Business 34902
Lecture 6

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No-arbitrage pricing of bonds

- Log price at t of a default-free bond maturing at $t+N$ with payoff of one,

$$p_{n,t} = \log \mathbb{E}_t \prod_{s=1}^{n} M_{t+s}$$

  and,

$$y_{n,t} = -\frac{1}{n} p_{n,t}$$
$$r_{n,t+1} = p_{n-1,t+1} - p_{n,t}$$
$$f_{n,t} = p_{n,t} - p_{n+1,t}$$

- Modeling default-free bond prices is all about modeling the SDF, nothing else.

Real and nominal SDF

- Typical equilibrium models (consumption-based), are models of the real SDF.
- Here we focus (mostly) on the nominal term structure of bond yields and the nominal SDF
- A bond with a fixed nominal payoff would be cash-flow risky in real terms: real payoff

$$\left(\prod_{s=1}^{n} \Pi_{t+s}\right)^{-1},$$

where Π is the inflation rate

Real and nominal SDF

- Valuation with real SDF M would then mean valuing risky real cash flows

$$P_{n,t} = \mathbb{E}_t\left[\left(\prod_{s=1}^{n} M_{t+s}\right)\left(\prod_{s=1}^{n} \Pi_{t+s}\right)^{-1}\right]$$

- Another way to read this:

$$\left(\prod_{s=1}^{n} M_{t+s}\right)\left(\prod_{s=1}^{n} \Pi_{t+s}\right)^{-1} = \prod_{s=1}^{n}\left(\frac{M_{t+s}}{\Pi_{t+s}}\right)$$

is the nominal SDF, which discounts a riskless nominal payoff of $1.

Extraction of zero-coupon yields

- U.S. Treasury bonds and many other government bonds around the world are coupon bonds.
- Pricing models we analyze below are typically formulated for zero-coupon bonds $\Rightarrow$ we must first extract zero-coupon bond yields from the empirically observed coupon bond yields.
- Fama-Bliss method (works well up to 5yr maturity): Example
  - Extract daily forward rates over 6 and 12-month horizon from (zero-coupon) T-bills with these maturities (constant between maturity dates)
  - Now suppose there is a Treasury Note available that matures in $t+17$ months paying coupons in months $t+5$, $t+11$ and $t+17$
    - Value $t+5$ and $t+11$ coupons with the daily forward rates extracted from the T-bills
    - Use the Notes' coupon and principal payment at $t+17$ to extract the daily forward rate for the period between $t+12$ and $t+17$ (to match market price of Treasury note)
  - Carry forward with the next available maturity note or bond

Common Factors in Yields

- As with equity factor models, a covariance matrix decomposition will be useful for bonds, too.
- For equities, we applied SDF to price (excess) returns, not asset cash flows
  - Cash flow uncertainty and many-period horizons would make pricing cash flows difficult
- For default-free bonds, there is no cash flow risk: We can normalize to payoff of \$1 and work directly with prices (or transformations of prices like yields, forward rates)
- Since stochastic shocks to bond prices come only from the SDF, bond prices are much more informative about (bond market) SDF than data on equity returns is about (equity market) SDF.
- Factors that drive SDF should show up in bond prices, yields, and forward rates.

Common Factors in Yields

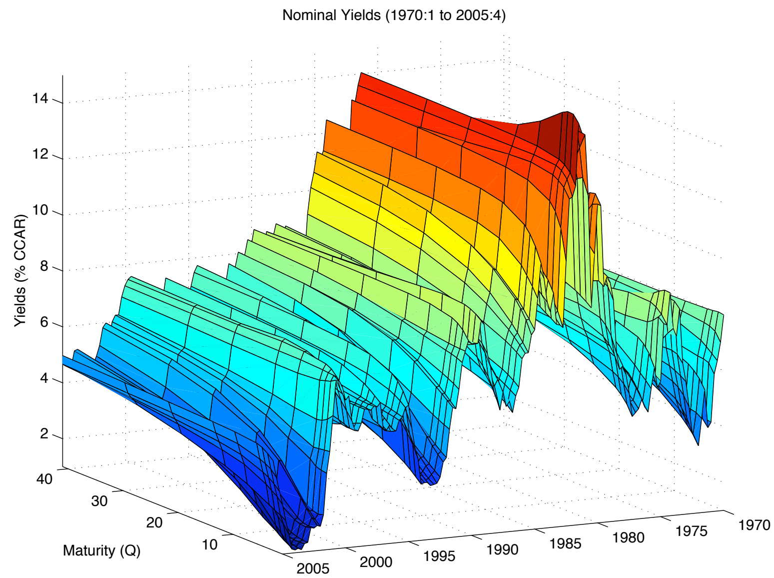


Figure 1: **Time series properties of the yield curve, 1970:1 to 2005:4.**

Common factors in yields

- Log yield changes $\Delta \boldsymbol{y}_t$ with covariance matrix $\boldsymbol{\Sigma}$
- Eigendecomposition of $\boldsymbol{\Sigma}$

