Honors ODE Team Project Report: Geodesics

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Abstract

This report gives a brief introduction to geodesics and derives the geodesic equation with Euler-Lagrange Equation by finding both action-minimizing and length-minimizing curves. To better understand the equation of motion, the report includes proofs of the existence and uniqueness of geodesics and explanations of equivalent action and symmetry. Besides of the theoretical aspects, the report also includes a discussion of various applications of geodesics, including geodesic patterns on different manifolds. Both analytical and numerical results are presented, a generic geodesic solver is provided, and a practical application is introduced.

1 Introduction and Objectives

A geodesic is the shortest path between two fixed points for motion constrained to lie in a surface. It is a straight line on the plane and is a great circle on the sphere. It itself has many interesting properties and works as a useful tool for exploring other geometries. In this report, we will derive geodesic equation, study its properties, and apply it to different geometries.

2 Derivation

2.1 Derivation of the Euler-Lagrange Equation

Suppose we want to minimize a path length

$$\Phi(\gamma) = \int_a^b |d\vec{\gamma}(t)|,$$

where $\gamma(t)$ is a vector valued function through space. This is defines a functional $\Phi(\gamma)$ taking a function $\gamma : I \to \mathbb{R}^n$ as its input and yielding a value in $\mathbb{R}$ as its output. A function like S that that takes a function as input and yields a number is called a functional. We may consider a variation to γ given by $(\gamma + h)(t) = \gamma(t) + h(t)$, where $h : I \to \mathbb{R}^n$ is small in its values and derivatives.

Definition. A functional Φ is called differentiable if $\Phi(\gamma + h) - \Phi(\gamma) = F + R$, where F depends linearly on h (i.e. $F_\gamma(h_1 + h_2) = F_\gamma(h_1) + F_\gamma(h_2)$), and $R_\gamma(h) = O(h^2)$ in the sense that for $|h| < \epsilon$ and $\left|\frac{dh}{dt}\right| < \epsilon$ (uniform bounds), we have $|R_\gamma| < C\epsilon^2$. The linear part of the increment, $F(h)$, is called the differential.

- If Φ is differentiable, its differential is **uniquely** defined. The differential of a functional is called its variation, and h is called a variation of the curve.

Theorem. Let $L = L(a, b, c)$ be a differentiable function of three variables. Then the functional

$$\Phi(x) = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t)dt$$

is differentiable and its derivative is given by the formula

$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] h \; dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Bigg|_{t_0}^{t_1}$$

Proof:

$$\begin{aligned}\Phi(x+h)-\Phi(x) &= \int_{t_0}^{t_1} L(x+h,\dot{(x+h)},t)dt - \int_{t_0}^{t_1} L(x,\dot{x},t)dt \\ &= \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right)dt + O(h^2) \Rightarrow \Phi \text{ differentiable.}\end{aligned}$$

Where $F(h)=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right)$ is linear and $R(h)=O(h^2)$. Further, by product rule,

$$\begin{aligned}F(h) &= \int_{t_0}^{t_1}\frac{\partial L}{\partial x}h\,dt + \left(\frac{\partial L}{\partial \dot{x}}\right)\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)hdt \\ &= \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]h\,dt + \left(\frac{\partial L}{\partial \dot{x}}h\right)\Bigg|_{t_0}^{t_1}.\end{aligned}$$

□

Definition. An extremal of a differentiable functional $\Phi(\gamma)$ is a curve γ such that $F(h)=0$ for all h.

It can be easily recognized that for differentiable functionals, if its value is either locally maximized or minimized on a function γ, then the extremal condition must be satisfied.

Theorem. The curve $x(t)$ is an extremal of the functional $\Phi(x)=\int_{t_0}^{t_1}L(x(t),\dot{x}(t),t)dt$ on the space of the curves passing through te points $x(t_0)=x_0$ and $x(t_1)=x_1$ iff

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0.$$

Proof. Lemma. If a continuous function $f(t)$, $t_0\le t\le t_1$, satisfies $\int_{t_0}^{t_1}f(t)h(t)dt=0$ for any continuous function $h(t)$ with $h(t_0)=h(t_1)=0$, then $f(t)\equiv 0$.

$$F(h)=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]h\,dt + \left(\frac{\partial L}{\partial \dot{x}}h\right)\Bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]h(t)dt.$$

