Planar Graphs Have Bounded Queue-Number

Vida Dujmović [1] Gwenaël Joret [2] Piotr Micek [3] Pat Morin [4]
Torsten Ueckerdt [5] David R. Wood [6]

[1]University of Ottawa [2]Université Libre de Bruxelles [3]Jagiellonian University
[4]Carleton University [5]Karlsruhe Institute of Technology [6]Monash University

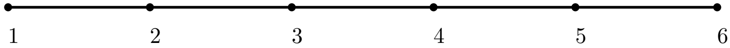
J. ACM 67/4:1-38, 2020

Geometry: Combinatorics and Algorithms Seminar
Shengzhe Wang
April 21, 2023

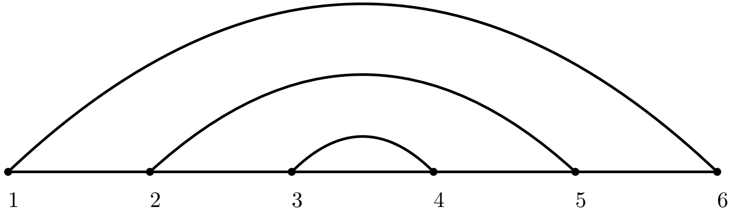
Queue Layout

A path graph with 6 vertices

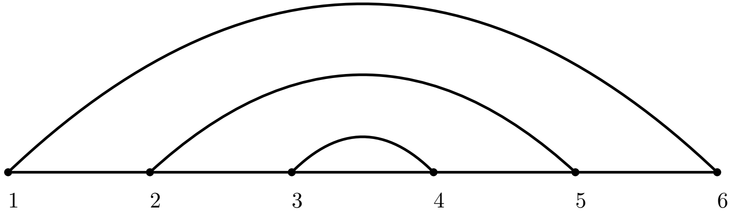
Queue Layout



A path graph with 6 vertices

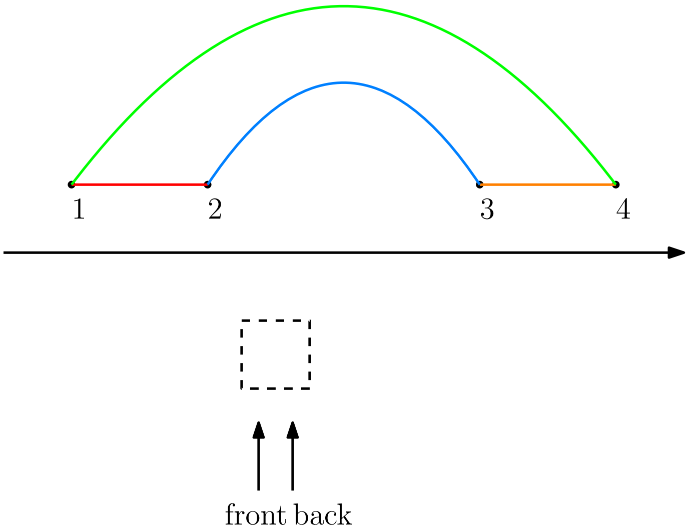


Queue Layout

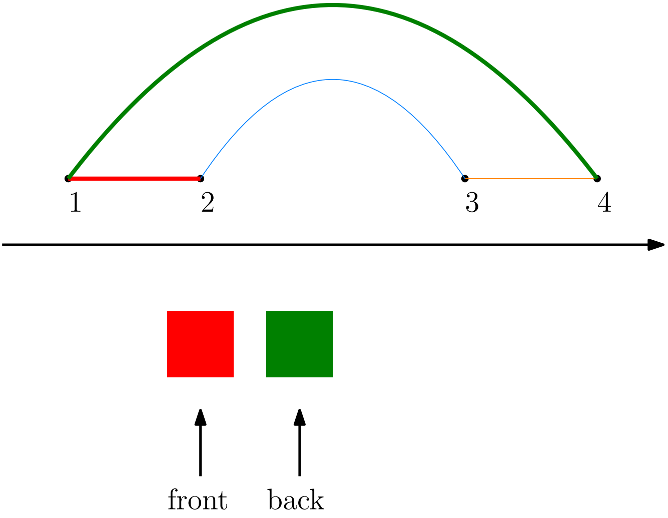


Not Valid Queue-Layout

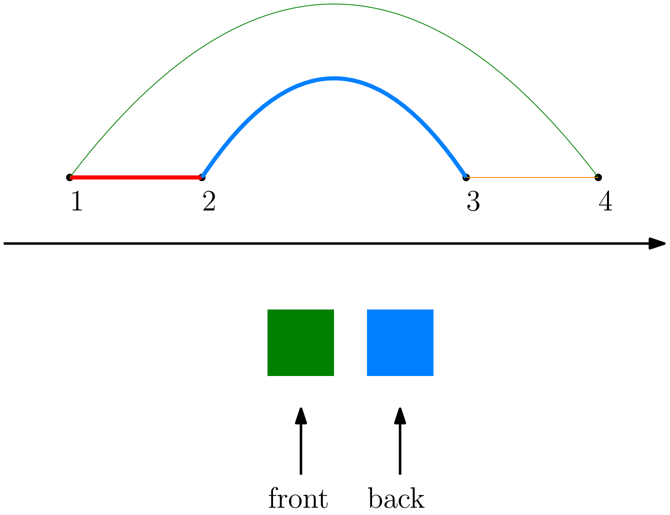
Queue Layout



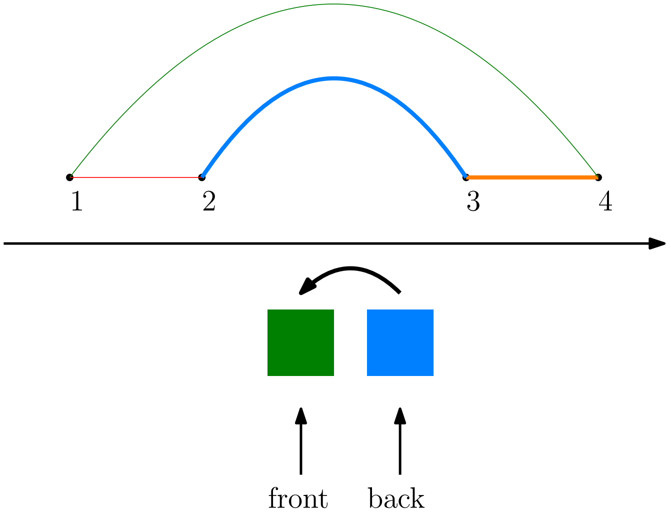
Queue Layout



Queue Layout



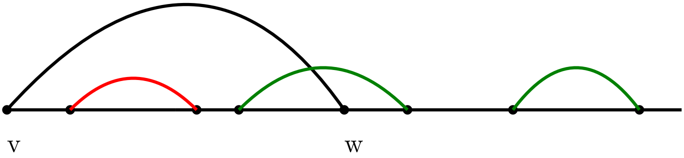
Queue Layout



Queue Layout

Definition: Queue

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, a queue of G is a set of edges $E' \subseteq E$ such that any disjoint edges $vw, xy \in E'$, w.l.o.g, $v \prec w, x \prec y$ and $v \prec x$, we have $w \prec y$.



Queue-Number

Definition: K-Queue Layout

Let $G = (V, E)$, consider a linear odering $\preceq$ of V, for an integer $k \geq 0$ a k-queue layout of G is a partition of E into $E_1, E_2, ..., E_k$ such that each E_i is a queue of G with respect to the ordering $\preceq$.

Queue-Number

Definition: K-Queue Layout

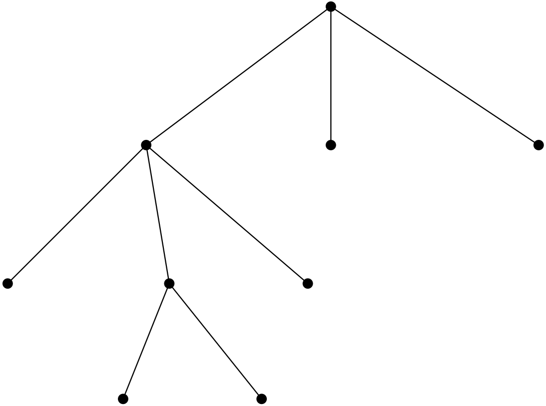
Let $G = (V, E)$, consider a linear odering $\preceq$ of V, for an integer $k \geq 0$ a k-queue layout of G is a partition of E into $E_1, E_2, ..., E_k$ such that each E_i is a queue of G with respect to the ordering $\preceq$.

Definition: Queue-Number

The queue-number of G, denoted by $\text{qn}(G)$, is the minimum integer k such that G has a k-queue layout for some ordering $\preceq$.