$$\boldsymbol{\Sigma} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$$

  where $\boldsymbol{\Lambda}$ is a diagonal matrix with eigenvalues of $\boldsymbol{\Sigma}$ on the diagonal, sorted by magnitude, and $\boldsymbol{Q}$ is an orthogonal matrix of eigenvectors.

- Constructing

$$\boldsymbol{f}_t = \boldsymbol{Q}'\Delta \boldsymbol{y}_t$$

  yields the principal component (PC) factors.

- Instead of changes yields, we could use changes forward rates, or levels of these variables and results would be very similar.

Common factors in yields

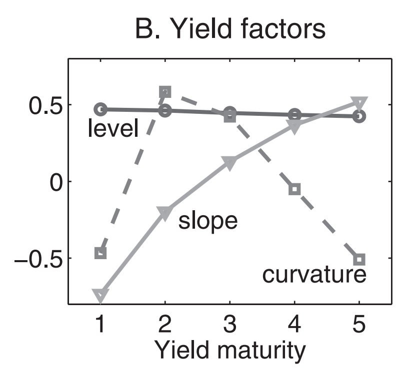
Table 2 ▪ Implied Zeroes: Relative Importance of Factors ▪ (Percent)

Maturity	Total Variance Explained	Proportion of Total Explained Variance Accounted for by: Factor 1	Factor 2	Factor 3
6 months	99.5	79.5	17.2	3.3
1 year	99.4	89.7	10.1	0.2
2 years	98.2	93.4	2.4	4.2
5 years	98.8	98.2	1.1	0.7
8 years	98.7	95.4	4.6	0.0
10 years	98.8	92.9	6.9	0.2
14 years	98.4	86.2	11.5	2.2
18 years	95.3	80.5	14.3	5.2
Average	**98.4**	**89.5**	**8.5**	**2.0**

from Litterman and Scheinkman (1991)

Common factors in yields

- The three factors are commonly labeled **level**, **slope**, and **curvature**. The labels reflect their weights on yields on forward rates – or yields – of different maturities



from Cochrane and Piazzesi (2005)

A simple term-structure model

- Example: one-factor homoskedastic constant price of risk affine model
- Consider a state variable, x_t, following a univariate AR(1),

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$

with $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$, and a log SDF,

$$m_{t+1} = -\delta_0 - \delta_1 x_t - \frac{1}{2}\sigma^2\lambda^2 - \lambda\sigma\varepsilon_{t+1}$$

  - λ is the "price of risk" that controls the sensitivity of the SDF to ε_{t+1} shocks and thus the volatility of the SDF
  - Note that $M_{t+1} = \exp(m_{t+1}) > 0$ (no arbitrage)
  - Restriction to small number of factors (one in this case) is the key (and ad-hoc) assumption that drives all empirical predictions

A simple term-structure model

- We then get the log price of a one-period zero-coupon bond,

$$\begin{aligned} p_{1,t} &= \log \mathbb{E}_t\left[M_{t+1}\right] \\ &= -x_t - \frac{1}{2}\lambda^2\sigma^2 + \frac{1}{2}\lambda^2\sigma^2 \\ &= -\delta_0 - \delta_1 x_t, \end{aligned}$$

- Hence, the log short-term interest rate $r_t = -p_{1,t} = \delta_0 + \delta_1 x_t$ reveals variation in the state variable.
- We can evaluate

$$p_{n,t} = \log \mathbb{E}_t\left[M_{t+1} P_{n-1,t+1}\right]$$

recursively to find prices of bonds at all maturities.

A simple term-structure model

- The recursion takes the form

$$p_{n,t} = a_n + b_n x_t$$

where

$$\begin{aligned} b_n &= -\delta_1 + b_{n-1}\phi^*, \\ a_n &= -\delta_0 + a_{n-1} + b_{n-1}\mu^* + \frac{1}{2}b_{n-1}^2\sigma^2, \\ \mu^* &= \mu - \lambda\sigma^2, \\ \phi^* &= \phi, \end{aligned}$$

with $a_0 = 0$ and $b_0 = 0$.

