NOTES: BD-SIMULATION ALGORITHM

A. CHEE, Y. LIU, T. TIAN

1. Introduction & Notation

1.1. **Preliminaries.** For a continuous-time Markov process $X = (X_t)_{t\geq 0}$ on a space E, its transition semigroup $P = (P_t)_{t\geq 0}$ is defined by $P_t f(x) = \mathbb{E}_x[f(X_t)]$ for $t \geq 0$, $x \in E$ and $f \in C_0(X)$. The process X is called *Feller* if P is strongly continuous, i.e. $t \mapsto P_t f$ is continuous in the uniform topology for any $f \in C_0(X)$, in which case P has a densely defined generator $(G, \mathcal{D}(G))$ such that

$$Gf = \lim_{t\to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(G). \tag{1.1}$$

1.2. **Processes on $\mathbb{R}$.**

(1) The *squared Bessel* (BESQ_m) process $X^{(B,m)}$ of index $m \geq 0$ is a diffusion process on $\mathbb{R}_+$ that is a solution to the SDE

$$dX_t = (1+m)dt + \sqrt{2X_t}\, dW_t \tag{1.2}$$

where W is a standard Weiner process.

It's transition semigroup, which we denote $Q^{(m)}$, has generator $B^{(m)}$ that satisfies

$$B^{(m)} f(x) = x f''(x) + (1+m) f'(x), \quad x \in \mathbb{R}_+, f \in C^2(\mathbb{R}_+). \tag{1.3}$$

We will denote $Q^{(0)} = Q$ and $B^{(0)} = B$. The BESQ_m processes are 1-self-similar:

$$(\alpha X_t^{(B,m)} \mid X_0^{(B,m)} = x) \overset{(d)}{=} (X_{\alpha t}^{(B,m)} \mid X_0^{(B,m)} = \alpha x), \quad \forall x \in \mathbb{R}_+, \alpha > 0. \tag{1.4}$$

The measure $\mu_m(dx) = \frac{x^m}{\Gamma(m+1)}\, dx$ ($\mu = \mu_0$) is invariant for $Q^{(m)}$ and $Q^{(m)}$ is self-adjoint in $L^2(\mu_m)$.

(2) The *Laguerre* process $X^{L,m}$ of index $m \geq 0$ is a diffusion process on $\mathbb{R}_+$ that is a solution to the SDE

$$dX_t = (1+m-X_t)dt + \sqrt{2X_t}\, dW_t \tag{1.5}$$

where W is a standard Weiner process.

It's transition semigroup, which we denote $K^{(m)}$, has generator $L^{(m)}$ that satisfies

$$L^{(m)} f(x) = x f''(x) + (1+m-x) f'(x), \quad x \in \mathbb{R}_+, f \in C^2(\mathbb{R}_+). \tag{1.6}$$

We will denote $K^{(0)} = K$ and $L^{(0)} = L$. The semigroup $K^{(m)}$ has stationary distribution $\epsilon_m(dx) = \frac{x^m}{\Gamma(m+1)} e^{-x}\, dx$ ($\epsilon = \epsilon_0$) and $K^{(m)}$ is self-adjoint in $L^2(\epsilon_m)$. The Laguerre processes are the stationary analog of the BESQ processes and satisfy

$$K_t^{(m)} = Q_{e^t-1}^{(m)} d_{e^{-t}} = d_{e^{-t}} Q_{1-e^{-t}}^{(m)} \quad t \geq 0, \text{on } L^2(\mathbb{R}_+). \tag{1.7}$$

1.3. **Processes in $\mathbb{Z}_+$.**

(1) The *discrete squared Bessel* (dBESQ$_m$) process $\mathbb{X}^{(B,m)}$ of index m is a birth-death process on $\mathbb{Z}_+$ that has rates

$$\lambda_n = n + m + 1, \quad \mu_n = n, \quad n \in \mathbb{Z}_+. \tag{1.8}$$

Let its transition semigroup and generator be denoted $\mathbb{Q}^{(m)}$ and $\mathbb{B}^{(m)}$, respectively. The measure $\mathfrak{m}_m(n) = \frac{\Gamma(n+m+1)}{n!\Gamma(m+1)}$ is invariant for $\mathbb{Q}^{(m)}$ and $\mathbb{Q}^{(m)}$ is self-adjoint in $\ell^2(\mathfrak{m}_m)$. Let us denote $\mathfrak{m} = \mathfrak{m}_0$.

