Tile Sets Generated by Hadamard Submatrices of Fourier Matrices

Troy Wiegand
with faculty mentor Dr. John Herr

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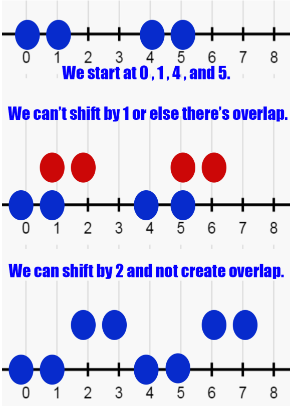
T can be decomposed into a finite set of shifts B and a period N_A such that,

$$T = B \oplus N_A\mathbb{Z}.$$

Tiling Example



Let's try to find T for $A = \{0, 1, 4, 5\}$

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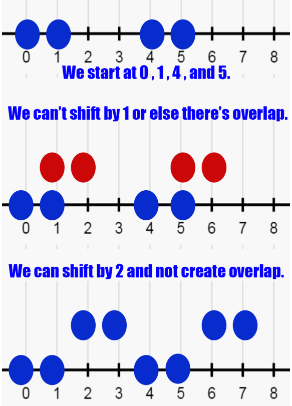

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Let start with B.

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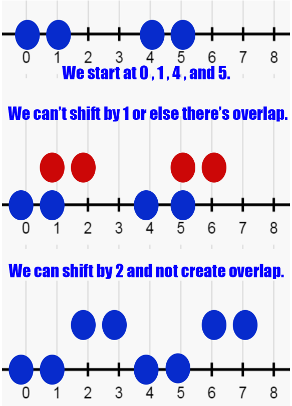

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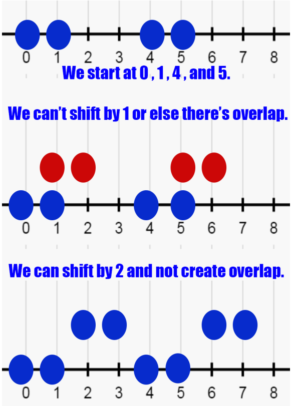

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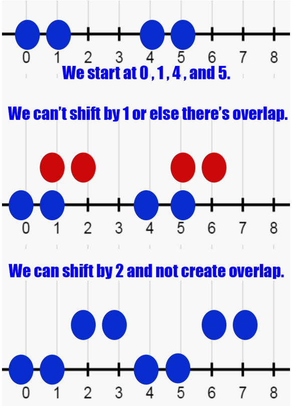

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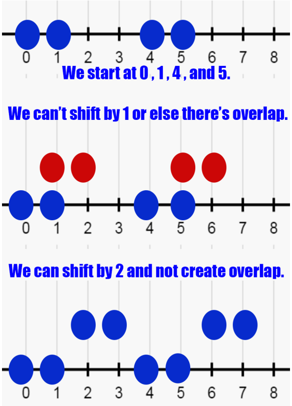

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$$N_A = 8$$

Therefore, $T = \{0, 2\} \oplus 8\mathbb{Z}$.

Notation for the CM Properties

In 1998 Ethan M. Coven and Aaron Meyerowitz discovered sufficient conditions for a set to tile.

Here is some more notation that will be necessary to understand the CM Properties:

- $A(x)$ is a polynomial of the form $\sum_{a \in A} x^a$ and $A(1) = |A|$
- S_A is the set of prime powers s such that sth cyclotomic polynomial $\Phi_s(x)$ divides A(x)

CM Properties

Theorem

Given

T1 $A(1)$ *needs to equal* $\prod_{s \in S_A} \Phi_s(1)$

T2 *If* $s_1, ..., s_m \in S_A$ *are powers of distinct primes, then* $\Phi_{s_1 \cdots s_m}$*divides* $A(x)$

If $|A|$ *has at most two prime factors, then* $A(x)$ *satisfies the two CM Properties (T1 and T2) if and only if* A *tiles the integers.*

CM Example

Example:

For $A = \{0, 1, 4, 5\}$, $A(x) = x^5 + x^4 + x + 1$ and $S_A = \{2, 8\}$

We need to show the two properties in order to show that this set tiles:

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T1 $A(1) = (1)^5 + (1)^4 + (1) + 1 = 4$ and $\prod_{s \in S_A} \Phi_s(1) = (x+1)(x^4+1)|_{x=1} = 4$. **T1** is true.