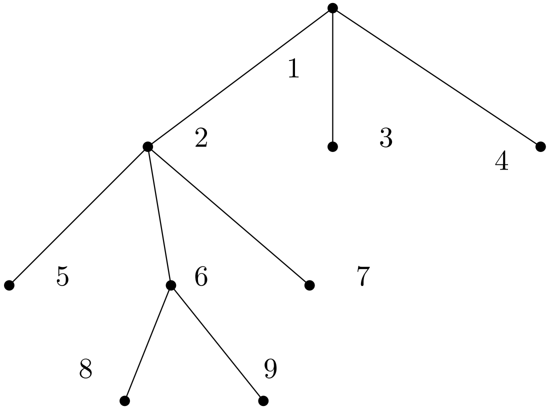
Queue-Number: Tree

What is the queue-number of a tree?



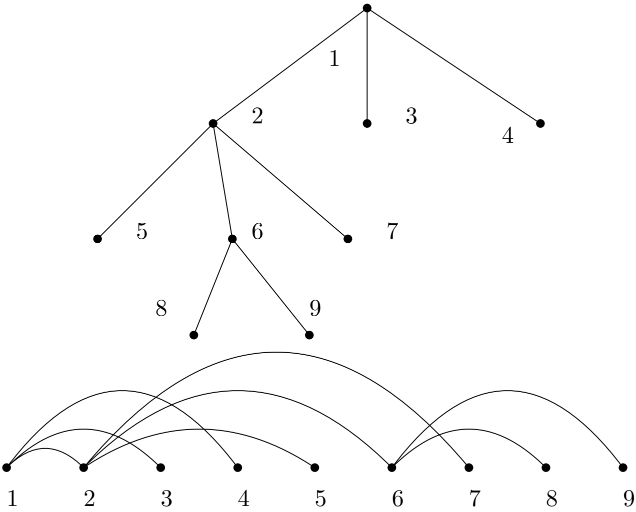
Queue-Number: Tree

What is the queue-number of a tree?

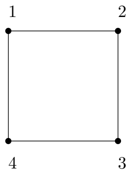


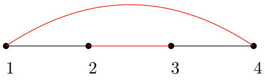
Queue-Number: Tree

What is the queue-number of a tree?

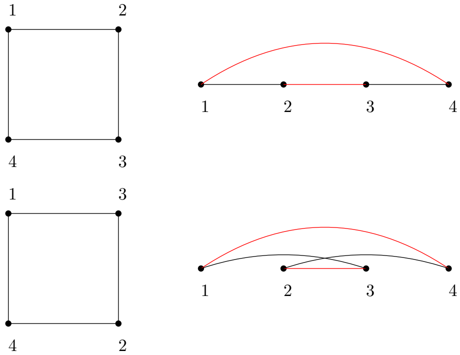


Queue Number: Cycle

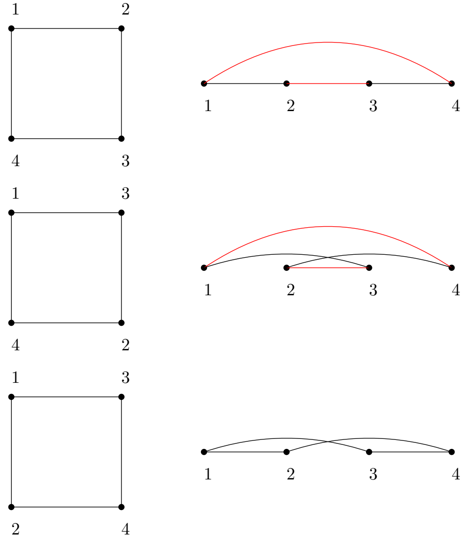




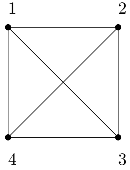
Queue Number: Cycle

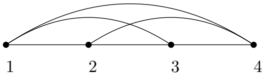


Queue Number: Cycle

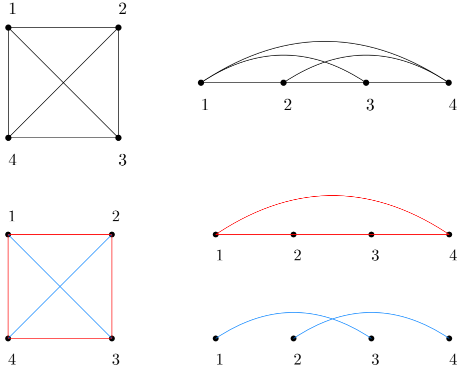


Queue Number: K_4

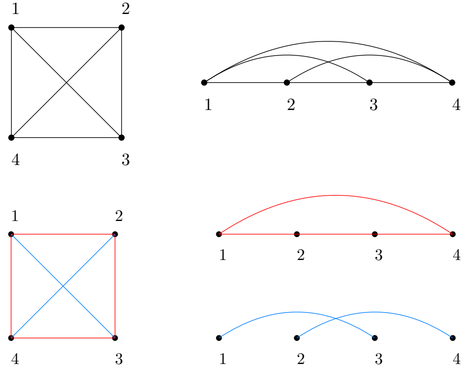




Queue Number: K_4



Queue Number: K_4



Theorem (Heath, Rosenberg, 1992)

The complete graph K_n has queue number $\lfloor \frac{n}{2} \rfloor$.

Treewidth

Do we have some tools to help bound the queue-number?

Treewidth

Do we have some tools to help bound the queue-number?

Theorem (Wiechert, 2017)

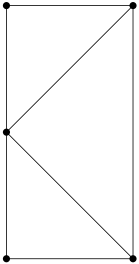
Every graph with treewidth k has queue-number at most $2^k - 1$.

Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.



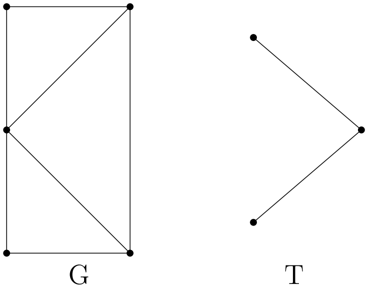
G

Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

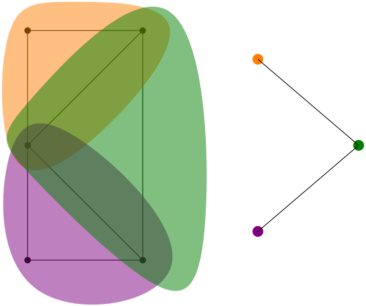


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

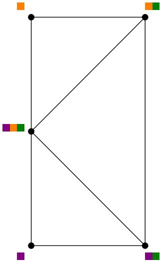


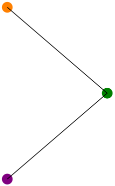
Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.



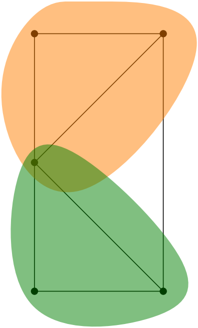


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

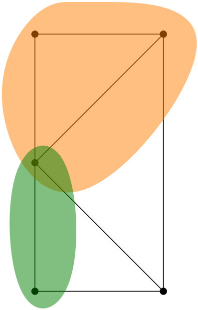


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

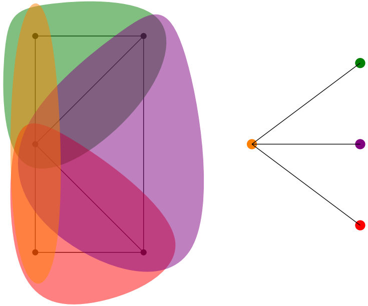


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

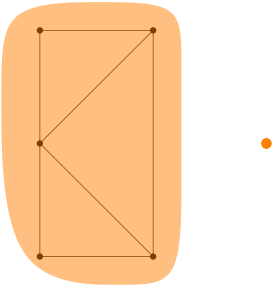


Tree-decomposition

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.



Treewidth

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

Definition: Width of Tree-decomposition

The width of a tree-decomposition of G is $\max_{x \in V(T)} |B_x| - 1$

Treewidth

Definition: Tree-decomposition

A tree-decomposition of a graph G is a pair (B, T). T is a tree and $B = \{B_x | x \in V(T)\}$ where each B_x is a subset of $V(G)$ for every vertex x in $V(T)$ such that

- $\forall \{v, w\} \in E(G)$, there exists $x \in V(T)$ with $v, w \in B_x$
- $\forall v \in V(G)$, the set $\{x | x \in V(T) \wedge v \in B_x\}$ induces a non -empty connected subtree of T.

Definition: Width of Tree-decomposition

The width of a tree-decomposition of G is $\max_{x \in V(T)} |B_x| - 1$

Definition: Treewidth

The treewidth of a graph G is the minimum width of all tree-decomposition of G.

Treewidth: Fixed-Parameter Tractability

- Computing a maximum independent set in a graph G is NP-hard.

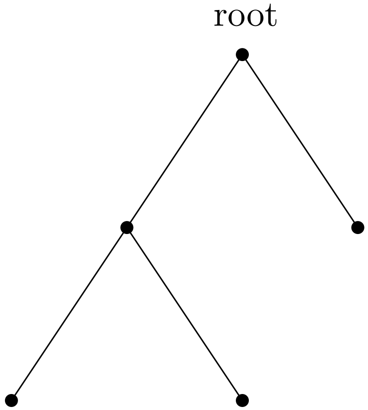
Treewidth: Fixed-Parameter Tractability

- Computing a maximum independent set in a graph G is NP-hard.
- If G has treewidth $\leq k$, then a maximum independent set in G can be computed in time $O(k^2 4^k |V|)$.

