NOTES: BD-SIMULATION ALGORITHM

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1. Introduction & Notation

1.1. **Preliminaries.** For a continuous-time Markov process $X = (X_t)_{t\geq 0}$ on a space E, its transition semigroup $P = (P_t)_{t\geq 0}$ is defined by $P_t f(x) = \mathbb{E}_x[f(X_t)]$ for $t \geq 0$, $x \in E$ and $f \in C_0(X)$. The process X is called *Feller* if P is strongly continuous, i.e. $t \mapsto P_t f$ is continuous in the uniform topology for any $f \in C_0(X)$, in which case P has a densely defined generator $(G, \mathcal{D}(G))$ such that

$$Gf = \lim_{t\to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(G). \tag{1.1}$$

1.2. **Processes on $\mathbb{R}$.**

(1) The *squared Bessel* (BESQ_m) process $X^{(B,m)}$ of index $m \geq 0$ is a diffusion process on $\mathbb{R}_+$ that is a solution to the SDE

$$dX_t = (1+m)dt + \sqrt{2X_t}\, dW_t \tag{1.2}$$

where W is a standard Weiner process.

It's transition semigroup, which we denote $Q^{(m)}$, has generator $B^{(m)}$ that satisfies

$$B^{(m)} f(x) = xf''(x) + (1+m)f'(x), \quad x \in \mathbb{R}_+, f \in C^2(\mathbb{R}_+). \tag{1.3}$$

We will denote $Q^{(0)} = Q$ and $B^{(0)} = B$. The BESQ_m processes are 1-self-similar:

$$(\alpha X_t^{(B,m)} \mid X_0^{(B,m)} = x) \overset{(d)}{=} (X_{\alpha t}^{(B,m)} \mid X_0^{(B,m)} = \alpha x), \quad \forall x \in \mathbb{R}_+, \alpha > 0. \tag{1.4}$$

The measure $\mu_m(dx) = \frac{x^m}{\Gamma(m+1)}\, dx$ is invariant for $Q^{(m)}$ and $Q^{(m)}$ is self-adjoint in $L^2(\mathfrak{m}_m)$.

(2) The *Laguerre* process $X^{L,m}$ of index $m \geq$ is a diffusion process on $\mathbb{R}_+$ that is a solution to the SDE

$$dX_t = (1+m-X_t)dt + \sqrt{2X_t}\, dW_t \tag{1.5}$$

where W is a standard Weiner process.

It's transition semigroup, which we denote $K^{(m)}$, has generator $L^{(m)}$ that satisfies

$$L^{(m)} f(x) = xf''(x) + (1+m-x)f'(x), \quad x \in \mathbb{R}_+, f \in C^2(\mathbb{R}_+). \tag{1.6}$$

We will denote $K^{(0)} = K$ and $L^{(0)} = L$. The semigroup $K^{(m)}$ has stationary distribution $\epsilon_m(dx) = \frac{x^m}{\Gamma(m+1)} e^{-x}\, dx$ ($\epsilon = \epsilon^{(0)}$) and $Q^{(m)}$ is self-adjoint in $L^2(\epsilon_m)$. The Laguerre processes are the stationary analog of the BESQ processes and satisfy

$$K_t^{(m)} = Q_{e^t-1} d_{e^{-t}} = d_{e^{-t}} Q_{1-e^{-t}} \quad t \geq 0, \text{on } L^2(\mathbb{R}_+). \tag{1.7}$$

1.3. **Processes in $\mathbb{Z}_+$.**

(1) The *discrete squared Bessel* (dBESQ$_m$) process $\mathbb{X}^{(B,m)}$ of index m is a birth-death process on $\mathbb{Z}_+$ that has rates

$$\lambda_n = n + m + 1, \quad \mu_n = n, \quad n \in \mathbb{Z}_+. \tag{1.8}$$

Let its transition semigroup and generator be denoted $\mathbb{Q}^{(m)}$ and $\mathbb{B}^{(m)}$, respectively. The measure $\mathfrak{m}_m(n) = \frac{\Gamma(n+m+1)}{n!\Gamma(m+1)}$ is invariant for $\mathbb{Q}^{(m)}$ and $\mathbb{Q}^{(m)}$ is self-adjoint in $\ell^2(\mathfrak{m}_m)$.

2. Simulation Algorithms and Formulae

2.1. **Base BESQ case.** Let Λ be the Poisson kernel operator

$$\Lambda f(x) = \sum_{n \in \mathbb{Z}_+} f(n) \frac{x^n}{n!} e^{-x}. \tag{2.1}$$

Then, Λ extends to $\ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$ and for $t \geq 0$,

$$Q_t \Lambda = \Lambda \mathbb{Q}_t \quad \text{on } \ell^2(\mathbb{Z}_+). \tag{2.2}$$

That is, if $f \in \ell^2(\mathbb{Z}_+)$ and $F = \Lambda f \in L^2(\mathbb{R}_+)$, then for a.e. $x \in \mathbb{R}_+$

$$\mathbb{E}_x[F(X_t^B)] = \mathbb{E}[f(\mathbb{X}_t^{(B)}) \mid X_0 \sim \text{Pois}(x)]. \tag{2.3}$$

2.2. **Base Laguerre case.** The Poisson kernel operator Λ extends to $\ell^2(\mathfrak{n}) \to L^2(\epsilon)$ and for $t \geq 0$,

$$L_t \Lambda = \Lambda \mathbb{K}_t \quad \text{on } \ell^2(\mathfrak{n}). \tag{2.4}$$

That is, if $f \in \ell^2(\mathfrak{n})$ and $F = \Lambda f$, then for a.e. $x \in \mathbb{R}_+$

$$\mathbb{E}_x[F(X_t^L)] = \mathbb{E}[f(\mathbb{X}_t^{(L)}) \mid X_0 \sim \text{Pois}(x)]. \tag{2.5}$$

3. Discussion

(1) We observe that it takes $O(e^t)$ jumps in the BESQ process to reach time t. Heuristically, this comes from considering the extreme scenario when $X_t^{(B)}$ trends upwards so $X_t^{(B)}(\tau_n) \sim n$ where τ_n is the nth jump time. Then, since the inter-jump times are on average $(X_t^{(B)}(\tau_n))^{-1} \sim n^{-1}$, we have $\tau_N \sim \sum_{n=1}^N n^{-1} \approx \log N$. Hence, we need $N \sim e^t$ for $\tau_N \sim t$.

This exponential time complexity can be overcome with the self-similarity of $X^{(B)}$, see (1.4), which can be written $d_\alpha Q_{\alpha t} = Q_t d_\alpha$, where $d_\alpha f(x) = f(\alpha x)$ is the dilation operator. This comes with the tradeoff that the simulation estimate variance is increased, but the variance increase should be polynomial in the dilation factor.

Alternatively, we propose to overcome the exponentially time complexity described above with simulating the BESQ process via the Laguerre process, using the relation (1.7), in which case we can similar exponentially large times for the BESQ process in linear time.

(2) .