Now $h(t_0)=h(t_1)=0$. If γ is extremal, then $F(h)=0$ for all h with $h(t_0)=h(t_1)=0$, thus

$$\int_{t_0}^{t_1}f(t)h(t)dt=0$$

where

$$f(t)=\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

for all such h. By the lemma, $f(t)\equiv 0$. Conversely, if $f(t)\equiv 0$, then clearly $F(h)\equiv 0$. □

2.2 Derivation of Geodesics

In Riemannian Space, we denote the distance between two points with metric

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu$$

where g is the metric tensor with $g_{ij}=\hat{e_i}\cdot\hat{e_j}$. For instance

$$\begin{aligned}&= dx^2+dy^2+dz^2 \quad \text{Cartesian}\\ &= dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \quad \text{Spherical}\end{aligned}$$

Locally, the metric tensor g describes an inner product dx^Tgdx on the linear tangent space T_pM that is isometric to the Euclidean embedding of M. By definitions of an inner product, g must be symmetric

and hence positive-definite.
Therefore, the integral that denote the length of the curve linking two points can be described as

$$\begin{aligned} l &= \int_a^b \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \\ &= \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} dt \end{aligned}$$

Minimizing the length gives us a geodesic. Consider the curve that describes action along the path.

$$\begin{aligned} S &= \int_a^b \frac{1}{2} (\frac{ds}{dt})^2 dt \\ &= \frac{1}{2} \int_a^b g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt \end{aligned}$$

Borrowing the result of Euler Lagrange Equation, we have

$$\begin{aligned} L &= \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} &- \frac{\partial L}{\partial x^\mu} = 0 \end{aligned}$$

$$\frac{d}{d\tau} [g_{\mu\nu} \frac{dx^\nu}{d\tau}] - \frac{1}{2} \partial_\mu g_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{1}$$

which is also commonly expressed using christoffel symbol

$$\frac{d^2 x^\sigma}{dt^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0 \tag{2}$$

,where

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}]$$

Here we reach the geodesic equation through minimizing action. Now let's show that it also minimizes the length between two points.
Use a Lagarange multiplier λ as a function of t.

$$S = \frac{1}{2} \int_a^b [g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \lambda^{-1}(t) + \lambda(t)] dt$$

Taking its derivative with respect to λ

$$\begin{aligned} \frac{\delta S}{\delta \lambda(t_1)} &= \lim_{\epsilon \to \infty} \frac{1}{2} \int_a^b [g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} (\lambda + \epsilon\delta(t_1 - t))^{-1}(t) + \lambda + \epsilon\delta(t_1 - t)] dt \\ &= -\lambda(t_1)^{-2} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}(t_1) + 1 \\ &= (|\dot{x}|^2 \lambda^{-2})|_{t_1} + 1 \end{aligned}$$

which is minimized when

$$\lambda = \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}$$

Plugging it back to S,

$$\begin{aligned} S &= \frac{1}{2} \int_a^b [g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} (\sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}})^{-1}(t) + (\sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}})] dt \\ &= l \end{aligned}$$

By parameterizing the curve with constant velocity, we show that the length minimizing curve also minimizes action.

3 Properties

3.1 Existence and Uniqueness

The geodesic equation allows us to explore some properties of geodesics.

Theorem 3.1. *(Existence and Uniqueness of Geodesics) Let M be a connected manifold. For any initial point $p \in M$, any $V \in T_pM$, and any $t_0 \in M$, there exist an open interval $I \subset \mathbf{R}$ containing t_0 and a geodesic $\Lambda : I \to M$ satisfying $\Lambda(t_0) = p, \dot{\Lambda}(t_0) = V$ and such geodesics agree in their common domain.*

Proof. Consider the neighborhood P around p with coordinate (x^i). If a curve Λ is a geodesic, for $t \in \mathbf{R}$, $\Lambda(t) = (x^1(t), \cdots, x^n(t))$ needs to satisfy

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

Geodesic equation is a second-order system of ordinary differential equations for the functions $x^i(t)$. To show existence and uniqueness of solution to a second order differential system of equations, we convert it to a first-order system through auxiliary variables $v^i = \frac{dx^i}{dt}$. The number of variables is doubled here.

$$\frac{dx^k(t)}{dt} = v^k(t)$$
$$\frac{dv^k(t)}{dt} + v^i(t)v^j(t)\Gamma^k_{ij}(x(t)) = 0$$