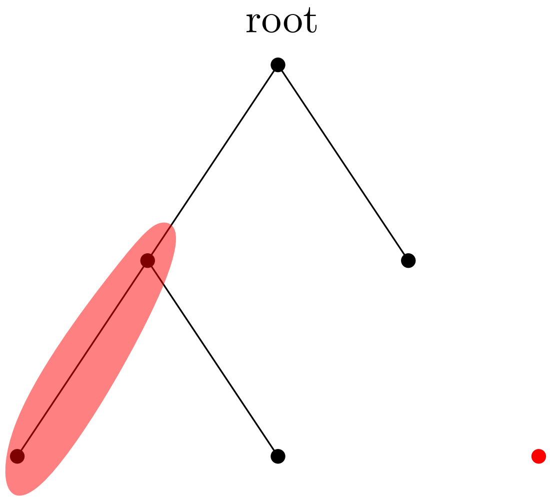
Treewidth: Fixed-Parameter Tractability

- Computing a maximum independent set in a graph G is NP-hard.
- If G has treewidth $\leq k$, then a maximum independent set in G can be computed in time $O(k^2 4^k |V|)$.
- Dynamic Programming on trees is relatively fast.

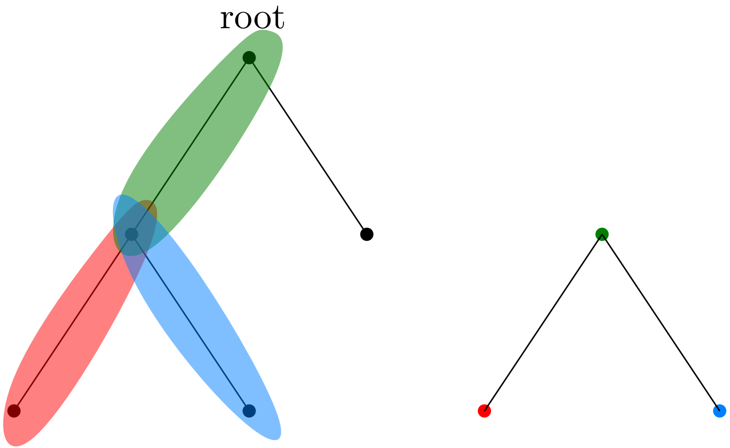
Treewidth: Tree



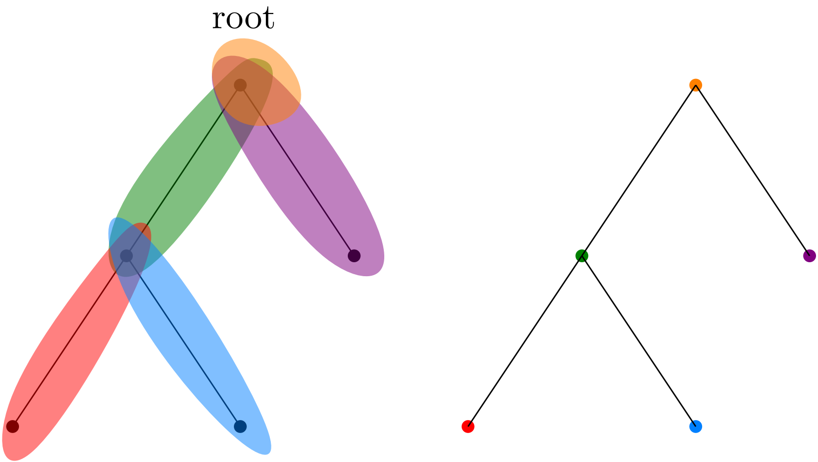
Treewidth: Tree



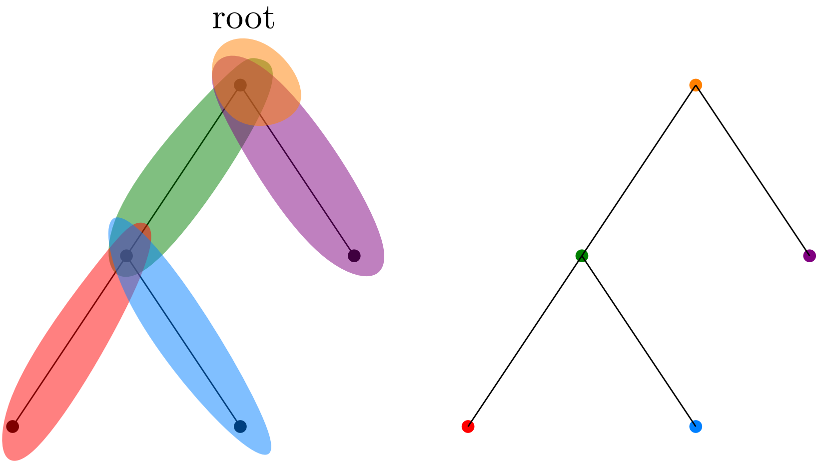
Treewidth: Tree



Treewidth: Tree



Treewidth: Tree



Tree has treewidth 1

Treewidth: Planar graph

Theorem (Wiechert, 2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Treewidth: Planar graph

Theorem (Wiechert, 2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Theorem (Alon, Seymour, Thomas, 1990)

Planar graph on n vertices has treewidth $O(\sqrt{n})$.

Treewidth: Planar graph

Theorem (Wiechert, 2017)

Every graph with treewidth k has queue-number at most $2^k - 1$.

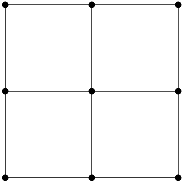
If planar graph has bounded treewidth, then planar graph has bounded queue-number.

Theorem (Alon, Seymour, Thomas, 1990)

Planar graph on n vertices has treewidth $O(\sqrt{n})$.

Theorem (Robertson, Seymour)

A grid graph with size $n \times n$ has treewidth n.



Partitions

Maybe we need more structure

Partitions

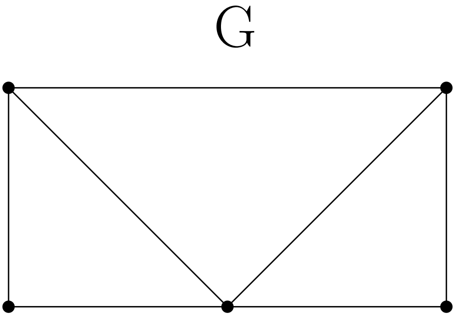
Maybe we need more structure

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

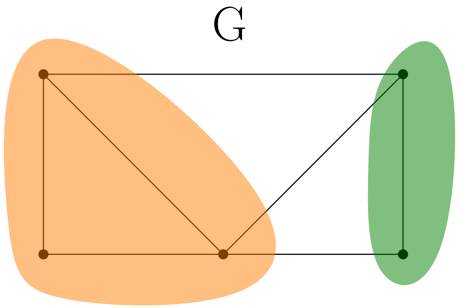
For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

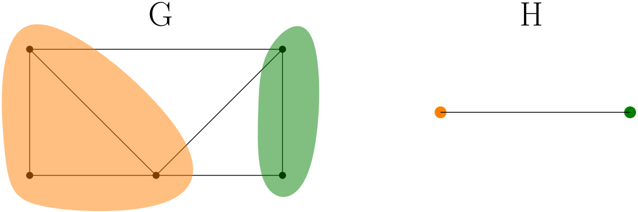
Partitions



Partitions



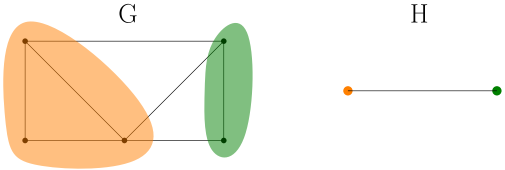
Partitions



Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.



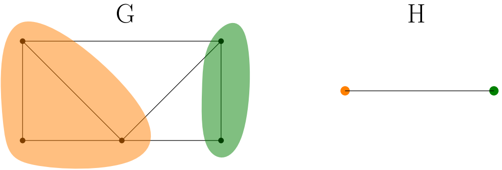
Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

An H-partition of a graph G is a pair (A, H), s.t. $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.



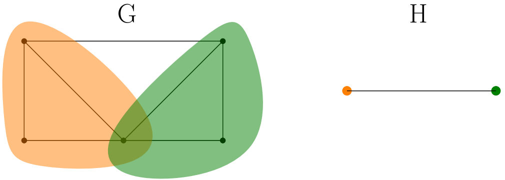
Partitions

Definition: Partition and Quotient

A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

An H-partition of a graph G is a pair (A, H), s.t. $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.