(2) The *discrete Laguerre* process $\mathbb{X}^{(L,m)}$ of index m is a birth-death process on $\mathbb{Z}_+$ that has rates

$$\lambda_n = n + m + 1, \quad \mu_n = 2n, \quad n \in \mathbb{Z}_+. \tag{1.9}$$

Let its transition semigroup and generator be denoted $\mathbb{K}^{(m)}$ and $\mathbb{L}^{(m)}$, respectively. The semigroup $\mathbb{K}^{(m)}$ has stationary distribution $\mathfrak{n}_m(n) = \frac{\Gamma(n+m+1)}{n!\Gamma(m+1)} \frac{1}{2^{n+m+1}}$ and $\mathbb{K}^{(m)}$ is self-adjoint in $\ell^2(\mathfrak{n}_m)$. Let us denote $\mathfrak{n} = \mathfrak{n}_0$.

1.4. **Intertwining Operators.** Given a measure μ^* on $\mathbb{R}_+$, we define the Hilbert space

$$L^2(\mu^*) = \{f : \mathbb{R}_+ \to \mathbb{R} \text{ measurable with } \int_0^\infty f^2(x)\mu^*(dx) < \infty\}. \tag{1.10}$$

Similarly, given a measure $\mathfrak{m}^*$ on $\mathbb{Z}_+$, we define the Hilbert space

$$\ell^2(\mathfrak{m}^*) = \{f : \mathbb{Z}_+ \to \mathbb{R} \text{ measurable with } \sum_{n=0}^{\infty} f^2(n)\mathfrak{m}^*(n) < \infty\}. \tag{1.11}$$

(1) Let $\Lambda : C_0(\mathbb{Z}_+) \to C_0(\mathbb{R}_+)$ be the Poisson kernel operator

$$\Lambda f(x) = \sum_{n \in \mathbb{Z}_+} f(n) \frac{x^n}{n!} e^{-x}, \quad x \in \mathbb{R}_+. \tag{1.12}$$

The Poisson kernel operator Λ extends to $\ell^2(\mathfrak{m}) \to L^2(\mu)$ and its dual $\Lambda^* : L^2(\mu) \to \ell^2(\mathfrak{m})$ is the Gamma kernel operator

$$\Lambda^* g(n) = \int_0^\infty g(x) \frac{x^n}{n!} e^{-x}\, dx. \tag{1.13}$$

Moreover, $\Lambda\Lambda^* = Q_1$, the BESQ-transition kernel at time 1 and $\Lambda^*\Lambda = \mathbb{Q}_1$.

(2) The Poisson kernel operator Λ also extends to $\ell^2(\mathfrak{n}) \to L^2(\epsilon)$ and its dual $\widehat{\Lambda} : L^2(\epsilon) \to \ell^2(\mathfrak{n})$ is given by

$$\widehat{\Lambda} g(n) = \int_0^\infty g(\frac{x}{2}) \frac{x^n}{n!} e^{-x}\, dx = \Lambda^* d_{\frac{1}{2}} g(x). \tag{1.14}$$

Moreover, $\Lambda\widehat{\Lambda} = K_{\ln 2}$ and $\widehat{\Lambda}\Lambda = \mathbb{K}_{\ln 2}$.

(3) The Bessel function $\mathcal{J}_\nu : \mathbb{R} \to \mathbb{R}$ of order $\nu \geq 0$ is defined

$$\mathcal{J}_\nu(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}. \tag{1.15}$$

The Hankel transform $\mathcal{H}_v : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is defined

$$\mathcal{H}_\nu f(x) = \int_0^\infty f(y) \mathcal{J}_n u(xy) \sqrt{xy}\, dy. \tag{1.16}$$

The Hankel transform is an involution, i.e. $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$ and an isometry in the sense

$$\langle f, g \rangle = \langle \mathcal{H}_\nu f, \mathcal{H}_\nu g \rangle, \quad \forall f, g \in L^2(\mathbb{R}_+). \tag{1.17}$$