Following the existence and uniqueness theorem for first-order ODEs (For any $(p, V) \in U \times \mathbf{R}^n$, there exist $\epsilon > 0$ and a unique solution $Y : (t_0 - \epsilon, t_0 + \epsilon) \to P \times \mathbf{R}^n$ to this system satisfying the initial condition $Y(t_0) = (p, V)$. Re-express components of Y as $Y(t) = (x^i(t), v^i(t))$, then the existence of the geodesic curve $\Lambda = (x^i(t)) \in U$ follows.
To prove the uniqueness claim, suppose $\Lambda, \Theta : I \to M$ are geodesics defined on an open interval that both satisfy the initial condition such that $\Lambda(t_0) = \Theta(t_0), \dot{\Lambda}(t_0) = \dot{\Theta}(t_0)$. Following the uniqueness of ODE, there exists some neighborhood that they agree. Let β be the supremum of numbers b such that they agree on $[t_0, b]$. If $\beta \in I$, then by continuity $\gamma(\beta) = \sigma(\beta), \dot{\gamma}(\beta) = \dot{\sigma}(\beta)$, and applying local uniqueness in a neighborhood of β, there exists a larger neighborhood that they agree, which contradicts our assumption. Following similar logic, they agree on all interval I. □

Following the claim of uniqueness, for any $p \in M, V \in T_pM$, there is a unique maximal geodesic $\Lambda : I \to M$ satisfying the initial condition (p, V) for some open interval I_i. Let I be the union all I_i. Notice that geodesics agree where they overlap, we define this maximal to be the geodesic Λ_V with initial (p, V)

3.2 Equivalence of Action

After exploring the existence of geodesics between two points, let us explore ways to create same equation of motions.

Theorem 3.2. *If two Lagrangians L_1, L_2 give the same equations of motion then there is a function $\Phi(q, t)$ such that*

$$L_1 - L_2 = \frac{d\Phi}{dt}$$

Proof. Suppose L_1, L_2 gives the same equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \Lambda(q, \dot{q}, \ddot{q}, t)$$

Define

$$\begin{aligned}
\Psi(q,\dot{q},t) &= L_1 - L_2 \\
\Lambda_1 - \Lambda_2 &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}}(L_1 - L_2) - \frac{\partial}{\partial q}(L_1 - L_2) \\
&= (\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q})\Psi \\
&= \frac{\partial^2 \Psi}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 \Psi}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial^2 \Psi}{\partial \dot{q}\partial t} - \frac{\partial \Psi}{\partial \dot{q}} \\
&= 0
\end{aligned}$$

By observing the order of $\Lambda_1 - \Lambda_2$, we can tell for this to be true for any $\ddot{q}, \dot{q}, q, t$, Ψ has to be linear in $\dot{q}$, which means for some f,g

$$\Psi = F(q,t)\dot{q} + G(q,t)$$

plugging it back to the differential equation,

$$\frac{\partial F}{\partial t} - \frac{\partial G}{\partial q} = 0$$

where

$$\Psi(\dot{q}, q, t) = \dot{q}\frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial t}$$

Therefore,

$$L_1 - L_2 = \frac{d\Phi}{dt}$$

□

3.3 Symmetry

Solving geodesic equations, a set of four coupled, second order ordinary differential equations, can be tricky on less simple metrics. One commonly used strategy is finding symmetry and conservation law such that we get our first integrals of the equations of motion. A symmetry implies the existence of a conserved quantity along the curve, which can help us find the geodesic easier.

4 Application

4.1 Analytically Solving Geodesics

Recall that the Euler-Lagrange Equation is,

$$\frac{\partial f}{\partial y} - \frac{d}{dy}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{3}$$

Writing $f = f(x, y, y')$, we can get

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} \\
&= \frac{\partial f}{\partial x} + \frac{d}{dy}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} \\
&= \frac{\partial f}{\partial x} + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)
\end{aligned}$$

Then we have a second form of Euler's Equation

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'}y'\right) = 0 \tag{4}$$

We have the following special cases:

- If $f = f(x, y')$, then $\frac{\partial f}{\partial y} = 0$, so from (3),

$$\frac{\partial f}{\partial y'} = C \tag{5}$$

.