- μ^*, ϕ^* control the mean and AC of x under *risk-neutral* dynamics.

Risk-neutral dynamics of the state variable

- Law of motion of x under the risk-neutral measure is

$$x_{t+1} = \mu^* + \phi^* x_t + \sigma\varepsilon_{t+1}.$$

- Using this risk-neutral law of motion to evaluate

$$\begin{aligned} p_{n,t} &= \log \mathbb{E}_t^* \left[\exp\left(-r_t\right) P_{n-1,t+1}\right] \\ &= \log \mathbb{E}_t^* \left[\exp\left(\sum_{j=0}^{n-1} -r_{t+j}\right)\right] \\ &= \log \mathbb{E}_t^* \left[\exp\left(\sum_{j=0}^{n-1} -\delta_0 - \delta_1 x_{t+j}\right)\right] \\ &= a_n + b_n x_t, \end{aligned}$$

where a_n and b_n are exactly those that we derived in the recursions above. QED.

Bond risk premia

- For pricing, we only need to know μ^*, not μ, but to assess risk premia, we also need to know μ.
- Let's look at risk premia of bonds in this model: First we need returns, then their covariance with the SDF
- Combining $r_{n,t+1} = p_{n-1,t+1} - p_{n,t}$ with the affine pricing relation we obtain

$$\begin{aligned} r_{n,t+1} - \mathbb{E}_t\left[r_{n,t+1}\right] &= p_{n-1,t+1} - p_{n,t} - \mathbb{E}_t\left[p_{n-1,t+1} - p_{n,t}\right] \\ &= p_{n-1,t+1} - \mathbb{E}_t\left[p_{n-1,t+1}\right] \\ &= b_{n-1}\left(x_{t+1} - \mathbb{E}_t\left[x_{t+1}\right]\right). \end{aligned}$$

i.e., perfectly correlated across all maturities. b_{n-1} reflects risk exposure of bond.

Bond risk premia

- Applying the SDF to bond returns, we have

$$\mathbb{E}_t\left[\exp\left(r_{n,t+1}\right) M_{t+1}\right] = 1.$$

- After taking logs,

$$\mathbb{E}_t\left[r_{n,t+1} + m_{t+1}\right] + \frac{1}{2}\operatorname{var}_t\left[r_{n,t+1} + m_{t+1}\right] = 0$$

- Now note that

$$\begin{aligned} \mathbb{E}_t\left[m_{t+1}\right] + \frac{1}{2}\operatorname{var}_t\left(m_{t+1}\right) &= \log \mathbb{E}_t\left[M_{t+1}\right] \\ &= -y_{1,t}. \end{aligned}$$

Bond risk premia

- Using the above fact, we get

$$\begin{aligned}\mathbb{E}_t\left(r_{n,t+1}\right) - y_{1,t} + \frac{1}{2}\mathrm{var}_t\left(r_{n,t+1}\right) &= -\mathrm{cov}_t\left(r_{n,t+1}, m_{t+1}\right)\\ &= b_{n-1}\lambda\sigma^2\\ &= b_{n-1}(\mu - \mu^*).\end{aligned}$$

One-factor model: identification

- If we want to estimate a term-structure model in reduced-form purely from yield data, we can only identify parameters if changing those implies observable changes in yields
- In our specification,

$$\begin{aligned}x_{t+1} &= \mu + \phi x_t + \sigma\varepsilon_{t+1}\\ m_{t+1} &= -\delta_0 - \delta_1 x_t - \frac{1}{2}\sigma^2\lambda^2 - \lambda\sigma\varepsilon_{t+1}\end{aligned}$$

  any change to μ and σ can be undone, in terms of pricing implications, by changing δ_0 and δ_1, λ accordingly $\Rightarrow$ underidentification

- We adopt a normalization to remove this redundancy. For example,

$$\begin{aligned}\mu &= \lambda\sigma^2\\ \delta_1 &= 1\end{aligned}$$

  which implies that $\mu^* = 0$.