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T2 The numbers 2 and 8 are not powers of distinct primes. Thus, **T2** is true.

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T2 The numbers 2 and 8 are not powers of distinct primes. Thus, **T2** is true.

Both of our properties are met, therefore this set $A = \{0, 1, 4, 5\}$ tiles.

$\blacksquare$

Determining Period and Shifting Set of A

Coven and Meyerowitz also found that the period and shifting set of A can be determined through use of $A(x)$ and S_A.

$$N_A := lcm(S_A)$$

$$B(x) := \prod_{s \in S_B} \Phi_s(x^{t(s)}),$$

where S_B range over the prime power factors of $N_A \notin S_A$ and $t(s)$ is the largest factor of N_A relatively prime to s.

Period and Shifting Set Example

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$$B(x) = \Phi_4(x^1) = 1 + x^2 \quad \text{and } B = \{0, 2\}.$$

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$S_B = \{4\}$ and $t(4) =$ largest factor of N_A relatively prime to 4

$$B(x) = \Phi_4(x^1) = 1 + x^2 \text{ and } B = \{0, 2\}.$$

Therefore, $T = \{0, 2\} \oplus 8\mathbb{Z}$.

What is a Hadamard Matrix?

A Hadamard Matrix H is a N by N square matrix whose entries all have a complex modulus of one and

$$H^*H = HH^* = N \cdot I_N,$$

where $*$ is the adjoint and I_N is the N by N Identity Matrix.

What is a Fourier Matrix?

Fourier Matrices are a subclass of Hadamard Matrices:

All entries of an M by M Fourier Matrix have the form:

$$f_{jk} = e^{2i\pi \frac{(j)(k)}{M}},$$

where j is the jth row and k is the kth column.

What Am I Doing?

Let's look at $\mathcal{F}_4$.

$$\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

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Let's only look at the 0th and 2th columns.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What Am I Doing?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let's remove two rows $\{2, 3\}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What Am I Doing?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let's remove two rows $\{2, 3\}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathcal{F}_2$$

What Am I Doing?

```
0    1
0    2
0    0
0    2
0    1
0    0
0    2
0    3
0    0
0    3
0    2
0    0
```

What Am I Doing?

```
0 1
0 2
0 0
0 2
0 1
0 0
0 2
0 3
0 0
0 3
0 2
0 0
```

```
0 2  has these rows that are the same:

0 1 CM Met   T=2Z

0 3 CM Met   T=2Z

```

Why Am I Doing It?

Theorem (Universal Tiling Conjecture (Dutkay and Jorgensen, 2013))

Let $p \in \mathbb{N}$. Let $\Gamma := \lambda_0 = 0, \lambda_1, \lambda_{p-1}$ be a subset of $\mathbb{R}$ with p elements. Assume Γ has a spectrum of the form $\frac{1}{p}A$ with $A \subset \mathbb{Z}$. Then for every finite family $A_1, A_2, ..., A_n$ of subsets of $\mathbb{Z}$ such that $\frac{1}{p}A_i$ is a spectrum for Γ for all i there exists a common tiling subset T of $\mathbb{Z}$ such that the set A_i tiles $\mathbb{Z}$ by T for all $i \in \{1, ..., n\}$.

Results

- All of the Hadamard submatrices from $\mathcal{F}_2$ up to $\mathcal{F}_{18}$ with common columns sets share a common tiling set T.
- The Hadamard submatrices of size 2, 4, and 5 from $\mathcal{F}_{20}$ with common columns sets share a common tiling set T.
- However, there can be multiple tiling sets amongst the same size of Hadamard submatrices generated from the same Fourier Matrix.

Further Study

We have developed multiple goals to help guide this project further:

- Write the Hadamard Submatrix Generator in a less interpreted language to be able to gather more submatrices to analyze
- Increase the capacity of the Tiling Check Program to handle larger submatrices when they are produced.

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Any Questions?