Combinatorics

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1 Basis

The best way to think about combinations and permutations is to think about all permutations of n things:

$$n! = P(n, n) \tag{1}$$

Then we think about discounting orderings that are the same:

$$\frac{n!}{n_1!n_2!\cdots n_m!} = \binom{n}{n_1, n_2, ..., n_m}, \quad \text{where } \sum_{k=1}^{m} n_k = n \tag{2}$$

This is referred to as **permutations of multisets**. These represent items in the set that are considered the same.

From this, we can derive all other types of combinations and permutations.

For standard **permutations**:

$$P(n, r) = \binom{n}{\underbrace{1, 1, ..., 1}_{r \text{ times}}, n - r} \tag{3}$$

In fact, since the sum of the n_k's must be n, there are always a number of implicit 1's to add up to n, e.g.:

$$\binom{n}{n_1, ..., n_m} = \binom{n}{n_1, ..., n_m, \underbrace{1, ..., 1}_{n-k \text{ times}}}, \quad \text{where } k = \sum_{i=1}^{m} n_k < n \tag{4}$$

The one exception is when there is only one term on the bottom, then there is an implicit $n - r$ term as well:

$$\binom{n}{r} = \binom{n}{r, n-r} \tag{5}$$

These are standard **combinations**.

2 Alternative Basis

Another way to think about combinations and permutations is to start with combinations:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{6}$$

Permutations can then be thought of as reordering n things:

$$P(n,n) = n! \tag{7}$$

To permute a subgroup, first *choose* the subgroup, then permute:

$$P(n,r) = \binom{n}{r} r! = \frac{n!}{\cancel{r!}(n-r)!}\cancel{r!} = \frac{n!}{(n-r)!}, \quad r \leq n \tag{8}$$

3 Identities

$$\binom{n}{0} = P(n,0) = 1 \tag{9}$$

$$\binom{0}{r} = P(0,r) = 0 \tag{10}$$

$$\binom{n}{1} = P(n,1) = n \tag{11}$$

$$\binom{n}{r} = \binom{n}{n-r} \tag{12}$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \tag{13}$$

4 Common Problems

4.1 Circular Permutations

Circular permutations:

$$P_C(n) = (n-1)! \tag{14}$$

4.2 Consecutive Items

k consecutive items:

$$P(n-k+1, r) \tag{15}$$

Count n minus k consecutive items, plus 1 representing the items as a group.

4.3 Separated Items

k items can't be together:

$$P(n-k, r) \cdot P(n-k+1, k) \tag{16}$$

Count all $n-k$ items without restriction. Then place k items in $n-k$ spaces after of each item, and 1 space before first item. Then the items

Table 1: Combinatorial Operations

Operation	Meaning
$+n$	Add n choices to an event
$-n$	Remove n choices from an event
$\times n$	Add an event with n choices
$\div n$	Remove an event with n choices

4.4 Integer Solutions to Linear Expressions

$$x_1 + x_2 + x_3 + \cdots + x_n = r$$

$$\binom{n+r-1}{r}$$

$$x_1 + x_2 + x_3 + \cdots + x_n \leq r$$

$$\binom{n+r}{r}$$

$$x_1 + x_2 + x_3 + \cdots + x_n < r$$

$$\binom{n+r-1}{r-1}$$

4.5 Lattice Paths

$$\binom{\Delta x + \Delta y}{\Delta x} = \binom{\Delta x + \Delta y}{\Delta y} \tag{17}$$

With n stops:

$$\binom{(x_2 - x_1) + (y_2 - y_1)}{(x_2 - x_1)} \cdots \binom{(x_n - x_{n-1}) + (y_n - y_{n-1})}{(x_n - x_{n-1})} \tag{18}$$