One-factor model: Identification approach

1. Time-series regression of r_t on r_{t-1} to identify σ^2
   - NB: Since $\phi^* = \phi$ here, we could identify δ_0 and ϕ^* and hence risk-neutral dynamics of x_t also from the time-series dynamics of r_t. But not when price of risk time-varying ($\phi^* \neq \phi$).
2. To estimate δ_0, ϕ^* from pooled cross-section and time-series of bond yields:

$$\min_{\delta_0, \phi^*} \sum_n^N \sum_t^T \left[y_{n,t} - \frac{1}{n}(-a_n - b_n x_t) \right]^2$$

   - Like regression of $y_{n,t}$ on x_t with intercept a_n, but with nonlinear restrictions that tie a_n and b_n to δ_0 and ϕ^*
   - Intercept and slope from one bond with $n > 1$ would be sufficient to identify δ_0 and ϕ^*, but more can be used.
   - That a second or more bonds could provide additional information is not strictly consistent with the single factor model: think of bond yields as having "measurement error" – really, pricing errors.

Multifactor affine models

- We now generalize the affine model to multiple state variables and time-varying prices of risk
- $K \times 1$ state vector follows first-order VAR,

$$\boldsymbol{x}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi} \boldsymbol{x}_t + \boldsymbol{\Omega} \varepsilon_{t+1}$$

  where $\varepsilon_{t+1} \sim \mathcal{N}(0, \boldsymbol{I}_K)$.
- Log SDF

$$m_{t+1} = -\delta_0 - \boldsymbol{\delta}_1' \boldsymbol{x}_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\Omega} \varepsilon_{t+1},$$

  where $\boldsymbol{\Sigma} = \boldsymbol{\Omega}\boldsymbol{\Omega}'$, with K time-varying prices of risk (generating heteroskedastic SDF)

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda}_1 \boldsymbol{x}_t.$$

Multifactor affine models

- Log one-period bond price

$$\begin{aligned} p_{1,t} &= \log \mathbb{E}_t \left[M_{t+1} \right] \\ &= \log \mathbb{E}_t \left[\exp \left(-\delta_0 - \boldsymbol{\delta}_1' \boldsymbol{x}_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\Omega} \boldsymbol{\varepsilon}_{t+1} \right) \right] \\ &= -\delta_0 - \boldsymbol{\delta}_1' \boldsymbol{x}_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda}_t + \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\Sigma} \boldsymbol{\lambda}_t \\ &= -\delta_0 - \boldsymbol{\delta}_1' \boldsymbol{x}_t \end{aligned}$$

- Prices of bonds with longer maturities

$$\begin{aligned} p_{n,t} &= \log \mathbb{E}_t \left[M_{t+1} P_{n,t+1} \right] \\ &= a_n + \boldsymbol{b}_n' \boldsymbol{x}_t, \end{aligned} \tag{1}$$

with a recursion, started at $\boldsymbol{b}_0 = 0$ and $a_0 = 0$,

Multifactor affine models

- and

$$\begin{aligned} \boldsymbol{b}_n' &= -\boldsymbol{\delta}_1' + \boldsymbol{b}_{n-1}' \boldsymbol{\Phi}^* \\ a_n &= -\delta_0 + a_{n-1} + \boldsymbol{b}_{n-1}' \boldsymbol{\mu}^* + \frac{1}{2} \boldsymbol{b}_{n-1}' \boldsymbol{\Sigma} \boldsymbol{b}_{n-1}, \end{aligned}$$

where

$$\boldsymbol{\Phi}^* = \boldsymbol{\Phi} - \boldsymbol{\Sigma} \boldsymbol{\Lambda}_1, \tag{2}$$

$$\boldsymbol{\mu}^* = \boldsymbol{\mu} - \boldsymbol{\Sigma} \boldsymbol{\lambda}_0, \tag{3}$$

- $\boldsymbol{\Phi}^*$ and $\boldsymbol{\mu}^*$ have interpretation as the parameters of the VAR of $\boldsymbol{x}$ under the risk-neutral distribution.

- As in our single-factor case, bond prices depend only on the risk-neutral VAR parameters.

