Elliptic Curves and Their Applications in Cryptography

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Fermat's Last Theorem

Pierre de Fermat:

> *It is impossible to write a cube as the sum of two cubes, a fourth power as the sum of two fourth powers, and, in general, any power beyond the second as the sum of two similar powers. For this I have discovered a truly wonderful proof but the margin is too small to contain it.*

1995–Andrew Wiles published the first successful proof.

Elliptic Curve Factorization

Pollard's $p-1$ algorithm can find prime factors p of a composite integer for which $p-1$ is smooth.

Elliptic curve factorization is a generalization of Pollard's $p-1$ algorithm using random elliptic curve groups over $\mathbb{Z}/p\mathbb{Z}$.

Definition

An elliptic curve is a projective algebraic curve with affine coordinates given by

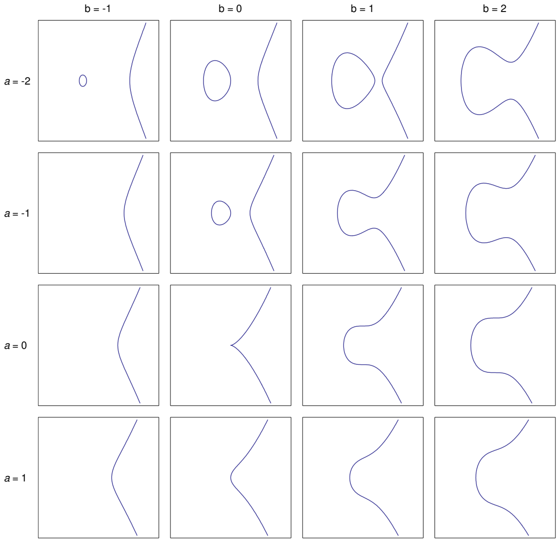
$$y^2 = x^3 + \alpha x + \beta$$

where

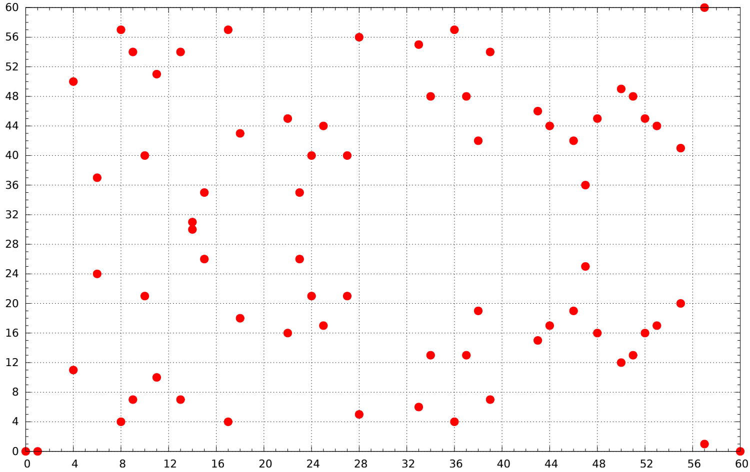
$$4\alpha^3 + 27\beta^2 \neq 0.$$

Elliptic curves over finite fields (usually $\mathbb{Z}/p\mathbb{Z}$) are of particular interest in cryptography.

Elliptic Curves



Elliptic Curve $y^2 = x^3 - x$ on $\mathbb{Z}/61\mathbb{Z}$



Affine Space

Definition

Given a finite field with q elements, $\mathbb{F}_q$, affine n-space over $\mathbb{F}_q$, denoted $A^n(\mathbb{F}_q)$, is the set of n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{F}_q$.

Definition

A **point** in $A^n(\mathbb{F}_q)$ is an n-tuple $(a_1, a_2, \ldots, a_n)$ for $a_i \in \mathbb{F}_q$.

Projective Space

Definition

Projective n-space over $\mathbb{F}_q$, denoted $P^n(\mathbb{F}_q)$, is the set of equivalence classes of nonzero elements of $A^{n+1}(\mathbb{F}_q)$ under the equivalence relation

$$(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n)$$

iff there exists a $0 \neq \lambda \in \mathbb{F}_q$ such that

$$a_i = \lambda b_i$$

for all $i = 0, 1, \ldots, n$.

Projective Space

Definition

A **point** in $P^n(\mathbb{F}_q)$, denoted $[a_0, a_1, \ldots, a_n]$, is the equivalence class containing $(a_0, a_1, \ldots, a_n)$.

While $A^2(\mathbb{F}_q)$ has q^2 points, $P^2(\mathbb{F}_q)$ has $q^2 + q + 1$ points.

Proof

The points in $P^2(\mathbb{F}_q)$ can be broken into 2 subsets:

- q^2 **finite points** of the form $[1, a_1, a_2]$ that map to $A^2(\mathbb{F}_q)$
- $q + 1$ **points at infinity** of the form $[0, a_0, a_1]$ with the structure of $P^1(\mathbb{F}_q)$

Algebraic Curves

Definition

An affine algebraic curve over $\mathbb{F}_q$ is defined by $f(x, y) = 0$ for an irreducible polynomial $f(x, y) \in \mathbb{F}_q[x, y]$.