For our purposes, it suffices to consider $\nu = 0$ and we denote $\mathcal{H} = \mathcal{H}_0$. In many applications, it is easier to work with the *reparametrized Bessel function*

$$J_0(x) = \mathcal{J}_0(2\sqrt{x}) = \sum_{m=0}^\infty \frac{(-x)^m}{m!m!}. \tag{1.18}$$

We can then define the *reparametrized Hankel transform*

$$Hf(x) = \int_0^\infty f(y) J_0(xy)\, dy. \tag{1.19}$$

By a substitution $z = \sqrt{y}$,

$$\begin{aligned} Hf(x) &= \int_0^\infty f(y) J_0(xy)\, dy \\ &= \int_0^\infty f(y) \mathcal{J}_0(2\sqrt{xy})\, dy \\ &= \int_0^\infty f(z^2) \mathcal{J}_0(2\sqrt{x}z)\, 2z\, dz \\ &= \frac{1}{x^{1/4}} \mathcal{H}F(2\sqrt{x}) \end{aligned}$$

where $F(x) = f(x^2)\sqrt{2x}$ provided $F \in L^2(\mathbb{R}_+)$. However,

$$\|F\|_2^2 = \int_0^\int f(x^2)^2 (2x)\, dx = \int_0^\infty f(y)^2\, dy = \|f\|_2^2$$

and

$$\|Hf(x)\|_2^2 = \int_0^\infty \mathcal{H}F(2\sqrt{x})^2 \frac{1}{\sqrt{x}}\, dx = \int_0^\infty \|\mathcal{H}F\|_2^2.$$

Hence, the reparametrized Hankel transform extends to $H : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ and is also an involution and an isometry. Moreover, for any $t > 0$,

$$Q_t f(x) = \int_0^\infty e^{-ty} J_0(xy) Hf(y)\, dy. \tag{1.20}$$

2. Simulation Algorithms and Formulae

2.1. **Base BESQ case.** The Poisson kernel operator Λ extends to $\ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$ and for $t \geq 0$,

$$Q_t \Lambda = \Lambda \mathbb{Q}_t \quad \text{on } \ell^2(\mathbb{Z}_+). \tag{2.1}$$

That is, if $f \in \ell^2(\mathbb{Z}_+)$ and $F = \Lambda f \in L^2(\mathbb{R}_+)$, then for a.e. $x \in \mathbb{R}_+$

$$\mathbb{E}_x[F(X_t^B)] = \mathbb{E}[f(\mathbb{X}_t^{(B)}) \mid X_0 \sim \mathrm{Pois}(x)]. \tag{2.2}$$

2.1.1. *Modifications:*

(1) (Self-similarity, see 1) Let t_0 be a reference time. For $t > 0$, by self-similarity, $Q_t = d_{t_0/t} Q_{t_0} d_{t/t_0}$. Thus, if $F \in L^2(\mathbb{R}_+)$ and $d_{t/t_0} F = \Lambda f_t$, then

$$\mathbb{E}_x[F(X_t^B)] = \mathbb{E}[f_t(\mathbb{X}_{t_0}^B) \mid \mathbb{X}_0^B \sim \mathrm{Pois}(x t_0/t)]. \tag{2.3}$$

(2) (Laguerre Connection, see 2) By the BESQ-Laguerre connection (1.7), for $t \geq 0$, $f \in \ell^2(\mathbb{Z}_+)$, $d_{t+1} F = \Lambda f_t$,

$$\mathbb{E}_x[F(X_t^B)] = \mathbb{E}[f_t(\mathbb{X}_{\log(t+1)}^L) \mid \mathbb{X}_0^B \sim \mathrm{Pois}(x)]. \tag{2.4}$$

(3) (Interweaving with delay) By duality and self-adjointness, for $t \geq 0$, $Q_{t+1} = \Lambda \mathbb{Q}_t \Lambda^*$. In particular, for $f \in L^2(\mathbb{R}_+)$,

$$\mathbb{E}_x[F(X_{t+1}^B)] = \mathbb{E}[F(\mathrm{Gamma}(\mathbb{X}_t^B)) \mid \mathbb{X}_0^B \sim \mathrm{Pois}(x)]. \tag{2.5}$$

