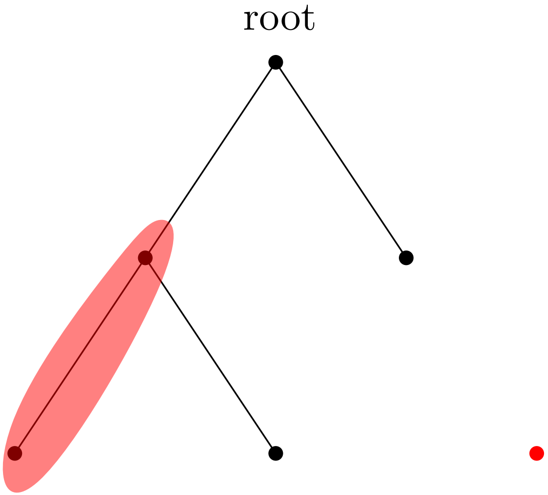

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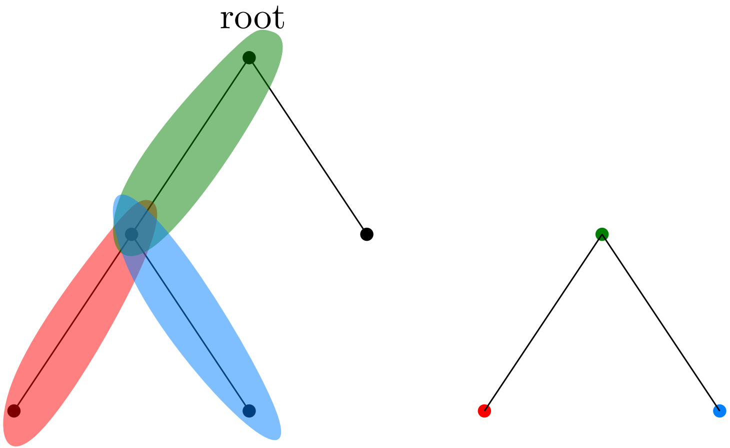

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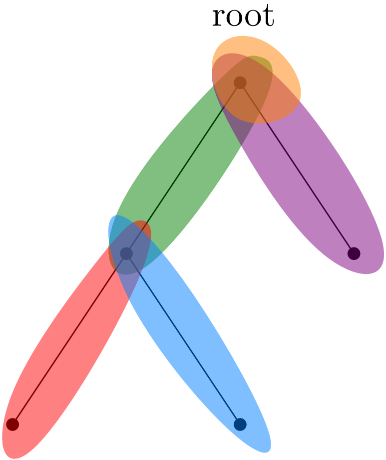

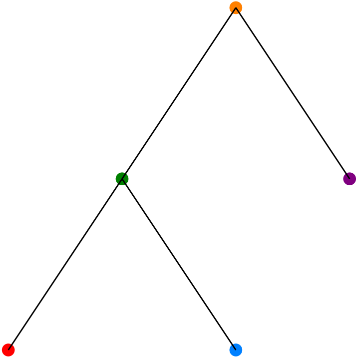

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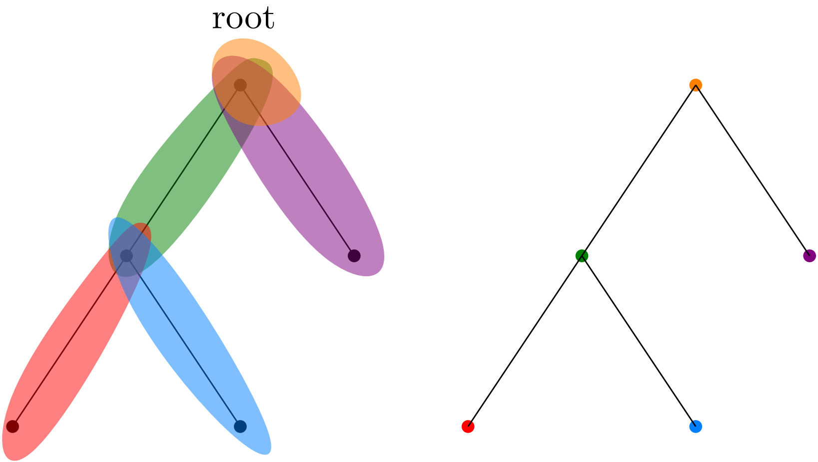

Tree has treewidth 1

Treewidth: Planar graph

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If planar graph has bounded treewidth, then planar graph has bounded queue-number.

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Planar graph on n vertices has treewidth $O(\sqrt{n})$.

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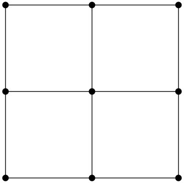
Theorem (Robertson, Seymour)

A grid graph with size $n \times n$ has treewidth n.



Partitions

Maybe we need more structure

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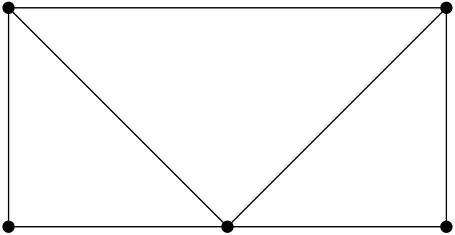
Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Partitions

G



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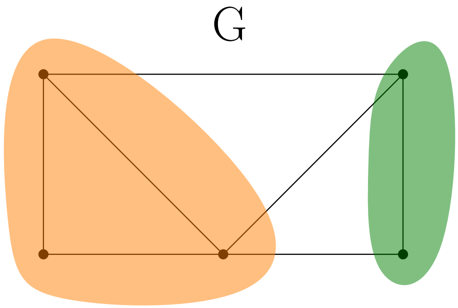

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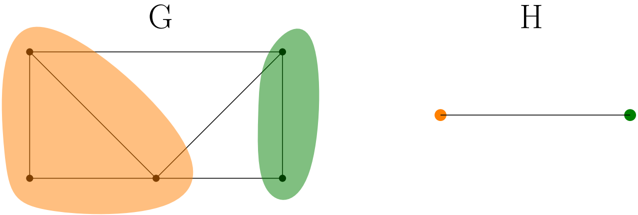
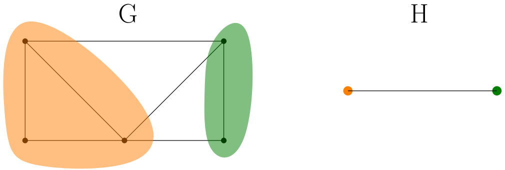


Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.