- If $f = f(y, y')$, then $\frac{\partial f}{\partial x} = 0$, so from (4),

$$f - \frac{\partial f}{\partial y'}y' = C \tag{6}$$

Recasting into curvilinear coordinates, then the surface $g(x, y, z) = 0$ has the form $x = x(u, v), y = y(u, v), z = z(u, v)$. Then

$$\begin{aligned}(ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= Pdu^2 + 2Qdudv + Rdv^2\end{aligned}$$

where

$$\begin{aligned}P &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ R &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\end{aligned}$$

Then the Laguargian becomes

$$L = \int_{u_1}^{u_2} f(u, v, v')du$$

for $f = f(u, v, v') = \sqrt{P + 2Qv' + 2R(v')^2}$ with

$$\frac{\frac{\partial P}{\partial v} + 2v'\frac{\partial Q}{\partial v} + (v')^2\frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + 2R(v')^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + 2R(v')^2}}\right) = 0 \tag{7}$$

Here we have the following special cases:

- If P, Q, R are explicit functions of u alone and $Q = 0$, then (7) yields $\frac{Rv'}{\sqrt{P+Rv'}} = C$ and therefore

$$v = C\int \frac{\sqrt{P}}{\sqrt{R^2 - C^2R}}du \tag{8}$$

- If P, R are explicit functions of u alone and $Q = 0$, then (7) yields $v'\frac{\partial f}{\partial v'} - f = C$ and therefore

$$u = C\int \frac{\sqrt{P}}{\sqrt{R^2 - C^2R}}dv \tag{9}$$

4.1.1 Geodesics on a plane

For a plane, $L = \int ds = \int_{x_1}^{x_2} \sqrt{1+(y')^2}dx$. Here $f(x,y,y') = \sqrt{1+(y')^2}$ is independent of y, so by (5),

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}} = C$$

The solution is in the form

$$y = C_1 x + C_2$$

Here the geodesics is a line in the plane.

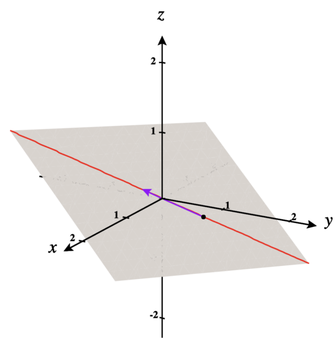

Figure 1: Geodesics on Plane

4.1.2 Geodesics on a sphere

For a sphere, we use the polar coordinates

$$x = r\sin\phi\cos\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\theta$$

The Lagrangian is

$$L = \int ds = \int_{\theta_1}^{\theta_2} \sqrt{P^2 + 2Q\phi' + R(\phi')^2}d\theta$$

where

$$P = r^2\sin^2\phi \qquad Q = 0 \qquad R = r^2$$

So by (9), this have the solution

$$\begin{aligned}\theta &= C\int \frac{\sqrt{P}}{\sqrt{R^2 - C^2R}}d\theta \\ &= -\sin^{-1}\frac{\cot\phi}{\sqrt{(r/C)^2-1}} + C_1\end{aligned}$$

The geodesics is

$$x\sin C_2 - y\cos C_2 - \frac{z}{\sqrt{(r/C_1)^2-1}} = 0$$

Notice that this is an expression of a great circle in the sphere.

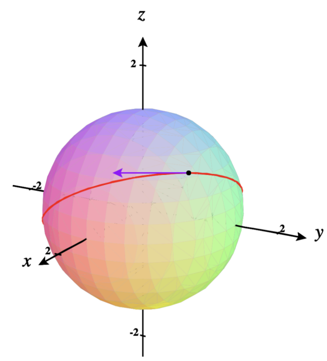

Figure 2: Geodesics on Sphere

4.1.3 Geodesics on a cylinder

For a cylinder, we use the coordinates

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Then

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (rd\theta)^2 + dz^2$$

and the Lagrangian is

$$L = \int ds = \int_{z_1}^{z_2} \sqrt{r^2\theta'^2 + 1}dz$$

Here $f = f(\theta')$. So by (8), we have solution

$$\theta = C_1 z + C_2$$

This is an expression of a spiral.

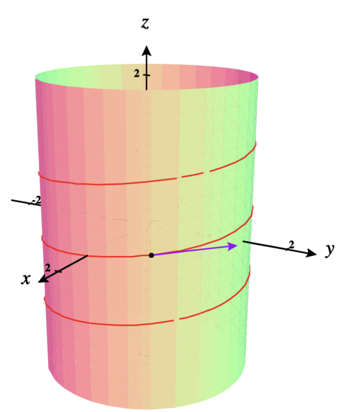

Figure 3: Geodesics on Cylinder

4.1.4 Geodesics on a cone

For a cones, we use the coordinate

$$x = r\sin\phi\cos\theta \qquad\qquad y = r\sin\phi\sin\theta \qquad\qquad z = r\cos\theta$$

Then

$$(ds)^2 = dr^2 = (r\sin\theta)^2 d\theta^2$$

and the Lagrangian is

$$L = \int ds = \int_{r_1}^{r_2} \sqrt{1 + (r\sin\phi)^2\theta'^2} dr$$

Here f is independent of θ. So from (8),

$$\frac{\partial f}{\partial \theta'} = C$$

and

$$\theta = \frac{1}{\sin\phi} \sec^{-1}\left(\frac{r\sin\phi}{C_1} + C_2\right)$$

This is also an expression of a spiral.