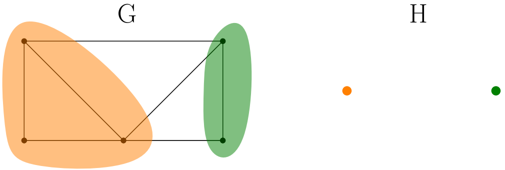
Partitions

Definition: Partition and Quotient

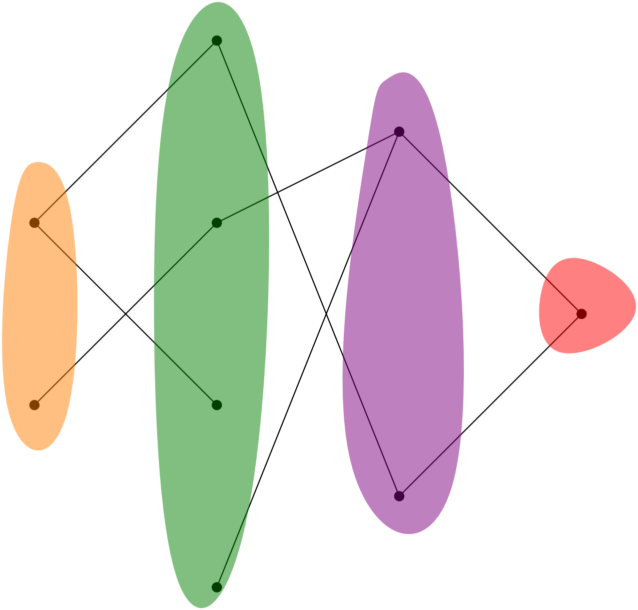
A partition of G is a set $\mathcal{P} = \{P_1, ..., P_n\}$ of non-empty subsets of $V(G)$ and each vertex of G is in exactly one element (part) of $\mathcal{P}$.
The quotient of $\mathcal{P}$ is a graph, denoted by $G/\mathcal{P}$, where each vertex v_i corresponds to P_i. For any two vertices v_i, v_j in $G/\mathcal{P}$, they are connected if and only if some vertex in P_i is connected to some vertex in P_j in graph G.

Definition: H-partition

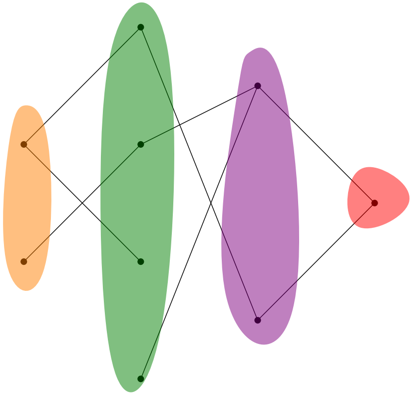
An H-partition of a graph G is a pair (A, H), s.t. $A = \{A_x | x \in V(H)\}$ is a partition of $V(G)$ and H is a graph isomorphic to the quotient G/A.

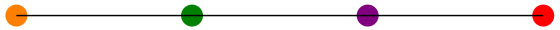


Layering



Layering





Layered width

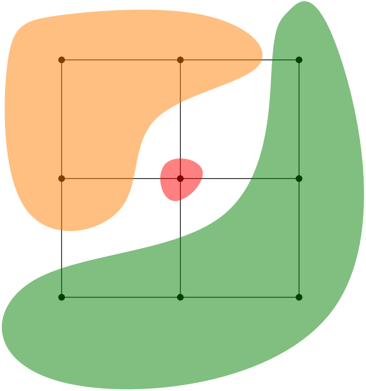
Definition: Layered width

The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, s.t. each element in $\mathcal{P}$ has at most ℓ vertices in each element of the path-partition.

Layered width

Definition: Layered width

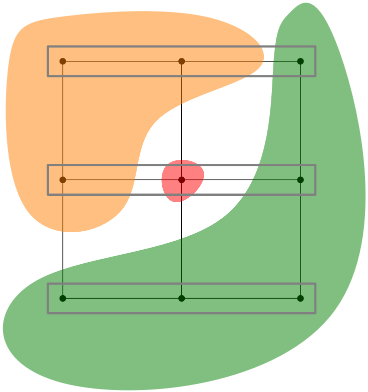
The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, s.t. each element in $\mathcal{P}$ has at most ℓ vertices in each element of the path-partition.



Layered width

Definition: Layered width

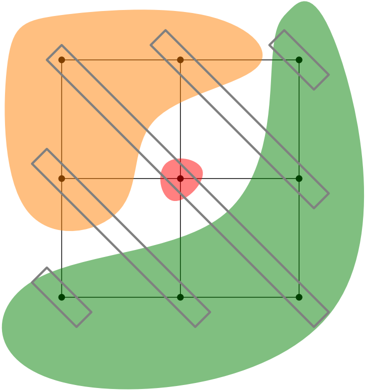
The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, s.t. each element in $\mathcal{P}$ has at most ℓ vertices in each element of the path-partition.



Layered width

Definition: Layered width

The layered width of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that there exists a path-partition (layering) of G, s.t. each element in $\mathcal{P}$ has at most ℓ vertices in each element of the path-partition.



Layered width: Tree

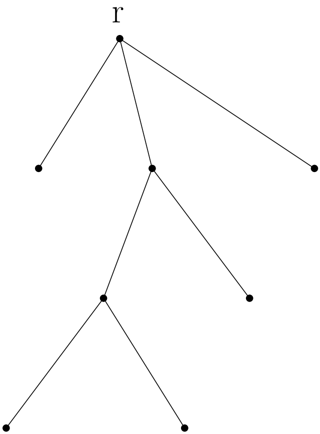
Definition: Vertical Path

Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.

Layered width: Tree

Definition: Vertical Path

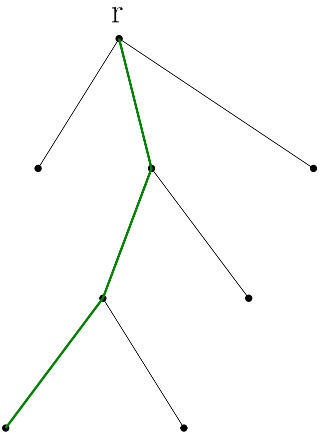
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

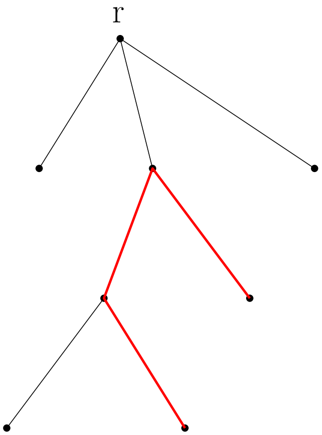
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

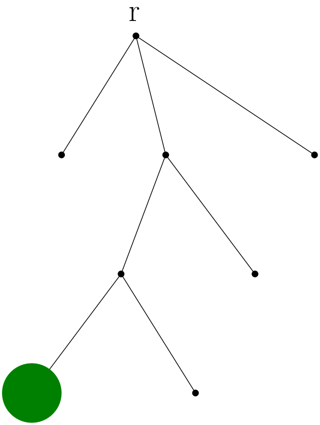
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

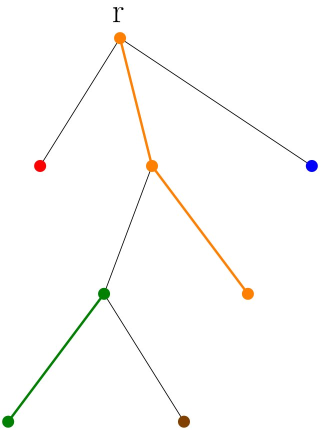
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

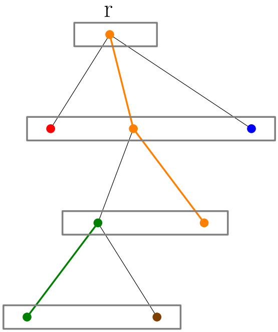
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

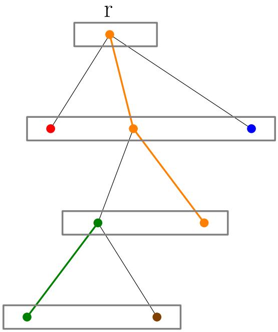
Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\mathrm{dist}_T(x_i, r) = d + i$.



Layered width: Tree

Definition: Vertical Path

Let T be a tree rooted at a vertex r, a non-empty path $(x_1, ..., x_p)$ in T is vertical if for some $d \geq 0$ and for all $1 \leq i \leq p$ we have $\text{dist}_T(x_i, r) = d + i$.