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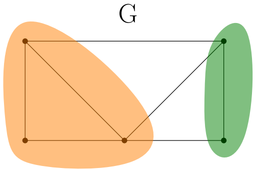
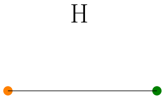
Definition: H-partition

An H-partition of a graph G is a pair (A, H), s.t. $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.





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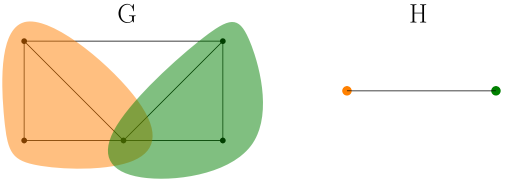

Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

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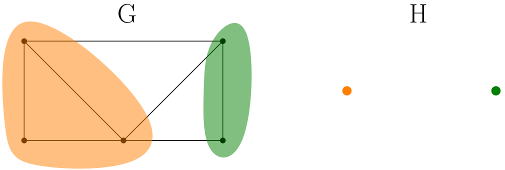
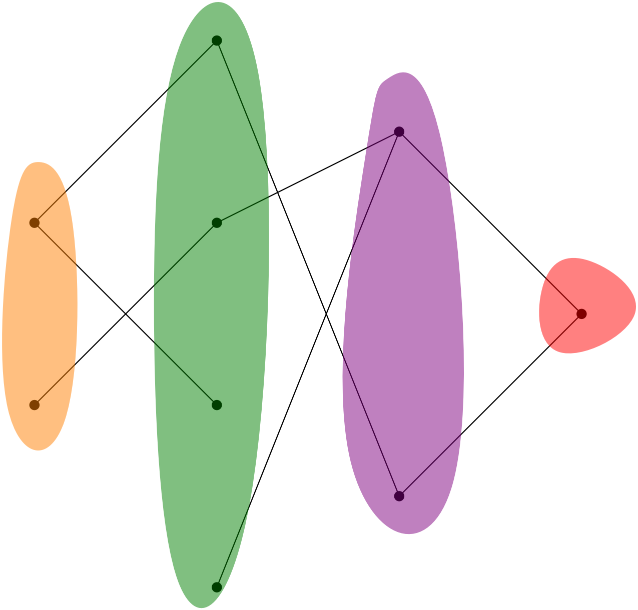


Layering



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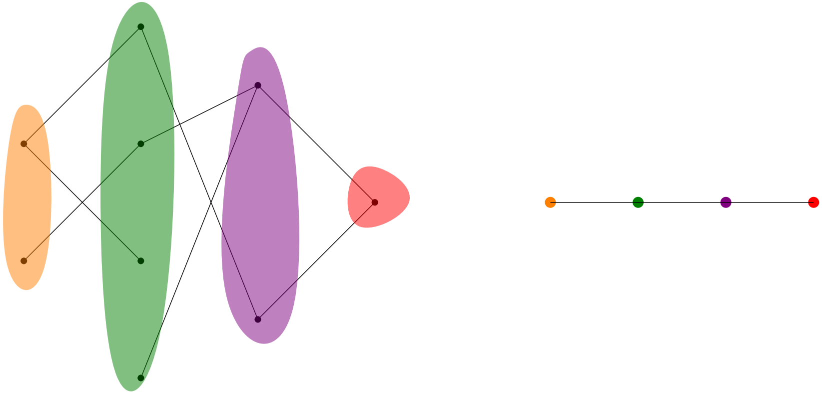

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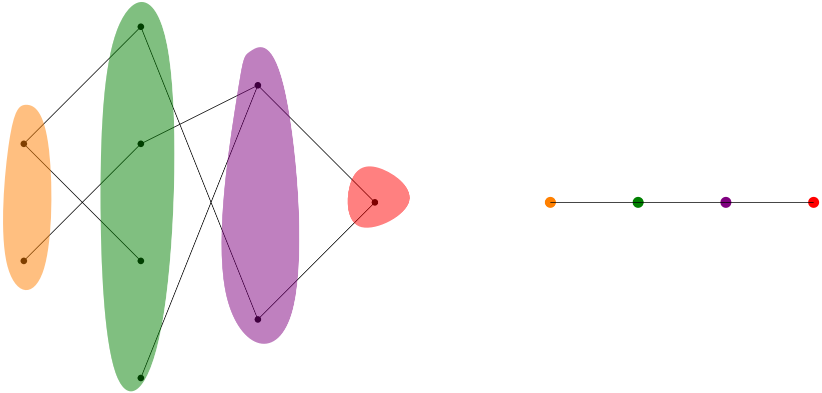

Layering is just a path-partition.

Layered width

Definition: Layered width

The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, s.t. each element in $\mathcal{P}$ has at most ℓ vertices in each element of the path-partition.

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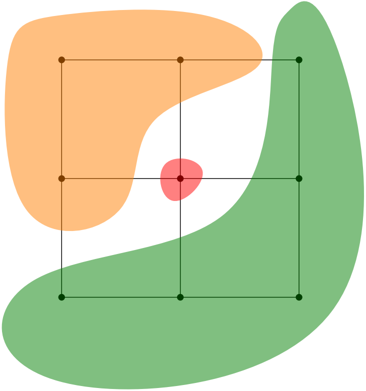

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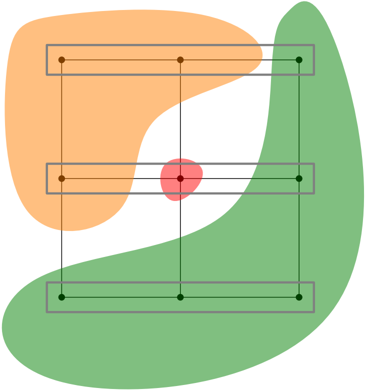

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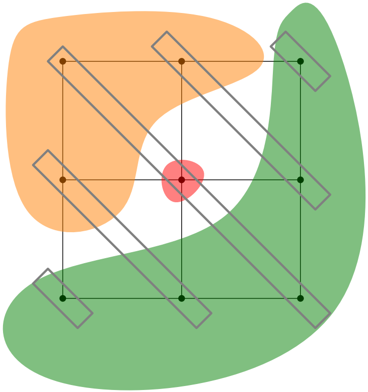

Layered width: Tree

Definition: Vertical Path

Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.

