Control Theory

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September 26, 2022

1 Systems

1.1 Linearity

$$f(ax + by) = af(x) + bf(y) \tag{1}$$

1.2 Time-Invariance

System is not directly dependent on time:

$$y(t) = F(x(t), t) = F(x(t)) \tag{2}$$

This also means that any shift to time in input translates to the same shift in the output.

$$x(t + \delta) \xrightarrow{F} y(t + \delta) \tag{3}$$

2 Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\ d\tau \tag{4}$$

3 Laplace Transform

$$F(s) = \mathscr{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt \tag{5}$$

4 Transfer Functions

$$\begin{aligned}
a_2\ddot{x} + a_1\dot{x} + a_0x &= b_2\ddot{y} + b_1\dot{y} + b_0y \\
\mathscr{L}\{a_2\ddot{x} + a_1\dot{x} + a_0x\} &= \mathscr{L}\{b_2\ddot{y} + b_1\dot{y} + b_0y\} \\
a_2s^2X(s) + a_1sX(s) + a_0X(s) &= b_2s^2Y(s) + b_1sY(S) + b_0Y(s) \\
G(s) = \frac{Y(s)}{X(s)} &= \frac{b_2s^2 + b_1s + b_0}{a_2s^2 + a_1s + a_0}
\end{aligned}$$

Notes:

- All initial conditions are zero
- $X(s) \neq a_2 s^2 + a_1 s + a_0$
- $Y(s) \neq b_2 s^2 + b_1 s + b_0$

$$\begin{aligned}
G(s) &= \frac{Y(s)}{X(s)} \\
Y(s) &= G(s)X(s) \\
\mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\{G(s)X(s)\} \\
y(t) &= (g * x)(t) \\
&= \int_{-\infty}^{\infty} g(\tau)x(t-\tau)\ d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)g(t-\tau)\ d\tau
\end{aligned}$$

5 Impulse Response

$$\begin{aligned}
Y(s) &= G(s) \\
\mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\{G(s)\} \\
y(t) &= g(t)
\end{aligned}$$

Table 1: Laplace Lookup

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
Multiplying by a Constant	$af(at)$	$aF(s)$
Time Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Frequency Scaling	$\frac{1}{a}f\left(\frac{t}{a}\right)$	$F(as)$
Time Shifting	$f(t-a)u(t-a)$	$e^{-at}F(s)$
Frequency Shifting	$e^{at}f(t)$	$F(s-a)$
Convolution	$(f * g)(t)$	$F(s)G(s) = G(s)F(s)$
	1	$\frac{1}{s}$
	t^n	$\frac{n!}{s^{n+1}}$
	$\sin at$	$\frac{a}{s^2+a^2}$
	$\cos at$	$\frac{s}{s^2+a^2}$
	$\sinh at$	$\frac{a}{s^2-a^2}$
	$\cosh at$	$\frac{s}{s^2-a^2}$