Definition

A projective algebraic curve over $\mathbb{F}_q$ is defined by $f(x, y, z) = 0$ for an irreducible homogeneous polynomial $f(x, y, z) \in \mathbb{F}_q[x, y, z]$.

Singular Points

Definition

A point P on an affine curve $f(x, y) = 0$ is called **singular** if

$$\left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right) = (0, 0)$$

Definition

An algebraic curve is called **nonsingular** or **smooth** if it contains no singular points.

Elliptic Curves

The **projective Weierstrass equation** of an elliptic curve is given by

$$y^2z + \alpha_1 xyz + \alpha_3 yz^2 = x^3 + \alpha_2 x^2 z + \alpha_4 xz^2 + \alpha_6 z^3.$$

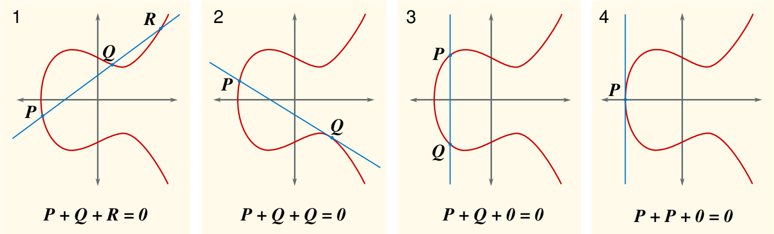
The affine **Weierstrass normal form** of an elliptic curve is

$$y^2 - x^3 - \alpha x - \beta = 0$$

with discriminant $\Delta = -16(4\alpha^3 + 27\beta^2)$.

Mappings

Additive Operation



Additive Operation

Theorem

For an elliptic curve $\mathcal{E}(\mathbb{F}_q)$ and $a, b, c, d \in \mathbb{F}_q$, let $P = (a, b)$ and $Q = (c, d)$ be two points on $\mathcal{E}$ such that $Q \neq -P = (a, -b)$. Define

$$M = \begin{cases} \dfrac{d-b}{c-a} & \text{if } a \neq c \\[2ex] \dfrac{3a^2+\alpha}{2b} & \text{if } a = c. \end{cases}$$

Then the point $P + Q$ is given by $R = (g, h)$ where

$$g = M^2 - a - c$$
$$h = Ma - Mg - b.$$

▸ Proof

Elliptic Curve Discrete Logarithm Problem

Given $P = (a, b)$ on $\mathcal{E}$, it is possible to efficiently compute $\overbrace{P + \cdots + P}^{n \text{ times}}$, denoted $[n]P$.

However, given P and $[n]P$, it can be *very* difficult to compute the value of n.

Efficient Computation of $[n]P$

Algorithm

1. *Write n in its binary form, i.e.* $n = n_0 + 2n_1 + 2^2 n_2 + \cdots + 2^t n_t$, *with* $n_i \in \{0,1\}$ *and* $n_t = 1$.
2. *Let* $P_0 = P$.
3. *For* $i = 1, \ldots, t$, *compute* $P_i = [2^i]P = [2]P_{i-1}$ *recursively.*
4. *Then,* $[m]P = \sum_{i=0}^{t} [n_i] P_i$.

While naïve application of the group operator requires n additions, this algorithm can be carried out in $2t \leq 2 \log n$ additions.

Order of $\mathcal{E}(\mathbb{F}_q)$

A curve with a small number of points would be vulnerable to cryptanalysis.

If an elliptic curve on $\mathbb{F}_p$ has exactly p points, the ECDLP can be transformed into addition in $\mathbb{Z}_p$.

Definition (Hasse-Weil Bound)

Let N be the number of points in $\mathbb{F}_q$ on an elliptic curve $\mathcal{E}$.

$$q + 1 - 2\sqrt{q} \leq N \leq q + 1 + 2\sqrt{q}.$$

Schoof's Algorithm

The number of points on an elliptic curve can be calculated efficiently using a deterministic polynomial time algorithm.

Schoof's Algorithm works by computing $q + 1 - N \pmod{p}$ for a large number of primes whose product is greater than $4\sqrt{q}$, then calculating $q + 1 - N$ by the Chinese Remainder Theorem.

With improvements by Atkin and Elkies, Schoof's Algorithm runs in $O(\log^4 q)$ time.

Pairings on Elliptic Curves

Definition

Given abelian groups G_1, G_2, G_3, a pairing $\omega : G_1 \times G_2 \to G_3$ maps every pair of elements in $G_1 \times G_2$ to some element in G_3.

A *cryptographically useful* pairing is also

- bilinear: if $g_1, g_1' \in G_1$ and $g_2, g_2' \in G_2$ then $\omega(g_1 g_1', g_2) = \omega(g_1, g_2)\omega(g_1', g_2)$ and $\omega(g_1, g_2 g_2') = \omega(g_1, g_2)\omega(g_1, g_2')$.
- nondegenerate: if $\omega(g_1, g_2) = 1$ for all $g_2 \in G_2$ then it follows that $g_1 = 1$.