A partition of layered width 1 where each part is a vertical path.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Planar Graph Decomposition

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has queue-number at most

$$3(2^8 - 1) + \left\lfloor \frac{3}{2} \right\rfloor = 766$$

Planar Graph Decomposition

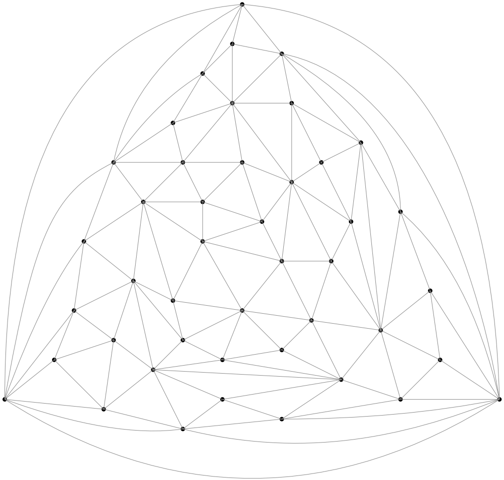
Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

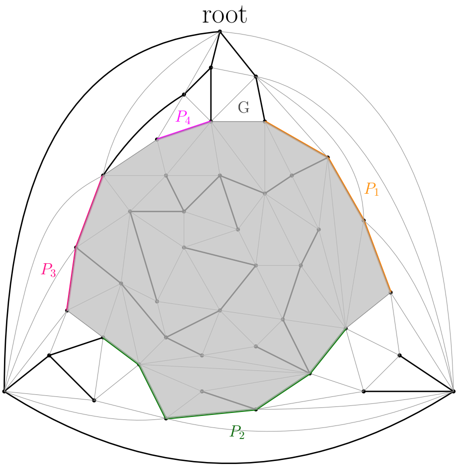
(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.



Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

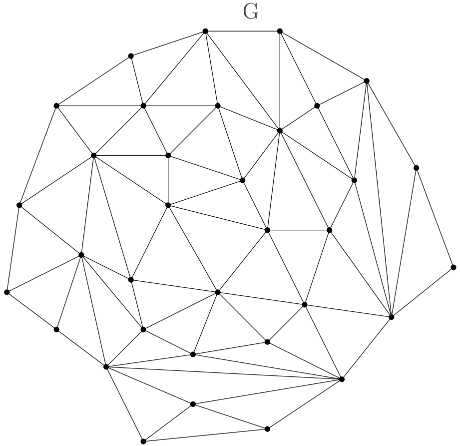
(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.



Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

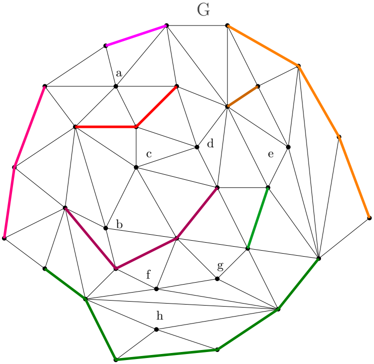
(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.

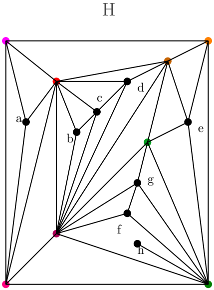


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.

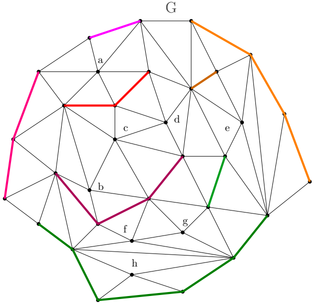


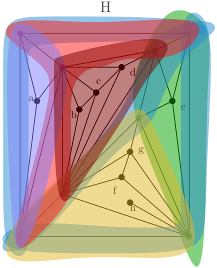


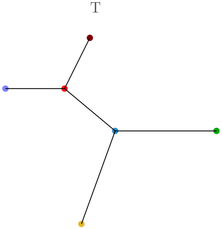
Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

(Simplified) Let G^+ be a maximal planar graph, we can find a subgraph G of G^+, then G has a desired partition.







Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

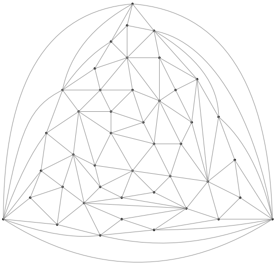
- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, \ldots P_k]$ and $1 \le k \le 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, \ldots, P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \le 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, \ldots, P_k$ in H is in B_x

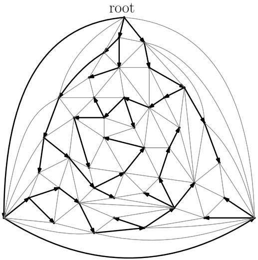


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, \ldots P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, \ldots, P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, \ldots, P_k$ in H is in B_x

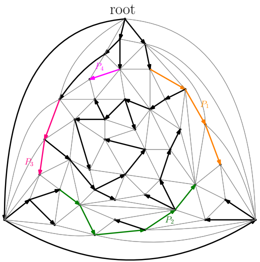


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

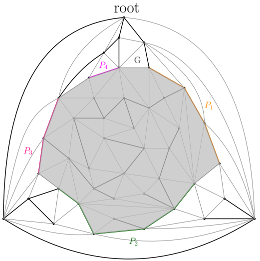


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

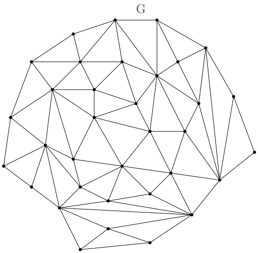


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

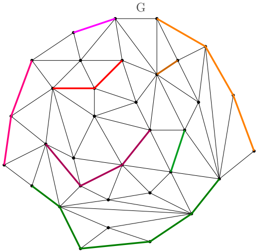


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

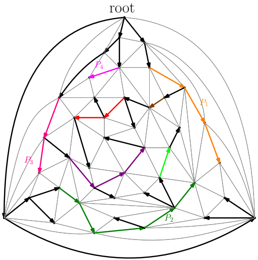


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, \ldots P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, \ldots, P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, \ldots, P_k$ in H is in B_x

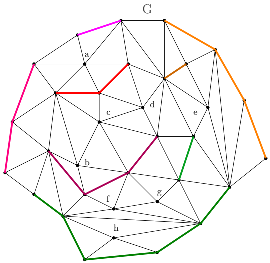


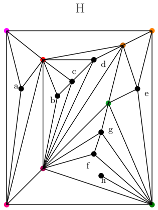
Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x



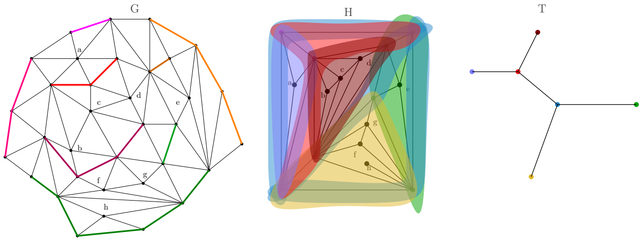


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

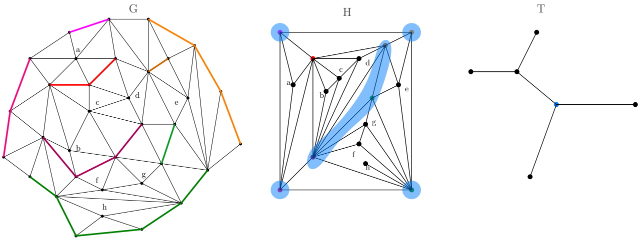


Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x



Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Note that three vertices on the outer cycle can always be three vertical paths of T.

Planar Graph Decomposition

Lemma (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Let G^+ be a maximal planar graph, let T be a spanning tree of G^+ rooted at vertex *root* on outerface of G^+. For any cycle F in G^+, which can be partitioned into at most 6 pairwise disjoint vertical paths of T, with $F = [P_1, ...P_k]$ and $1 \leq k \leq 6$. Let G be the internally triangulated subgraph of G^+ which consists of all edges and vertices of G^+ contained in F and the interior of F, then G has a partition $\mathcal{P}$ into vertical paths of T, and $P_1, ..., P_k \in \mathcal{P}$, and the quotient graph $H = G/\mathcal{P}$ has a tree-decomposition (B, T) that

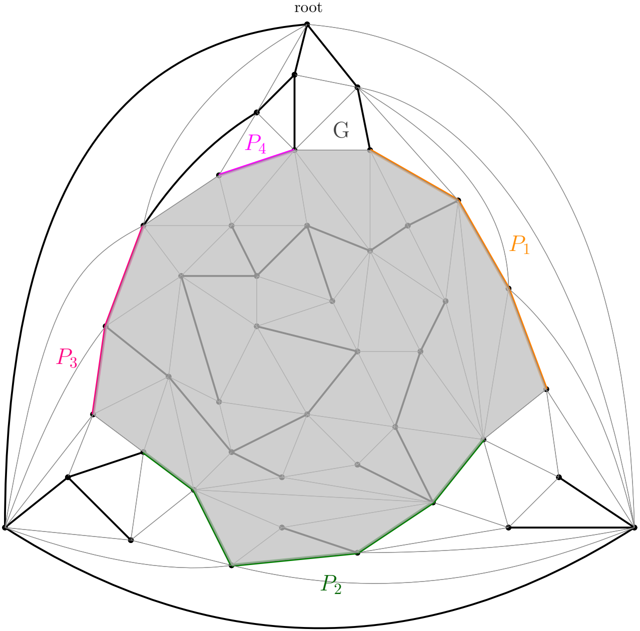
- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Note that three vertices on the outer cycle can always be three vertical paths of T.