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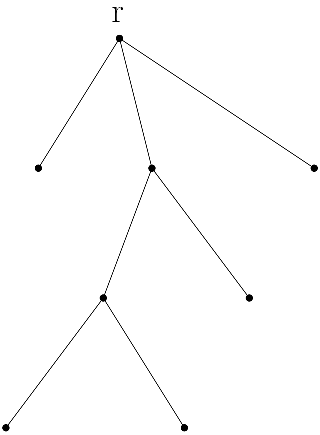

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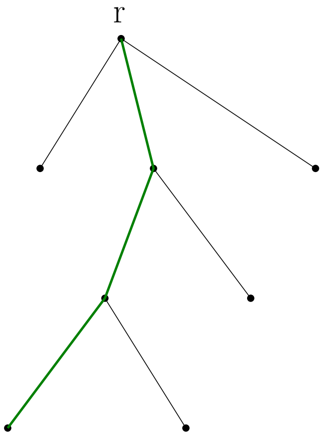

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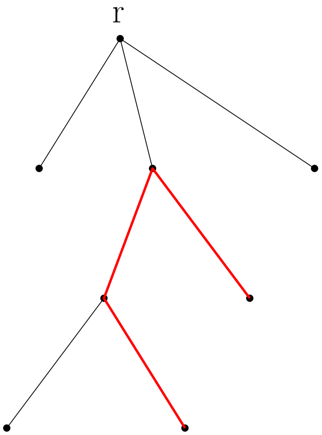

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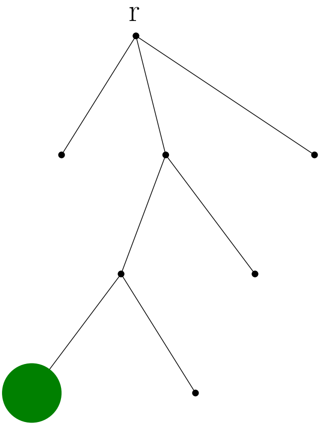

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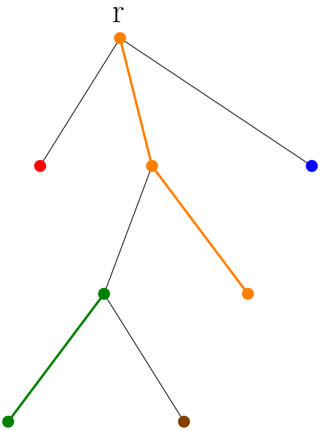

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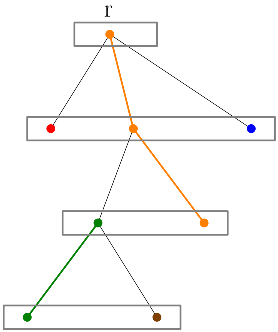

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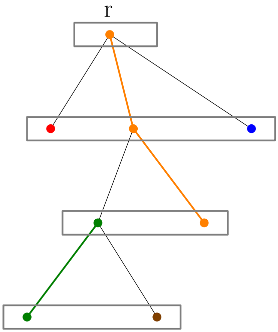

A partition of layered width 1 where each part is a vertical path.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

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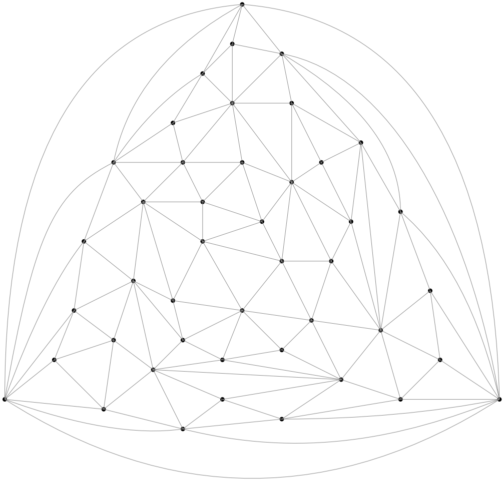

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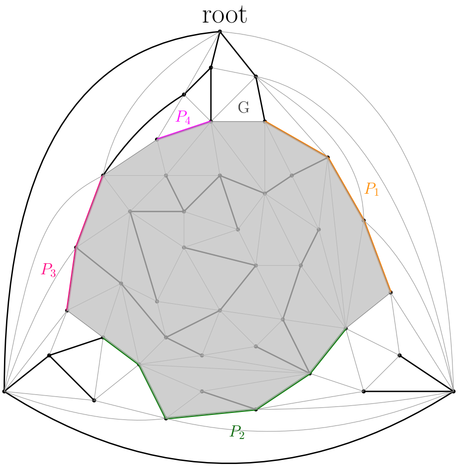

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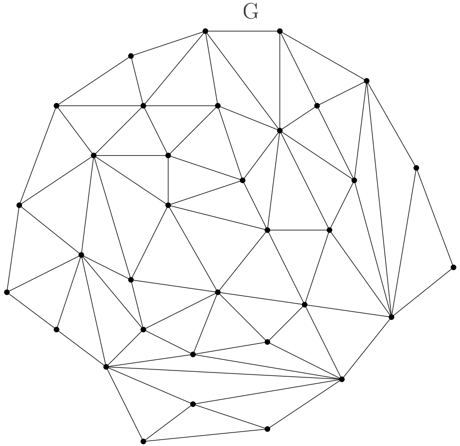