Multifactor affine models

- The wedge between the parameters of the state variable law of motion under the risk-neutral and physical distributions reflect risk premia.
- We can see this by examining expected excess returns:

$$\begin{aligned}\mathbb{E}_t\left[r_{n,t+1} - y_{1,t}\right] &+ \frac{1}{2}\mathrm{var}_t(r_{n,t+1}) \\ &= \boldsymbol{b}_{n-1}'\boldsymbol{\Sigma}\boldsymbol{\lambda}_0 + \boldsymbol{b}_{n-1}'\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\boldsymbol{x}_t \\ &= \boldsymbol{b}_{n-1}'(\boldsymbol{\mu} - \boldsymbol{\mu}^*) + \boldsymbol{b}_{n-1}'(\boldsymbol{\Phi} - \boldsymbol{\Phi}^*)\boldsymbol{x}_t \qquad (4)\end{aligned}$$

- Thus, for fitting the model to prices, all we need is $\boldsymbol{\mu}^*$ and $\boldsymbol{\Phi}^*$. But for assessing risk premia, we also need to know $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}$.
- Expected excess returns vary with $\boldsymbol{x}_t$ unless $\boldsymbol{\Lambda}_1 = 0$ and hence $\boldsymbol{\Phi} = \boldsymbol{\Phi}^*$

Multifactor affine models: Interpretation of no-arbitrage restrictions

- With K small, the affine pricing relation provides fairly tight restrictions on prices.
- Because of these restrictions that tie together the price movements of bond with different maturity, the literature often speaks of "powerful" nature of no-arbitrage restrictions in term structure models.
- But keep in mind that these are really no-arbitrage restriction jointly with the ad-hoc assumptions on the number of factors and the functional form of the SDF that provides these tight restrictions.
- No arbitrage on its own doesn't provide much in terms of pricing restrictions (e.g., with arbitrary number of factors)

Multifactor affine models: Identification

- As in single-factor case, the general specification of the model we set up above is not identified based on bond yield data only
- Literature has often used ad-hoc parameter restrictions that (hopefully) ensure identification
- Joslin, Singleton and Zhu (2011RFS) find a normalization that ensures identification (but does not restrict model)
  - $\boldsymbol{\mu}^* = 0$
  - $\boldsymbol{\delta}_1 = (1, 1, ..., 1)'$
  - $\boldsymbol{\Omega}$ lower triangular
  - $\boldsymbol{\Phi}^*$ diagonal

Multifactor affine models: Backing out the state vector

- To construct the yields implied by the model, and evaluate the fit to search for parameter values, we need to back out the latent state vector $\boldsymbol{x}_t$
- Suppose we observe K bond yields with maturities $n(1), ..., n(K)$. Collect the yields in the vector $\boldsymbol{y}_t$. The affine model implies
$$\boldsymbol{y}_t = -\boldsymbol{a}_y - \boldsymbol{B}_y \boldsymbol{x}_t \tag{5}$$
where $\boldsymbol{a}_y$ and $\boldsymbol{B}_y$ contain the appropriate a_n/n and $\boldsymbol{b}_n/n$ for the maturities of the bonds.
  - Division by n because we look at yields here: $y_{h,t} = -p_{n,t}/n$

Multifactor affine models: Backing out the state vector

- Given parameter values, we know $\boldsymbol{a}_y$ and $\boldsymbol{B}_y$, and if $\boldsymbol{B}_y$ is invertible, we can back out the unobservable factors from the observable yields by inverting the affine pricing relation

$$\boldsymbol{x}_t = -\boldsymbol{B}_y^{-1}(\boldsymbol{y}_t + \boldsymbol{a}_y)$$

  More on this invertibility condition in the next session

- Which K yields? If the model provided a perfect description of the yield curve, it would not matter which K yields we picked: Singularity!

- In reality, model does not fit perfectly: pricing errors ("measurement errors")

- Typical approach: Regard K yields, or K linear combinations of yields as "measured without error" and use those to back out state vector while the other yields are assumed to be "measured with error" and not perfectly fit.

Multifactor affine models: Extracting the state vector from PCs

- Nice for interpretability: Use K principal components as the linear combinations of yields that are measured without error

- Let $\boldsymbol{Q}$ denote the eigenvectors associated with the K highest eigenvalues from an eigendecomposition of var$(\Delta \boldsymbol{y}_t)$
  - Potentially a little better: use VAR innovations

- PC factors

$$\boldsymbol{f}_t = \boldsymbol{Q}'\boldsymbol{y}_t = -\boldsymbol{Q}'\boldsymbol{a}_y - \boldsymbol{Q}'\boldsymbol{B}_y\boldsymbol{x}_t$$

- Then we can back out state vector from the PC factors:

$$\boldsymbol{x}_t = -\boldsymbol{B}_{PC}^{-1}\left(\boldsymbol{f}_t + \boldsymbol{Q}'\boldsymbol{a}_y\right)$$

  where $\boldsymbol{B}_{PC} = \boldsymbol{Q}'\boldsymbol{B}_y$.