Similarly, for $t \geq 0$, $K_{t+\ln 2} = \Lambda \mathbb{K}_t \widehat{\Lambda}$. Fix $t \geq 0$ and let $s = \ln(\frac{t}{2} + 1)$. We have

$$\mathbb{E}_x[F(X_{t+1}^B)] = K_{s+\ln 2} d_{t+2} F(x) = \Lambda \mathbb{K}_s \Lambda^* d_{\frac{t}{2}+1} F(x) \tag{2.6}$$

$$= \mathbb{E}[d_{\frac{t}{2}+1} F(\mathrm{Gamma}(\mathbb{X}_s^L)) \mid \mathbb{X}_0^L \sim \mathrm{Pois}(x)]. \tag{2.7}$$

2.2. **Base Laguerre case.** The Poisson kernel operator Λ extends to $\ell^2(\mathfrak{m}) \to L^2(\epsilon)$ and for $t \geq 0$,

$$K_t \Lambda = \Lambda \mathbb{K}_t \quad \text{on } \ell^2(\mathfrak{m}). \tag{2.8}$$

That is, if $f \in \ell^2(\mathfrak{m})$ and $F = \Lambda f$, then for a.e. $x \in \mathbb{R}_+$

$$\mathbb{E}_x[F(X_t^L)] = \mathbb{E}[f(\mathbb{X}_t^{(L)}) \mid X_0 \sim \mathrm{Pois}(x)]. \tag{2.9}$$

2.2.1. *Modifications:*

(1) (Interweaving with delay) For $t \geq 0$, using $K_{t+\ln 2} = \Lambda \mathbb{K}_t \widehat{\Lambda}$, we have

$$\mathbb{E}_x[F(X_{t+\ln 2}^L)] = \mathbb{E}[d_{\frac{1}{2}} F(\mathrm{Gamma}(\mathbb{X}_t^L)) \mid \mathbb{X}_0^L \sim \mathrm{Pois}(x)]. \tag{2.10}$$

2.3. **Spectral Simulation of $\mathbf{BESQ}^{(0)}$.** By the formula (1.20),

$$Q_t f(x) = \frac{1}{t}\mathbb{E}[\mathcal{J}_0(xZ)Hf(Z)] \tag{2.11}$$

where $Z \sim \mathrm{Exp}(t)$.

3. Discussion

(1) We observe that it takes $O(e^t)$ jumps in the BESQ process to reach time t. Heuristically, this comes from considering the extreme scenario when $X_t^{(B)}$ trends upwards so $X_t^{(B)}(\tau_n) \sim n$ where τ_n is the nth jump time. Then, since the inter-jump times are on average $(X_t^{(B)}(\tau_n))^{-1} \sim n^{-1}$, we have $\tau_N \sim \sum_{n=1}^N n^{-1} \approx \log N$. Hence, we need $N \sim e^t$ for $\tau_N \sim t$.

This exponential time complexity can be overcome with the self-similarity of $X^{(B)}$, see (1.4), which can be written $d_\alpha Q_{\alpha t} = Q_t d_\alpha$, where $d_\alpha f(x) = f(\alpha x)$ is the dilation operator. This comes with the tradeoff that the simulation estimate variance is increased, but the variance increase should be polynomial in the dilation factor. It follows that simulation will be of complexity will be constant in time.

(2) Alternatively, we propose to overcome the exponentially time complexity described above with simulating the BESQ process via the Laguerre process, using the relation (1.7), in which case we can similar exponentially large times for the BESQ process in linear time. In particular, for $t \geq 0$, $f \in \ell^2(\mathbb{Z}_+)$, $d_{t+1}F = \Lambda f_t$,

$$\mathbb{E}_x[F(X_t^{(B)})] = \mathbb{E}[f_t(\mathbb{X}^L_{\log(t+1)}) \mid X_0 \sim \mathrm{Pois}(x)]. \tag{3.1}$$

Since, the states for the discrete Laguerre on the RHS will be of size $O(tx)$ on average, noting that mean reversion will drive the process towards 1, the time complexity of simulation will be $O(tx \log t)$ with high probability.