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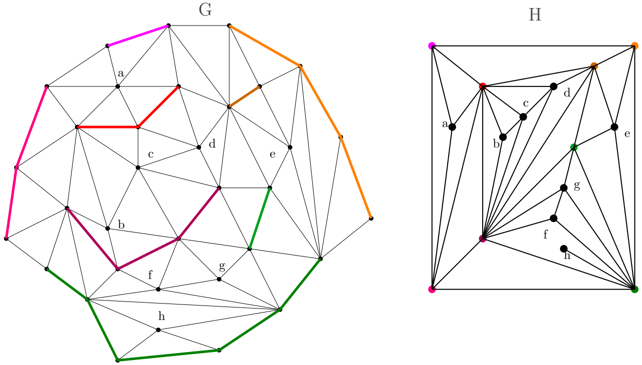

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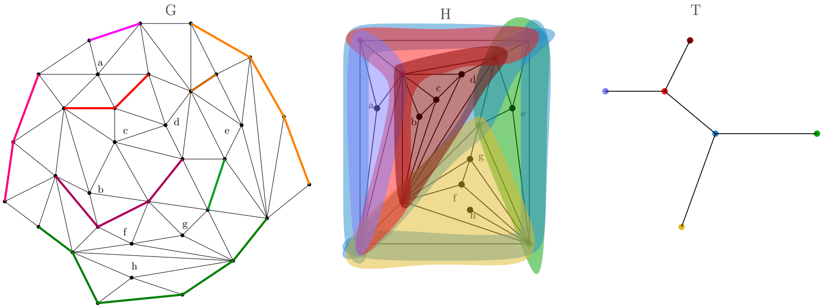

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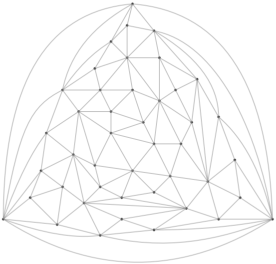

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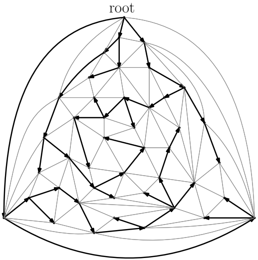

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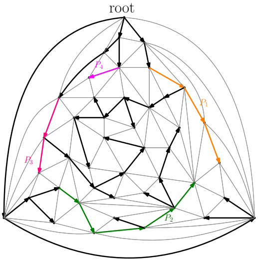

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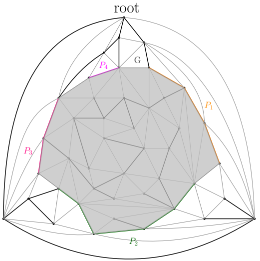

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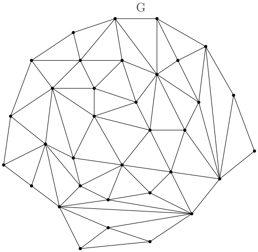

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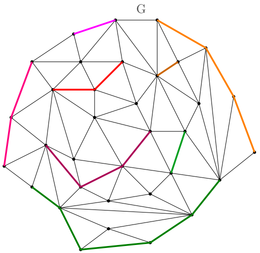

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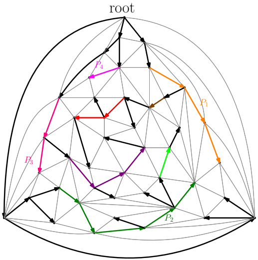

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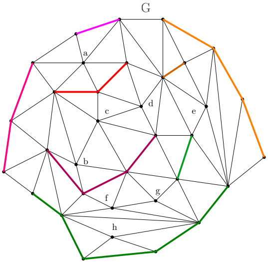

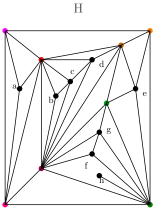

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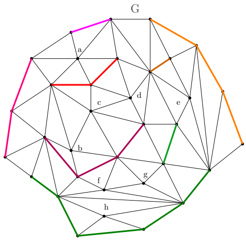

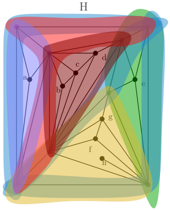

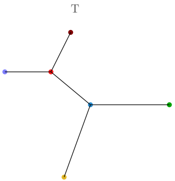

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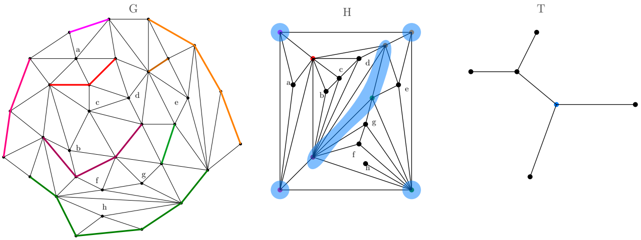
Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ... P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x



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Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

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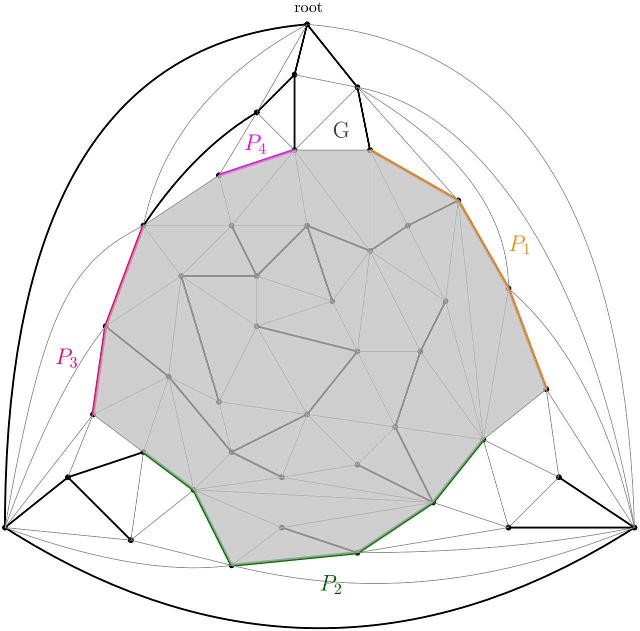

