Combinatorics Problems

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Contents

1 Circular Permutations

Circular permutations:

$$P_C(n) = (n-1)! \tag{1}$$

2 Consecutive Items

k consecutive items:

$$P(n-k+1, r) \tag{2}$$

Count n minus k consecutive items, plus 1 representing the items as a group.

3 Separated Items

k items can't be together:

$$P(n-k,r)\cdot P(n-k+1,k) \tag{3}$$

Count all $n-k$ items without restriction. Then place k items in $n-k$ spaces after of each item, and 1 space before first item. Then the items

4 Integer Solutions to Linear Expressions

When all $x_i \geq 0$:

$$x_1+x_2+x_3+\cdots+x_n=r \implies \binom{n+r-1}{r} \tag{4}$$

$$x_1+x_2+x_3+\cdots+x_n \leq r \implies \binom{n+r-1+1}{r} \tag{5}$$

$$x_1+x_2+x_3+\cdots+x_n < r \implies \binom{n+r-1}{r-1} \tag{6}$$

For $x_1 \geq a_1, x_2 \geq a_2, ..., x_n \geq a_3$:

$$z_1+z_1+\cdots+z_n=r-a \implies \binom{n+(r-a)-1}{(r-a)} \tag{7}$$

$$\begin{aligned} z_1 &= x_1-a_1 \\ z_2 &= x_2-a_2 \\ &\vdots \\ z_n &= x_n-a_n \end{aligned}$$

$$a=\sum_{i=1}^{n} a_i$$

All these modifications can stack.

For $x_1 \leq a_1, x_2 \leq a_2, ..., x_n \leq a_3$:

5 Lattice Paths

$$\binom{\Delta x + \Delta y}{\Delta x} = \binom{\Delta x + \Delta y}{\Delta y} \tag{8}$$

With n stops:

$$\begin{aligned}
\Delta x_{1,2} &= x_2 - x_1, & \Delta y_{1,2} &= y_2 - y_1 \\
\Delta x_{2,3} &= x_3 - x_2, & \Delta y_{2,3} &= y_3 - y_2 \\
&\vdots & &\vdots \\
\Delta x_{n-1,n} &= x_n - x_{n-1}, & \Delta y_{n-1,n} &= y_n - y_{n-1}
\end{aligned}$$

$$\binom{\Delta x_{1,2} + \Delta y_{1,2}}{\Delta x_{1,2}} \binom{\Delta x_{2,3} + \Delta y_{2,3}}{\Delta x_{2,3}} \cdots \binom{\Delta x_{n-1,n} + \Delta y_{n-1,n}}{\Delta x_{n-1,n}} \tag{9}$$

Note: there are $n-1$ factors, since they represent the trips *between* points.

6 Finding Binomial Coefficients

What is the coefficient of $x^a y^b$ in $(x+y)^n$?

Start with binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{10}$$

We want the coefficient when $k = a$ and $n - k = b$, so:

$$\binom{n}{a}$$

6.1 A Complicated Example

What is the coefficient of $x^{17} y^6$ in $(x^4 y - 3xy^2)^5$?

The binomial theorem is:

$$\begin{aligned}
(x^4y - 3xy^2)^5 &= \sum_{k=0}^{5} \binom{5}{k} (x^4y)^k (-3xy^2)^{5-k} \\
&= \sum_{k=0}^{5} \binom{5}{k} x^{4k} y^k (-3)^{5-k} x^{5-k} y^{10-2k} \\
&= \sum_{k=0}^{5} \binom{5}{k} (-3)^{5-k} x^{4k} x^{5-k} y^k y^{10-2k} \\
&= \sum_{k=0}^{5} \binom{5}{k} (-3)^{5-k} x^{3k+5} y^{-k+10}
\end{aligned}$$

So, we want the coefficient when $3k + 5 = 17$, i.e., when $k = 4$:

$$\binom{5}{4} (-3)^{5-4} = -15 \tag{11}$$

7 Finding Multinomial Coefficients

What is the coefficient of $x^a y^b z^c$ in $(x + y + z)^n$?

$$\binom{n}{a, b, c}$$

7.1 A Complicated Example

What is the coefficient of $x^3 y^6 z^4$ in $(2x + y - z^2)^{11}$?

$$\begin{aligned}
(2x + y - z^2)^{11} &= \sum \binom{11}{k_1, k_2, k_3} (2x)^{k_1} y^{k_2} (-x^2)^{k_3} \\
&= \sum \binom{11}{k_1, k_2, k_3} 2^{k_1} (-1)^{k_3} x^{k_1} y^{k_2} z^{2k_3}
\end{aligned}$$

We want $k_1 = 3$, $k_2 = 6$, and $2k_3 = 4 \implies k_3 = 2$:

$$\binom{11}{3, 6, 2} 2^3 (-1)^2$$

8 Strings not Including Substring

How many words with 10 distinct letters don't contain "CAT", "ATE", or "MOUSE"?

Let c_1 be "must contain 'CAT"', c_2 be condition "must contain 'ATE"', and c_3 be condition "must contain 'MOUSE"'.

$$\begin{aligned}
\bar{N} &= N - N(c_1) - N(c_2) - N(c_3) \\
&+ N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3) \\
&- N(c_1 c_2 c_3) \\
&= \binom{26}{10} 10!, && \text{Any 10 letter wod} \\
&- 2\binom{23}{7} 8!, && \text{Arrange 7 + group for "CAT"/"ATE"} \\
&- \binom{21}{5} 6!, && \text{Same for "MOUSE"} \\
&+ \binom{22}{6} 7!, && \text{Same for "CATE"} \\
&+ \binom{18}{2} 4!, && \text{2 groups for "CAT" and "MOUSE"} \\
&+ 0, && \text{Can't have "ATE" and "MOUSE"} \\
&- 0, && \text{Same}
\end{aligned}$$