Multifactor affine models: Covariance matrix of state vector innovations

- Parameters:
  - Fixed by JSZ normalization: $\boldsymbol{\mu}^*$, $\boldsymbol{\delta}_1$
  - To estimate: δ_0, $\boldsymbol{\Sigma} = \boldsymbol{\Omega}\boldsymbol{\Omega}'$, $\boldsymbol{\Phi}^*$ (diagonal)

- For $\boldsymbol{\Sigma}$ estimation it would be helpful to bring in information from $\boldsymbol{\Sigma}_{PC} = \text{var}(\Delta \boldsymbol{f}_t)$, rather than just trying to fit $\boldsymbol{\Sigma}$ to match cross-section of yield levels.

- Since the $\boldsymbol{b}_n$ and hence $\boldsymbol{B}_{PC}$ do not depend on $\boldsymbol{\Sigma}$, we can compute $\boldsymbol{B}_{PC}$ first and then

$$\boldsymbol{\Sigma} = \boldsymbol{B}_{PC}^{-1}\boldsymbol{\Sigma}_{PC}(\boldsymbol{B}_{PC}')^{-1}$$

  which we need to compute the a_n and $\boldsymbol{a}_y$.

- We can then estimate the model by minimizing sum of squared errors in fitting yields

Multifactor affine models: Estimation – Summary

1. Obtain $\boldsymbol{Q}$ via eigendecomposition of $\text{var}(\Delta \boldsymbol{y}_t)$, construct $\boldsymbol{f}_t$, and estimate $\boldsymbol{\Sigma}_{PC} = \text{var}(\Delta \boldsymbol{f}_t)$.
2. Fix guess for parameters $(\delta_0, \boldsymbol{\Phi}^*)$ and calculate $\boldsymbol{B}_y$ and $\boldsymbol{B}_{PC} = \boldsymbol{Q}'\boldsymbol{B}_y$
3. Calculate $\boldsymbol{\Sigma} = \boldsymbol{B}_{PC}^{-1}\boldsymbol{\Sigma}_{PC}(\boldsymbol{B}_{PC}')^{-1}$ and then $\boldsymbol{a}_y$.
4. Back out $\boldsymbol{x}_t = -\boldsymbol{B}_{PC}^{-1}\left(\boldsymbol{f}_t + \boldsymbol{Q}'\boldsymbol{a}_y\right)$
5. Calculate model-implied yields $\hat{\boldsymbol{y}}_t = -\boldsymbol{a}_y - \boldsymbol{B}_y\boldsymbol{x}_t$
6. Evaluate sum of squared pricing errors $\Rightarrow$ back to step 2 to minimize

Multifactor affine models: Regression interpretation

- Estimation based on the least-squares criterion

$$\min_{\delta_0, \boldsymbol{\Phi}^*} \sum_{n}^{N} \sum_{t}^{T} \left[y_{n,t} - \frac{1}{n}(-a_n - \boldsymbol{b}_n' \boldsymbol{x}_t) \right]^2$$

  is like running a regression of the $y_{n,t}$ on $\boldsymbol{x}_t$ with maturity-specific intercept $-a_n$ and slopes $-\boldsymbol{b}_n$.

- This in turn is like a regression of $y_{n,t}$ on the PCs $\boldsymbol{f}_t$ with correspondingly rotated intercept and slope.

- Affine term-structure model imposes restrictions across maturities on the $-a_n$ and slopes $-\boldsymbol{b}_n$.

- The higher the number K of factors in the SDF, the more flexibility in allowing a_n and $\boldsymbol{b}_n$ to vary across maturities

Multifactor affine models: Regression interpretation

- If we fit the model only to the K PCs, we would fit the data perfectly for any parameter values, as we can always invert the model to back out a state vector that makes the model perfectly fits the data

- So we need at least one bond or portfolio of bonds that is not exactly the same linear combination of bonds as the PCs: With one additional bond we could then back out the $K + 1$ parameters from the estimates of intercept $-a_n$ and K slopes $-\boldsymbol{b}_n$ for this bond.

- But more bonds are better, of course. (Again, as in one-factor case, with "measurement error" interpretation why more bonds provide additional information).