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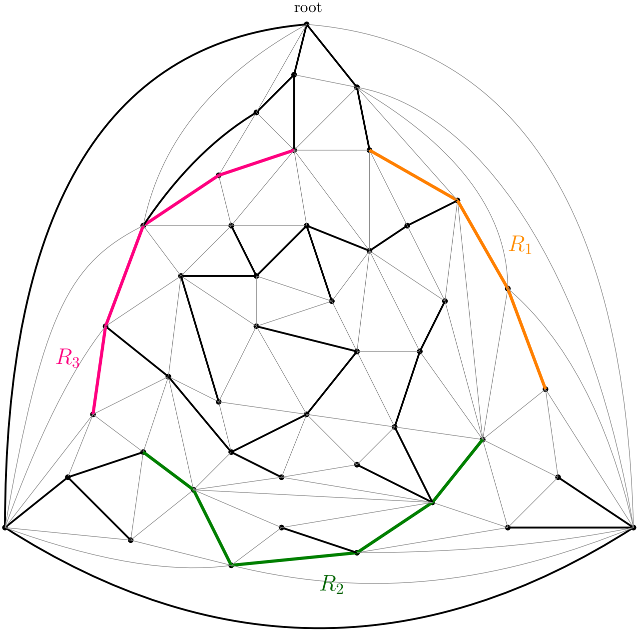

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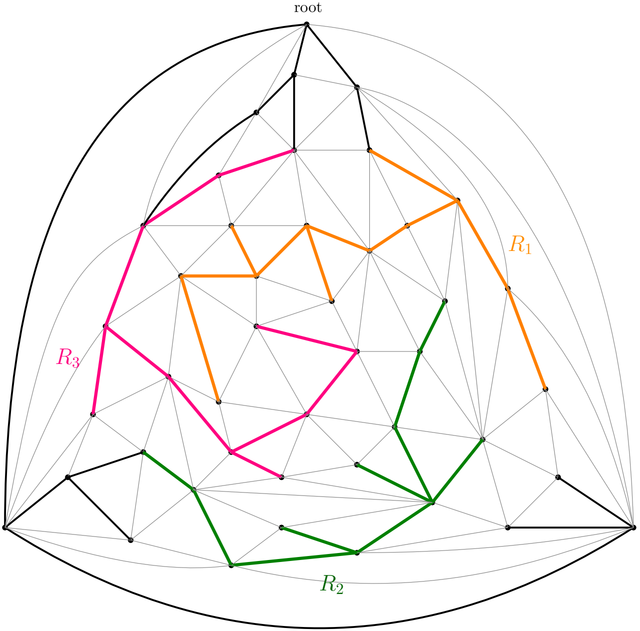

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Lemma (Sperner's Lemma)

Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.

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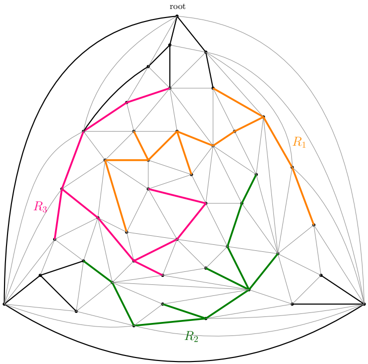

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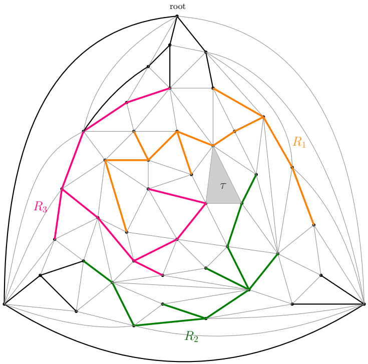

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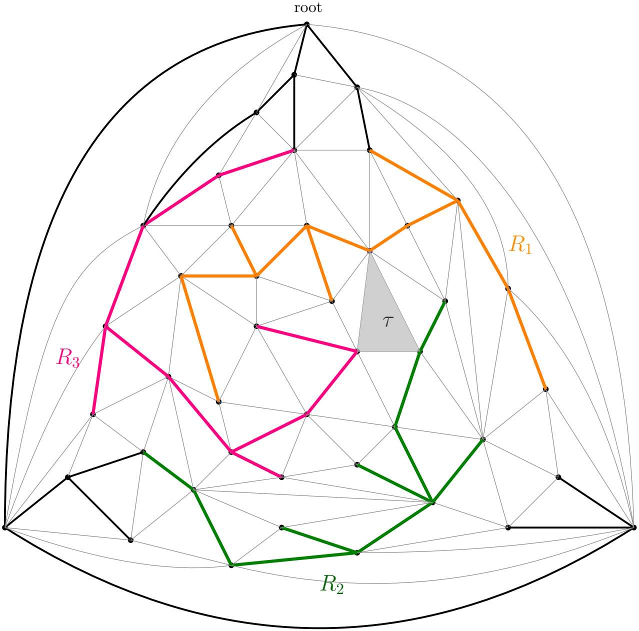

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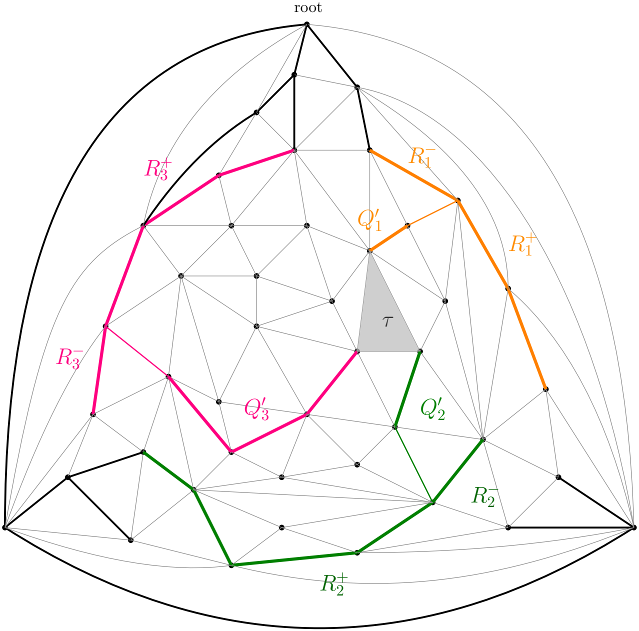