Assume that all parties agree in advance on a choice of elliptic curve $\mathcal{E}$ on finite field $\mathbb{F}_p$, $P \in \mathcal{E}(\mathbb{F}_p)$, and paring ω on $\mathcal{E}$.

The chosen parameters are assumed to be public knowledge and to possess the properties appropriate for security.

Let $M \in \mathcal{E}(\mathbb{F}_p)$ be a message (encoded as a point on $\mathcal{E}(\mathbb{F}_p)$) that Alice would like to send to Bob.

Elliptic Curve Diffie-Hellman Exchange

Setup

- *Alice chooses a random secret* $a \in \mathbb{Z}$ *and sends* $A = [a]P$ *to Bob over an insecure channel.*
- *In the same way, Bob chooses a random secret* $b \in \mathbb{Z}$ *and sends* $B = [b]P$ *to Alice.*

Algorithm

- *Alice computes* $Q = [a]B = [ab]P$.
- *Bob computes* $Q = [b]A = [ba]P$.

Elgamal Encryption

Setup

Bob chooses a private key $b \in \mathbb{Z}$ *and computes his public key* $B = [b]P$. *Bob is free to publish* B.

Encryption

Alice chooses a random $k \in \mathbb{Z}$ *and computes* $C_1 = [k]P$, $C_2 = M + [k]B$ *and sends* (C_1, C_2) *to Bob.*

Decryption

Bob computes

$C_2 - [a]C_1 = M + [k]B - [a][k]P = M + [ka]P - [ak]P = M.$

Tripartite Key Exchange

Setup

Alice, Bob, and Carl each choose a random secret $a, b, c \in \mathbb{Z}$, *respectively. They compute and share* $A = [a]P$, $B = [b]P$, *and* $C = [c]P$ *over insecure channels.*

Algorithm

Each computes the shared secret as follows using his/her respective secret

- *Alice:* $\omega(B, C)^a$
- *Bob:* $\omega(A, C)^b$
- *Carl:* $\omega(A, B)^c$

since $\omega(B, C)^a = \omega(A, C)^b = \omega(A, B)^c = \omega(P, P)^{abc}$

Identity-based Encryption

Let Trent act as the trusted authority.

Setup

1. *Trent chooses a master secret* $s \in \mathbb{Z}$ *and publishes* $S = [s]P$.
2. *Bob encodes his identity (e.g. username, email, etc.) as* $b \in \mathbb{Z}$. *Anyone can compute Bob's public key* $B = [b]P$.
3. *Bob requests his private key* $E = [s]B = [sb]P$ *from Trent.*

Identity-based Encryption

Encryption

Alice chooses a random secret $t \in \mathbb{Z}$ *and sends* $(U, V) = ([t]P, M + \omega(B, S)^t)$ *to Bob.*

Decryption

Observe that

$$\omega(E, U) = \omega([s]B, [t]P) = \omega(B, P)^{st} = \omega(B, [s]P)^t = \omega(B, S)^t.$$

Bob computes $M = V - \omega(E, U) = V - \omega(B, S)^t$.

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Number of Points in $P^2(\mathbb{F}_q)$

It is clear that the number of points in $A^n(\mathbb{F}_q)$ is q^n.

Proof.

Observe that

$$P^2(\mathbb{F}_q) = \{[1,a,b] | a,b \in \mathbb{F}_q\} \cup \{[0,1,a] | a \in \mathbb{F}_q\} \cup \{[0,0,1]\}$$

It follows that the number of points in $P^2(\mathbb{F}_q)$ is

$$q^2 + q + 1.$$

$\square$

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Mappings Between Projective and Affine Spaces

Let K be a field and define H as the points at infinity of $P^n(K)$. Mappings λ and ϕ between affine and projective spaces are defined as follows.

$$\lambda : A^n(K) \to P^n(K)$$
$$\lambda(a_1, a_2, \ldots, a_n) = [1, a_1, a_2, \ldots, a_n]$$

$$\phi : P^n(K) - H \to A^n(K)$$
$$\phi([b_0, b_1, \ldots, b_n]) = \left(\frac{b_1}{b_0}, \frac{b_2}{b_0}, \ldots, \frac{b_n}{b_0}\right)$$

Back

Correctness of Additive Operation on $\mathcal{E}$

Let ℓ, the line passing through P and Q, be given by $y = Mx - C$ with M as defined for the addition operation and $N = b - Ma$. Also let $S = (g', h')$ be the third intersection of $\mathcal{E}$ and ℓ. Substituting the equation for ℓ into that of $\mathcal{E}$,

$$(Mx + N)^2 = x^3 + \alpha x + \beta$$

which expands to

$$f(x) = x^3 - M^2x^2 + (\alpha - 2MN)x + \beta - N^2 = 0.$$

Correctness of Additive Operation on $\mathcal{E}$

Since a, c, g' are the x coordinates of p, Q, S, respectively,

$$f(x) = (x-a)(x-c)(x-g')$$

and by expanding and comparing coefficients,

$$g' = M^2 - a - c$$
$$h' = Mg' + N$$

so

$$P + Q = S = (g', -h').$$

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