Example: Cochrane and Piazzesi (2008), "Decomposing the Yield Curve"

- Empirical implementation of affine models is typically done with at least three factors.
- Researchers who use principal components as factors often include the first three (level, slope, and curvature).
- Cochrane and Piazzesi (2008) add a fourth factor (x in their notation) that seems to be important for forecasting excess returns even though it does not play much of a role in explaining bond yield variation. They let variation in $\mathbf{\Lambda}_t$ be driven solely by this fourth return-forecasting factor.
- Note that CP don't use the JSZ normalization. So set of estimated parameters a bit different. Not clear whether identified (i.e., whether another combination of model parameters might produce the same fit)

Loadings: restricted and unrestricted (Cochrane and Piazzesi 2008)

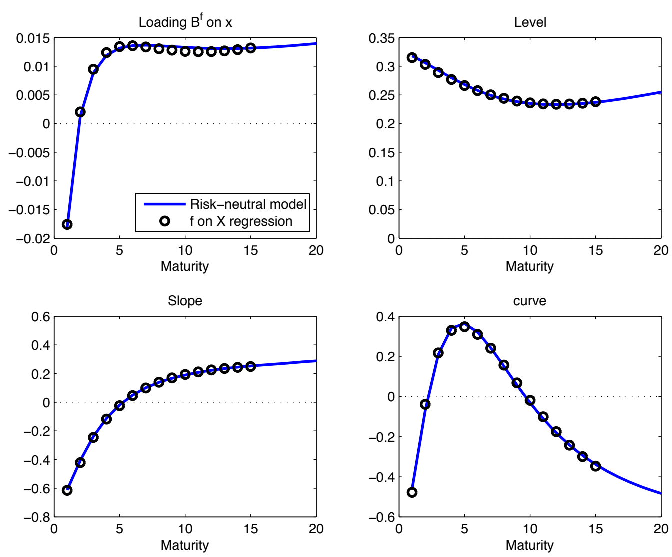


Figure 5: Affine model loadings, B^f in $f^{(n)} = A^f + B^{f\prime}X_t$. The line gives the loadings of the affine model, found by searching over parameters $\delta_0, \delta_1, \mu^*, \phi^*$. The circles give regression coefficients of forward rates on the factors.

Example: Estimated four-factor model (Cochrane and Piazzesi 2008)

	$100\times\mu$	x	level	slope	curve	mse	R^2
Risk-neutral:	μ^*			ϕ^*			
x	-1.39	0.35	-0.02	-1.05	8.19	25.4	
level	0.68	0.03	0.98	-0.21	-0.22	4.77	
slope	0.00	0.00	-0.02	0.76	0.77	1.73	
curve	0.00	0.00	-0.01	0.02	0.70	0.65	
Actual:	μ			ϕ			
x	0	*0.61*	-0.02	-1.05	8.19	27.0	
level	0	*-0.09*	0.98	-0.21	-0.22	3.65	
slope	0	*-0.00*	-0.02	0.76	0.77	1.74	
curve	0	*0.00*	-0.01	0.02	0.70	0.65	

Table 4. Estimates of model dynamics, μ and ϕ in $X_{t+1} = \mu + \phi X_t + v_{t+1}$. |eig| gives the eigenvalues of ϕ, in order. mse is mean-squared error.

(Note that their model is restricted to let only the x factor affect prices of risk)

Multifactor affine models: Prices of risk and Sharpe ratios of PCs

- To illustrate, focus on constant price of risk case where $\boldsymbol{\Lambda}_1 = 0$ and $\boldsymbol{\Phi} = \boldsymbol{\Phi}^*$
- With $\boldsymbol{\mu}^* = 0$ (JSZ), which implies $\boldsymbol{\mu}^*_{PC} = 0$, we get the prices of risk for each PC factor as

$$\boldsymbol{\lambda}_{0,PC} = \boldsymbol{\Sigma}_{PC}^{-1}\boldsymbol{\mu}_{PC}$$

  from VAR intercepts $\boldsymbol{\mu}_{PC}$.

- To get Sharpe ratios of PC factors, calculate

$$\boldsymbol{\Sigma}_{PC}^{\frac{1}{2}}\boldsymbol{\lambda}_{0,PC}$$

Real term structure in equilibrium models

- Example: Representative agent CRRA SDF

$$m_{t+1} = \log \delta - \gamma \Delta c_{t+1}$$

- Let Δc_t follow an AR(1),

$$\Delta c_{t+1} = \theta_c + \phi_c \Delta c_t + \sigma_\varepsilon \varepsilon_{t+1}$$

with $\varepsilon_{t+1} \sim \text{IID } \mathcal{N}(0, 1)$.