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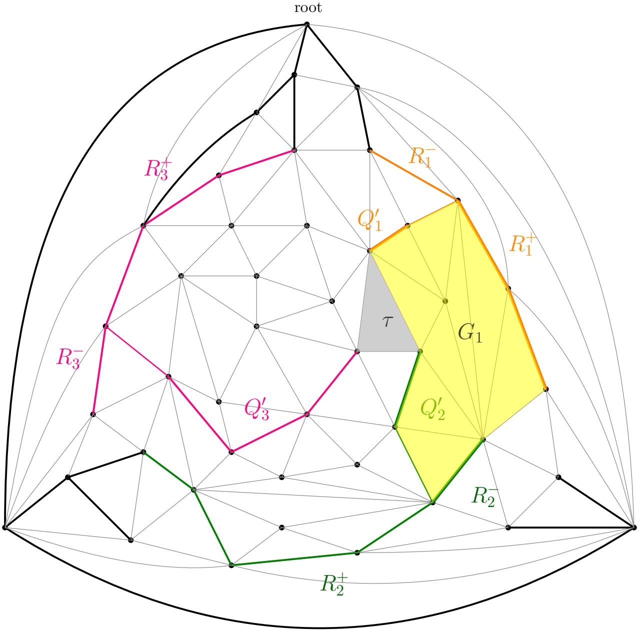

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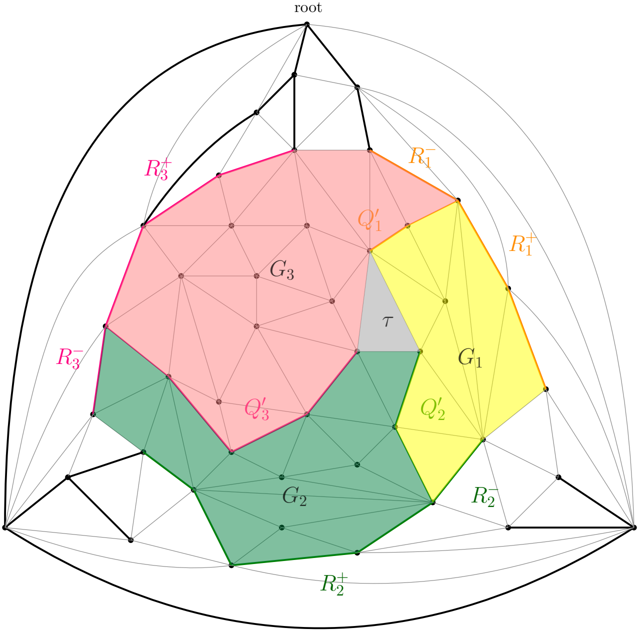
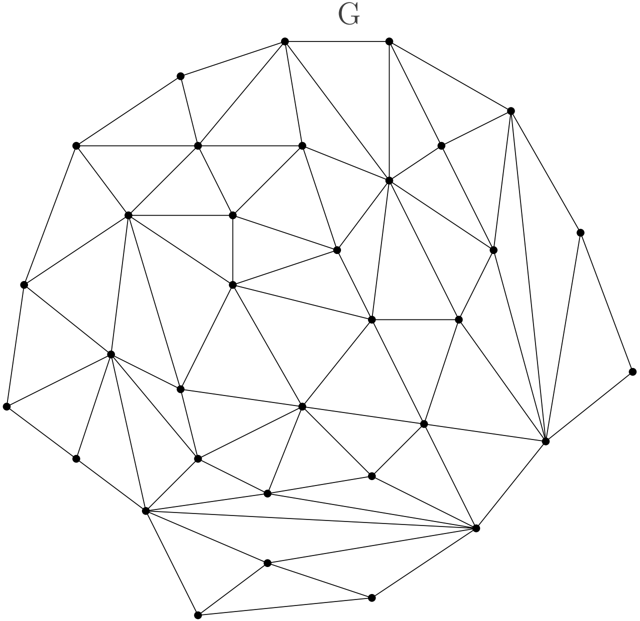


Planar Graph Decomposition: Construction



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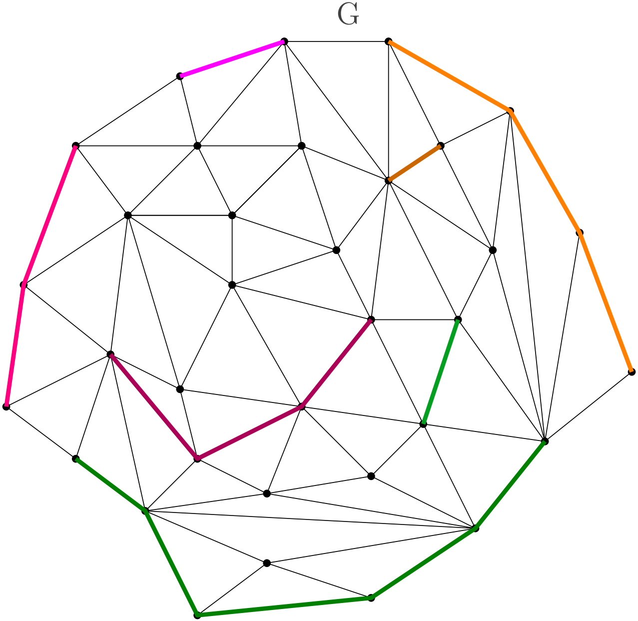

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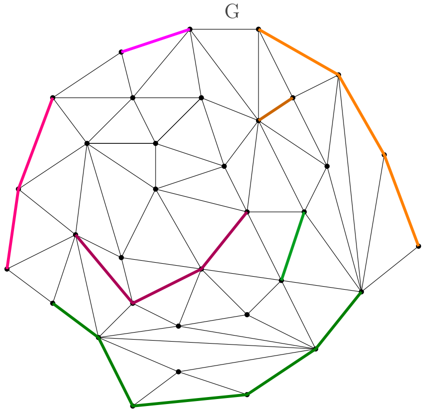