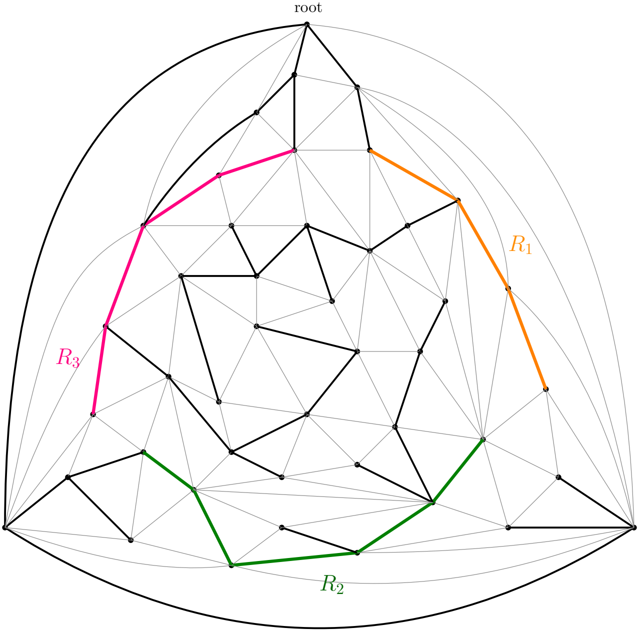
Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

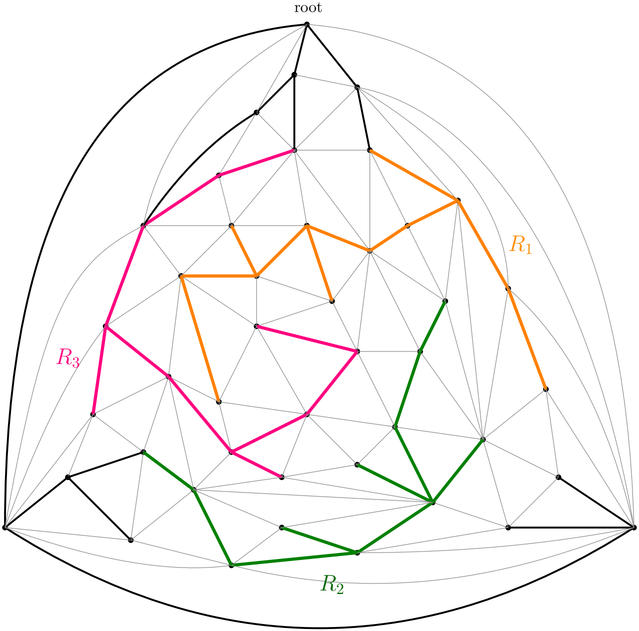
Planar Graph Decomposition: Induction



Planar Graph Decomposition: Induction



Planar Graph Decomposition: Induction



Planar Graph Decomposition: Induction

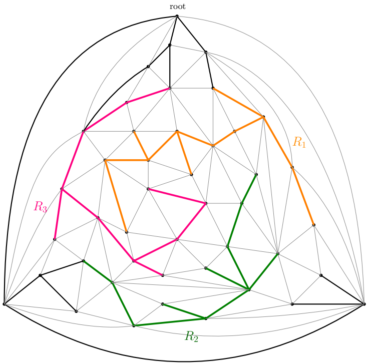
Lemma (Sperner's Lemma)

Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.

Planar Graph Decomposition: Induction

Lemma (Sperner's Lemma)

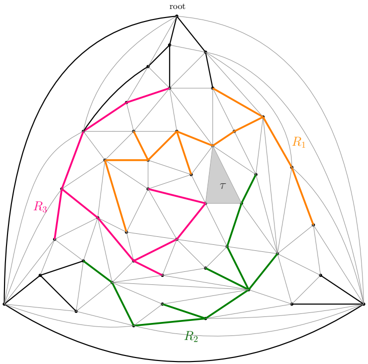
Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.



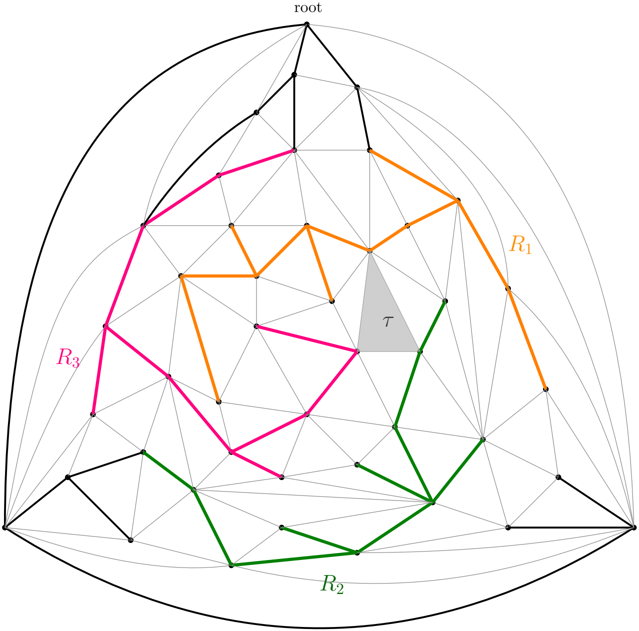
Planar Graph Decomposition: Induction

Lemma (Sperner's Lemma)

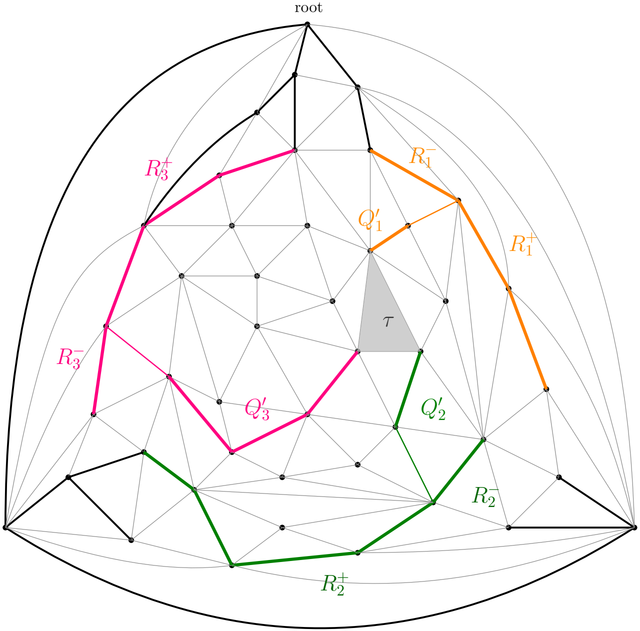
Let G be an internally triangulated graph whose vertices are colored 1,2,3 with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is colored i. Then G contains an internal face whose vertices are colored 1,2,3.