- Then,

$$m_{t+1} = \log \delta - \gamma \theta_c - \gamma \phi_c \Delta c_t - \gamma \sigma_\varepsilon \varepsilon_{t+1}$$

Real term structure in equilibrium models

- We can map this into our one-factor affine SDF

$$m_{t+1} = -\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda^2\sigma^2 - \lambda\sigma\varepsilon_{t+1}$$

by setting

$$\begin{aligned} x_t &= \Delta c_t \\ \sigma &= \sigma_\varepsilon \\ \delta_0 &= -\log \delta + \gamma \theta_c - \frac{1}{2}\gamma^2\sigma_\varepsilon^2 \\ \delta_1 &= \gamma \phi_c \\ \phi &= \phi^* = \phi_c \\ \lambda &= \gamma \end{aligned}$$

Real term structure in equilibrium models

- Recall risk premium ,

$$\mathbb{E}_t\left[r_{n,t+1}\right] - y_{1,t} + \frac{1}{2}\operatorname{var}(r_{n,t+1}) = b_{n-1}\lambda\sigma^2$$

  where, using the above parameter mapping,

$$\lambda = \gamma$$
$$b_n = -\gamma\phi_c + b_{n-1}\phi_c$$

- As long as $\phi_c \neq 0$, interest rates can vary in this model: long-term bonds are risky.
- $\phi_c > 0$: negative risk premium, more so the higher γ or the higher σ_c^2
  - SDF is low following a positive shock to the short rate when long-term bond prices fall: long-term bonds are a hedge
- $\phi_c < 0$: positive risk premium, more so the higher γ or the higher σ_c^2

Evidence on Real term structure

- Empirical data indicates that consumption growth is, if anything, positively autocorrelated, i.e., $\phi_c > 0 \Rightarrow$ Negative risk premium
- But the earlier evidence we discussed suggests that the *nominal* term structure, at least, is upward sloping on average. What about *real* term structure?
- Evidence from inflation-indexed bonds: Flatter, perhaps flat? But in times where inflation-index bonds are available, nominal term structure also quite flat (consistent with small inflation risk)

Evidence on Real term structure

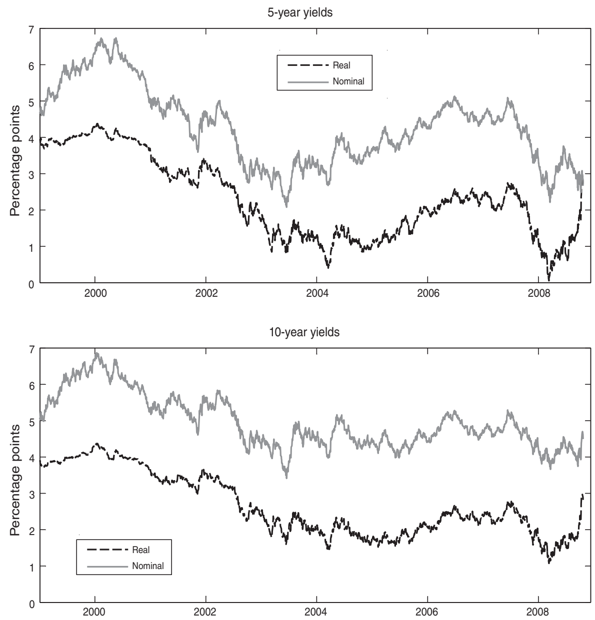


FIGURE 5. ZERO-COUPON YIELDS: TIPS AND NOMINAL

Source: Gürkaynak, R.S., Sack, B. and Wright, J.H., 2010. The TIPS yield curve and inflation compensation. American Economic Journal: Macroeconomics, 2(1), pp.70-92.

Link to nominal term structure

- If we wanted the structural model to make predictions about the nominal term structure, we would need to add assumptions on the stochastic behavior of inflation rates.
- As noted earlier, the log nominal SDF is

$$m_{t+1} = m_{real,t+1} - \pi_{t+1}$$

  i.e., a two-factor SDF.
- Taking conditional expectations of the nominal SDF $\exp(m_{t+t})$ now involves covariance terms of $m_{real,t+1}$ and π_{t+1}. These covariance terms capture inflation risk.
  - With our one-factor real SDF: Long-term nominal bonds can command positive risk premium (relative to real bonds) if π_{t+1} high in recessions