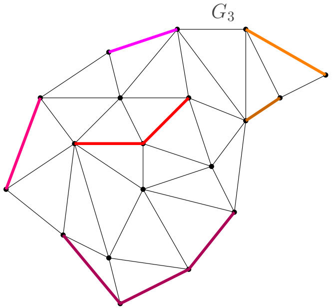

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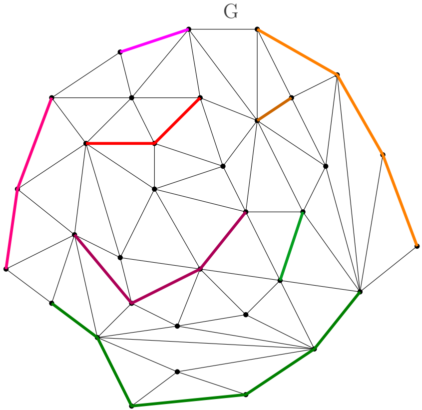

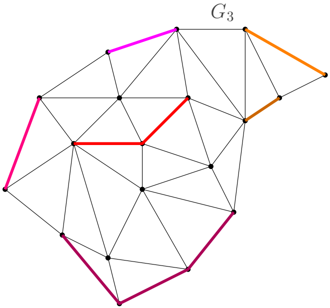

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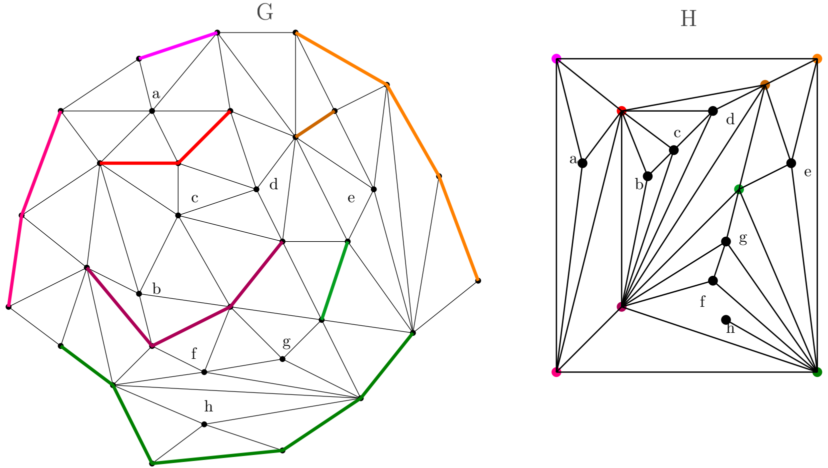

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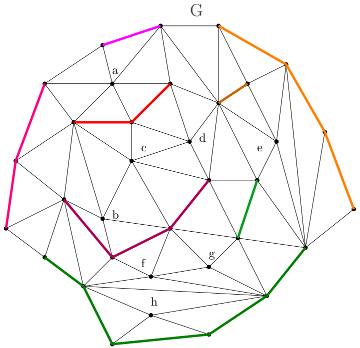

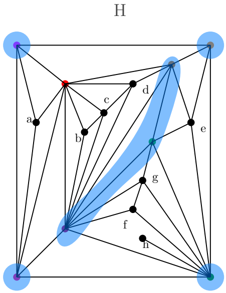

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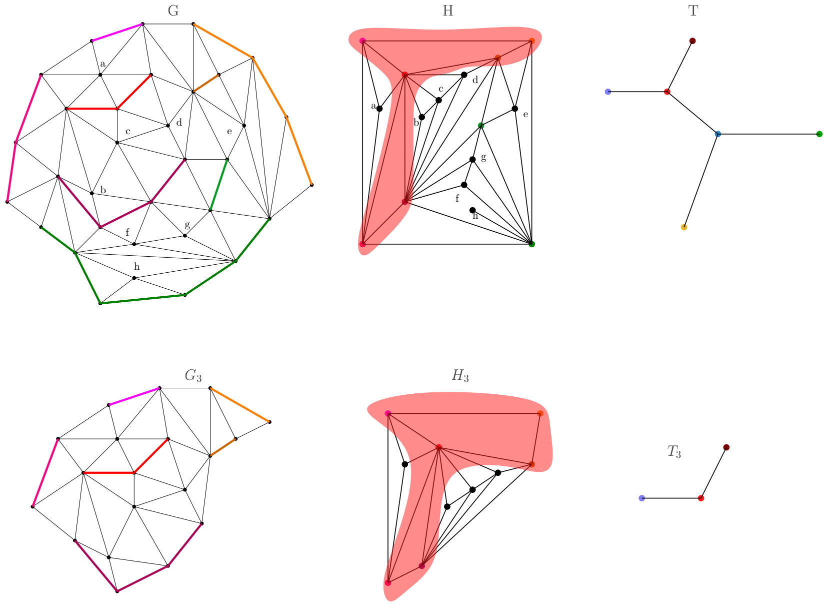

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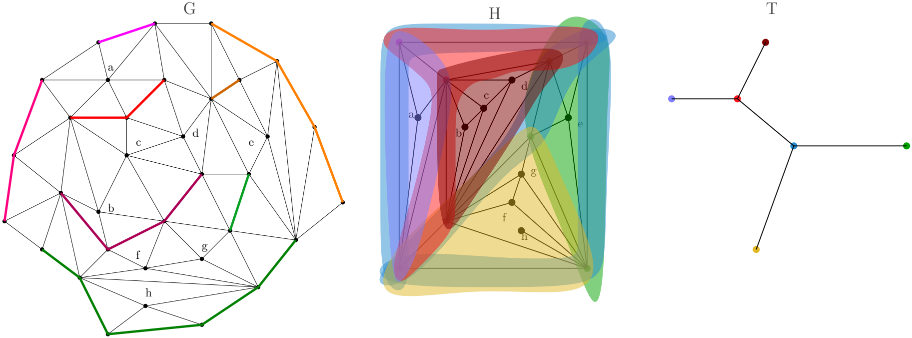

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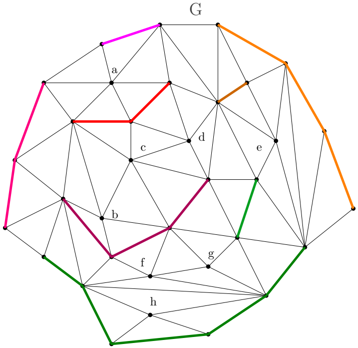