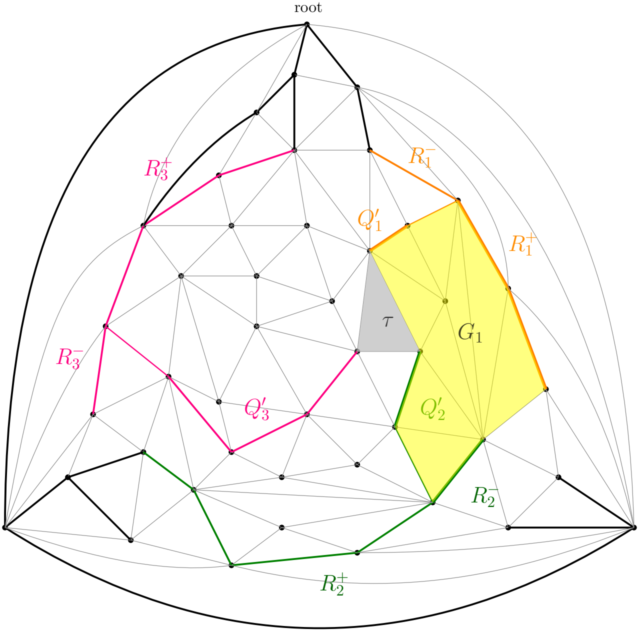
Planar Graph Decomposition: Induction



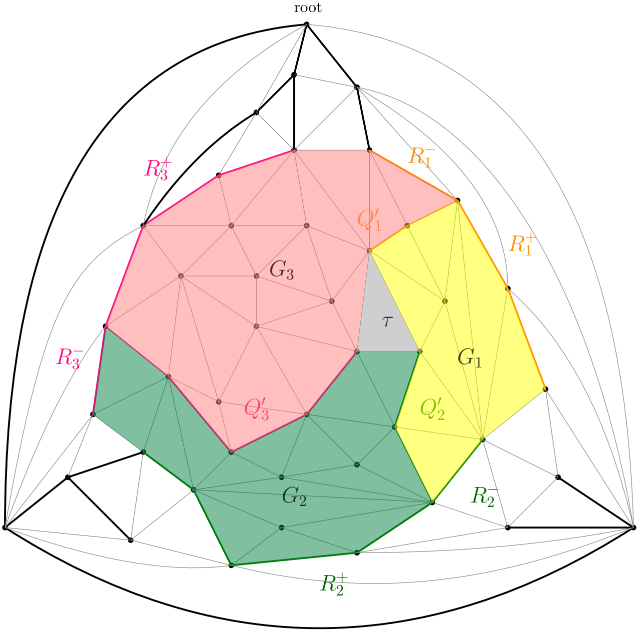
Planar Graph Decomposition: Induction



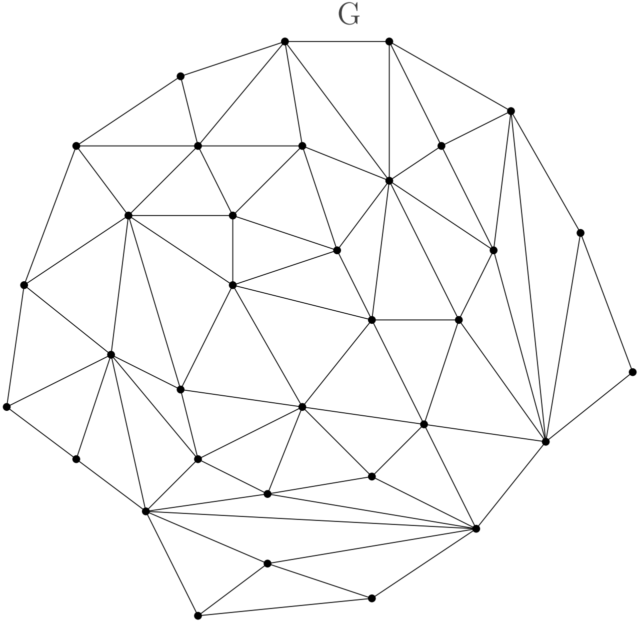
Planar Graph Decomposition: Induction



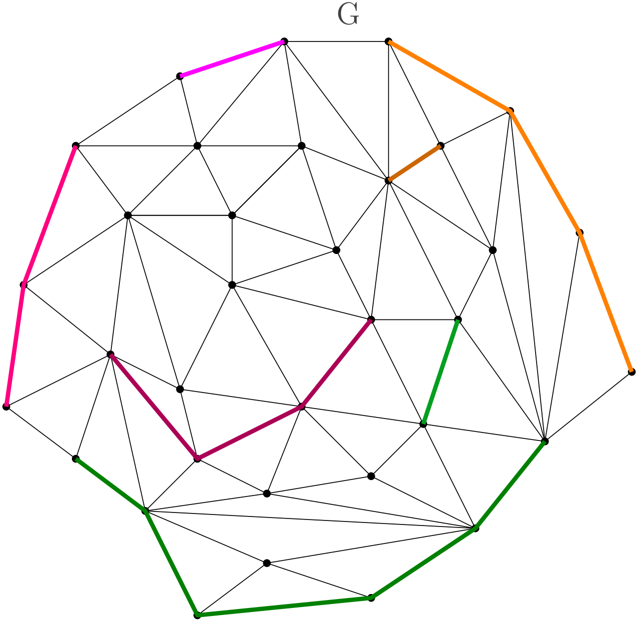
Planar Graph Decomposition: Induction



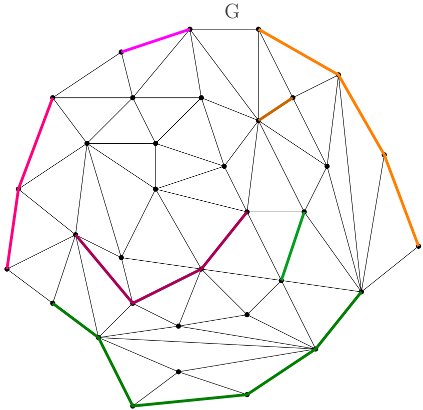
Planar Graph Decomposition: Construction

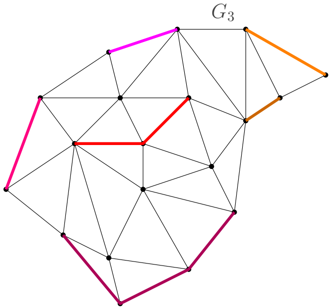


Planar Graph Decomposition: Construction

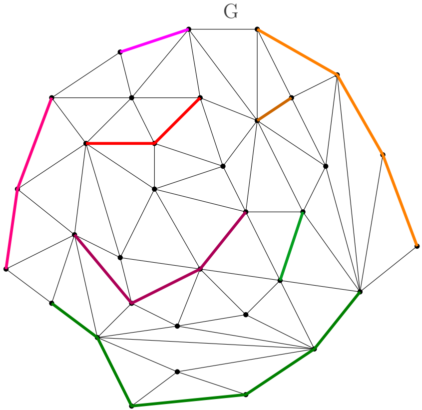


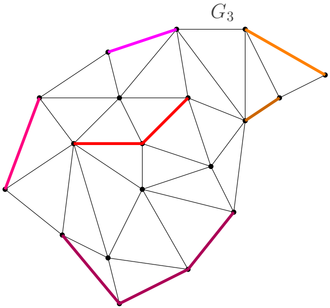
Planar Graph Decomposition: Construction



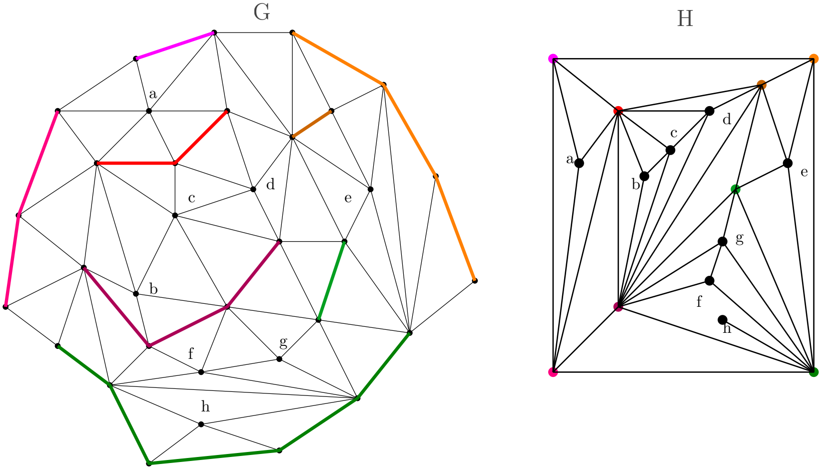


Planar Graph Decomposition: Construction

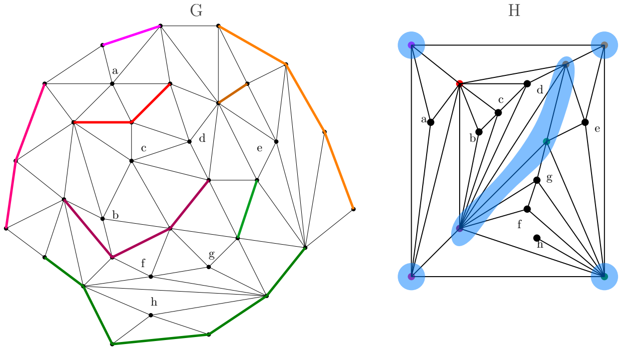




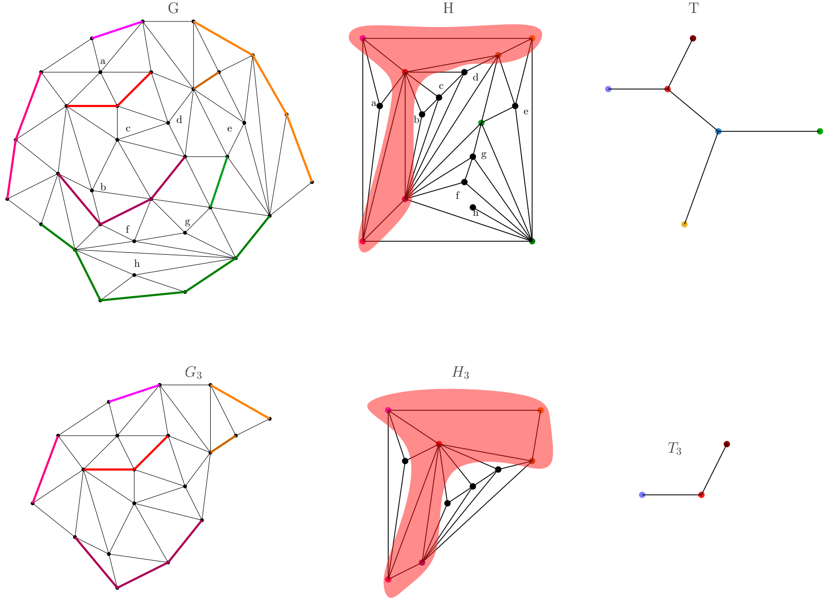
Planar Graph Decomposition: Construction



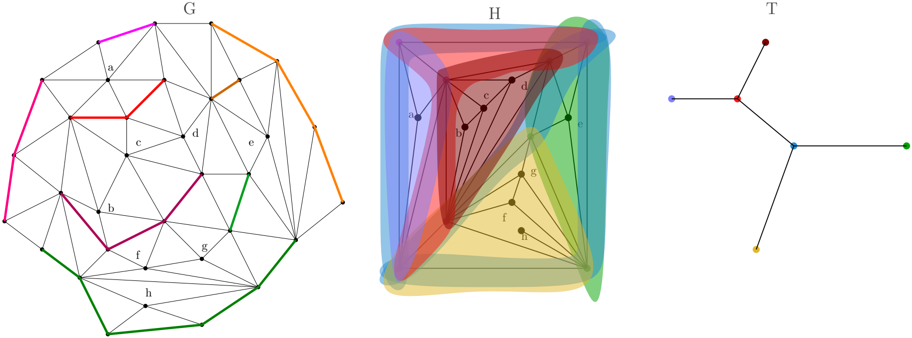
Planar Graph Decomposition: Construction



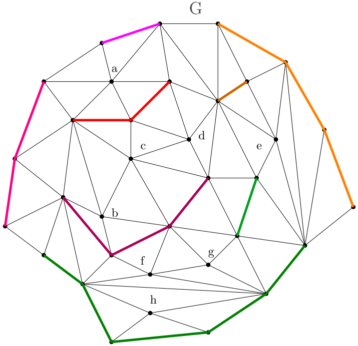
Planar Graph Decomposition: Construction

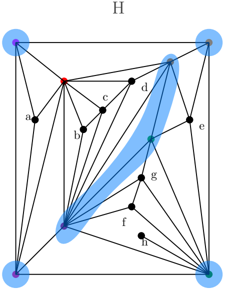


Planar Graph Decomposition: Construction



Planar Graph Decomposition: Correctness

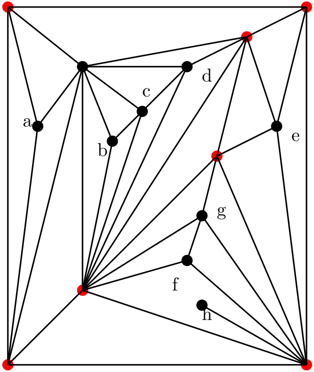




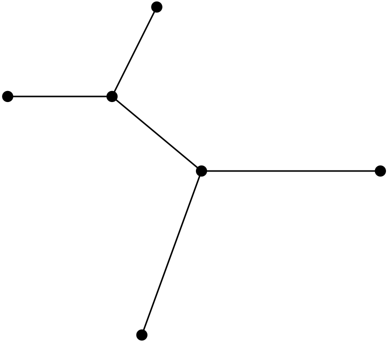
- $|B_x| \leq 9$ for any $x \in V(T)$
- $\exists x \in V(T)$, all vertices correspond to $P_1, ..., P_k$ in H is in B_x

Planar Graph Decomposition: Correctness

H

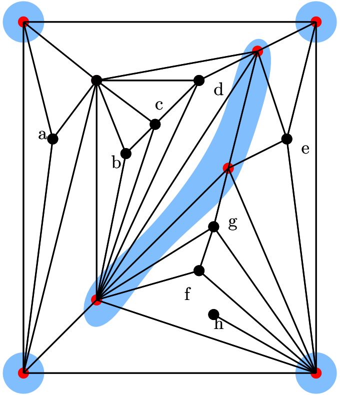


T

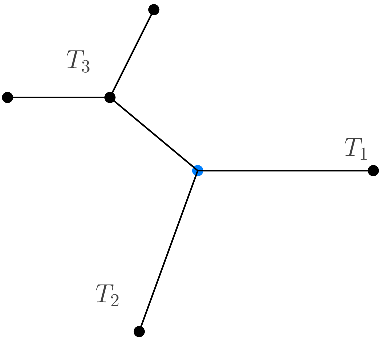


Planar Graph Decomposition: Correctness

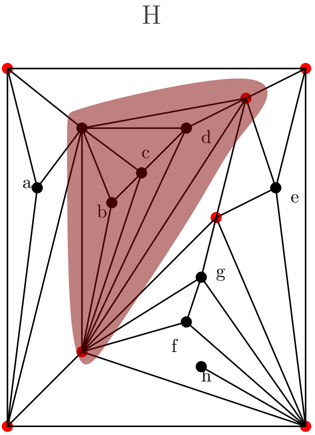
H

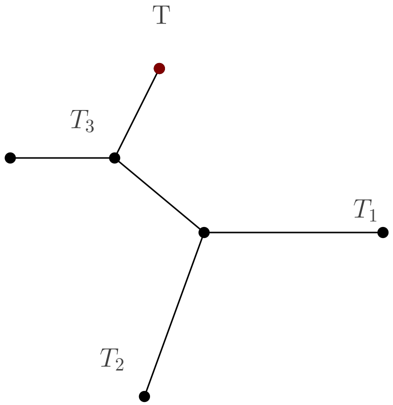


T

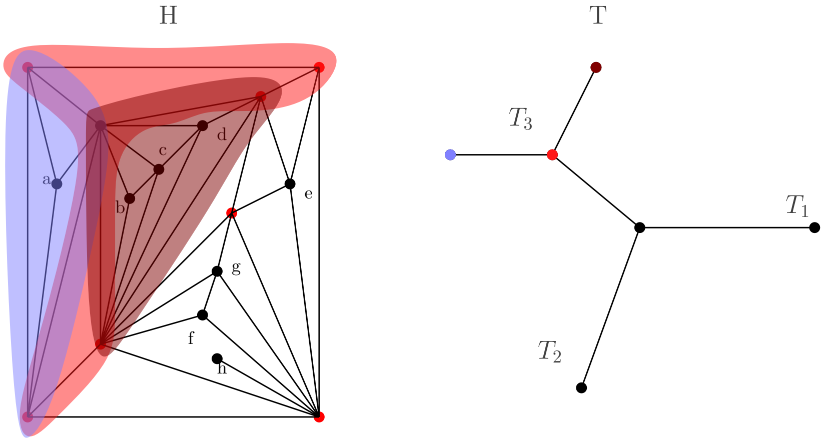


Planar Graph Decomposition: Correctness

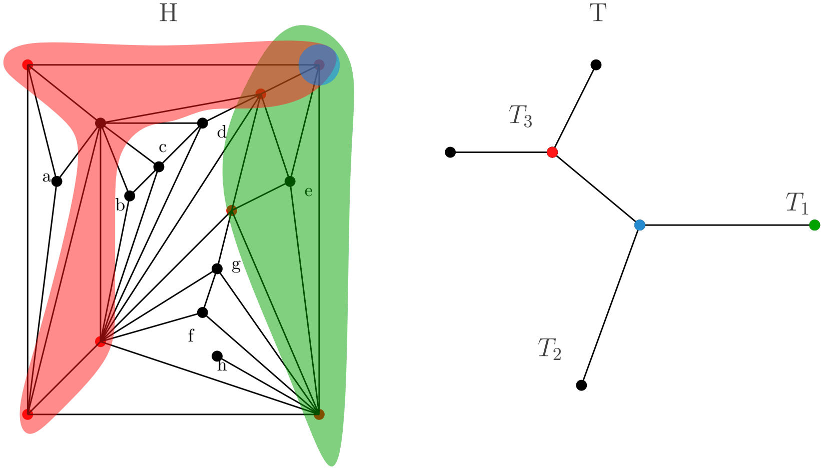




Planar Graph Decomposition: Correctness



Planar Graph Decomposition: Correctness



Wrap-Up

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Wrap-Up

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Wrap-Up

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $H = G/\mathcal{P}$ has treewidth at most 8.

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

For a graph G, if G has an H-partition of layered width ℓ and H has treewidth k, then

$$qn(G) \leq 3\ell(2^k - 1) + \left\lfloor \frac{3}{2}\ell \right\rfloor$$

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood, 2020)

Every planar graph G has queue-number at most

$$3(2^8 - 1) + \left\lfloor \frac{3}{2} \right\rfloor = 766$$

Open Problems

- Maximum queue-number of planar graphs
  - The constant bound can be lower to 48 with similar decomposition.
  - With method in such a direction, that is the optimum result.

Open Problems

- Maximum queue-number of planar graphs
  - The constant bound can be lower to 48 with similar decomposition.
  - With method in such a direction, that is the optimum result.
- Largest class of graphs with bounded queue-number
  - The decomposition method also applies to the proper minor-closed class of graphs.
  - Every proper minor-closed class of graphs has bounded queue-number.

Open Problems

- Maximum queue-number of planar graphs
  - The constant bound can be lower to 48 with similar decomposition.
  - With method in such a direction, that is the optimum result.
- Largest class of graphs with bounded queue-number
  - The decomposition method also applies to the proper minor-closed class of graphs.
  - Every proper minor-closed class of graphs has bounded queue-number.
- Relation between treewidth and queue-number
  - For graph with treewidth k
  - Best known lower bound for queue-number is $k + 1$.
  - Best known upper bound for queue-number is $2^k - 1$.
- ...

Portal