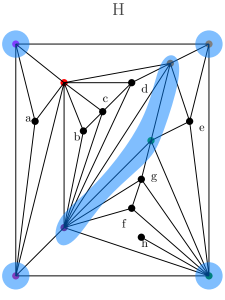

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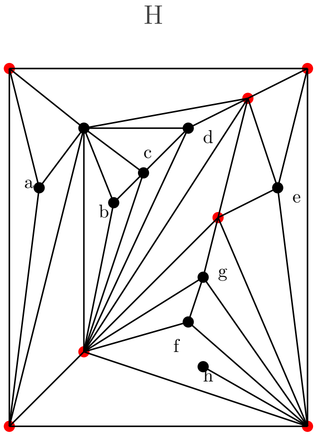

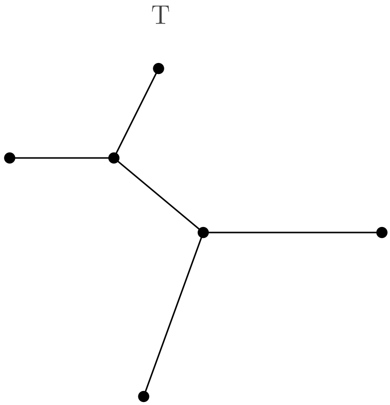

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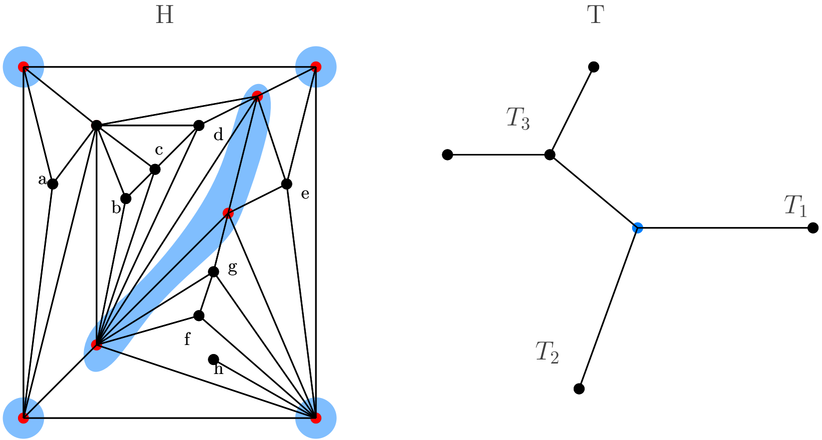

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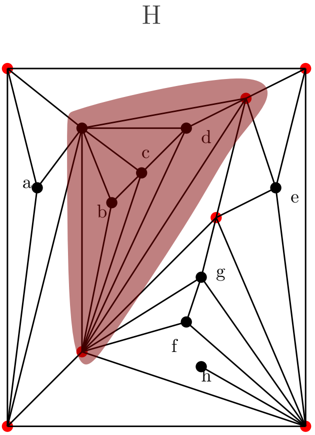

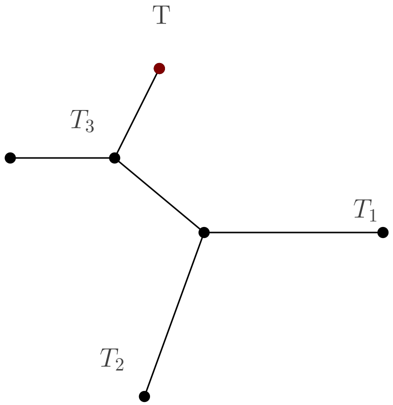

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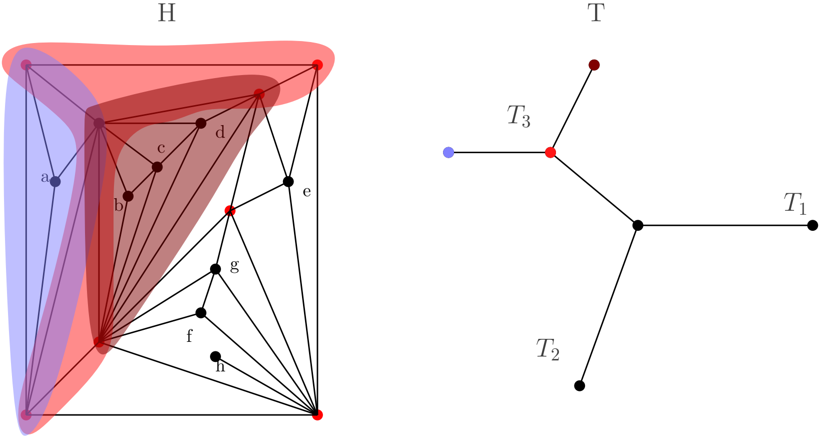

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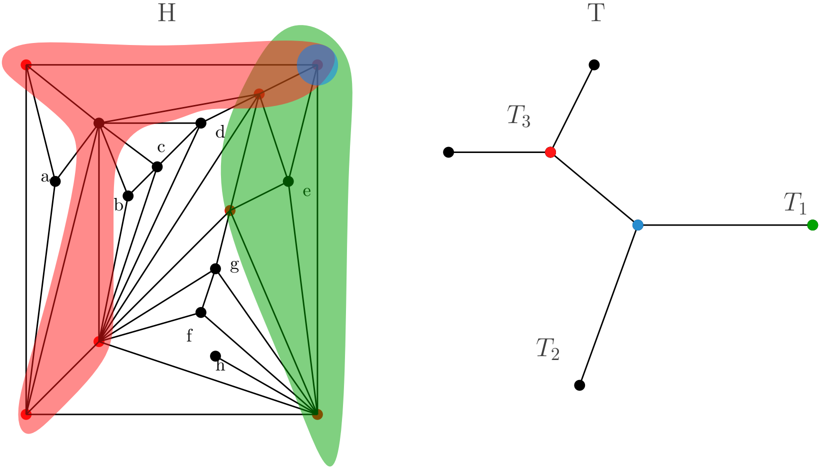

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Every planar graph G has queue-number at most

$$3(2^8 - 1) + \left\lfloor \frac{3}{2} \right\rfloor = 766$$

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- ...

Portal