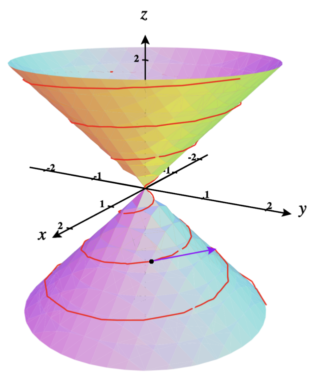

Figure 4: Geodesics on Cone

4.1.5 Geodesics on a hyperbolic paraboloid

For a hyperbolic paraboloid, we use the coordinate

$$x = \sinh\theta \quad y = \cosh\theta \quad z = \phi$$

Then

$$(ds)^2 = (\cosh^2\theta + \sinh^2\theta)d\theta^2 + d\phi^2 = \cosh 2\theta d\theta^2 + d\phi^2$$

and the Lagrangian is

$$L = \int ds = \int_{\theta_1}^{\theta_2} \sqrt{\cosh 2\theta + v'^2} d\theta$$

Here f is independent of ϕ, so from (3),

$$\frac{\partial f}{\partial \phi'} = C$$

Therefore

$$\phi = C \int \sqrt{\cosh 2\theta} d\theta$$

There is no closed form of this integral, but it turns out that this could be put in the for of an incomplete elliptic integral of the second kind, with

$$\phi = -iE(i\theta \mid 2) + C$$

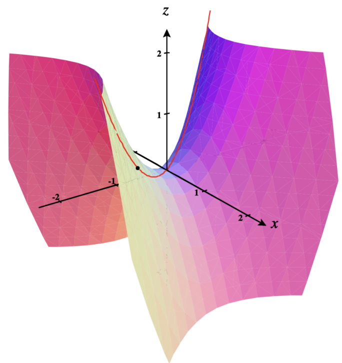

Figure 5: Geodesics on Hyperbolic Paraboloid

4.2 Geodesics on surfaces of revolutions

Let's take our attention to a more general case, where the surface is generated by a cross section revolved around the z axis. For such a surface we can express it in cylindrical coordinates (ϕ, r, z). It is holonomic in coordinates z, ϕ, with $r = r(z)$. Lagrangian of path γ parameterized by t:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\left(\left(1 + \left(\frac{\partial z}{\partial r}\right)^2\right)\dot{r}^2 + r^2\dot{\phi}^2\right).$$

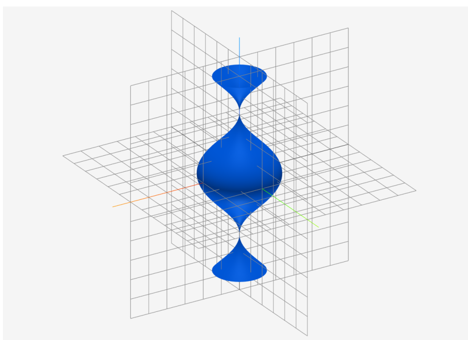

Figure 6: A revolutionary surface parameterized by $r(z) = (1 + cos(z))$

As L is cyclic in ϕ,

$$\frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} = c_1$$

for some constant c.

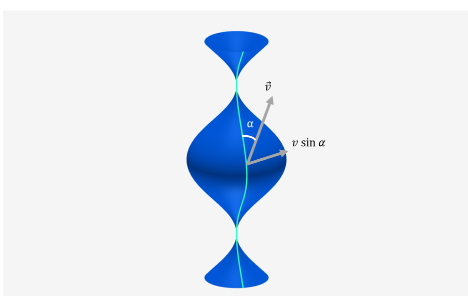

Figure 7: Same surface with the meridian (cyan) at $\phi = 2.38$

Denote by α the angle of the orbit with a meridian (the intersection of the rotated plane $\phi = \phi_0$ with the surface of rotation). Then $v_\phi = r\dot{\phi} = |v| \sin\alpha$. This is because the radial velocity $r\dot{\phi}$ is the component of velocity $\vec{v}$ that is perpendicular to the inclination basis vector along the meridian. Now as discussed in section 2.2, this parameterization of $\gamma(t)$ obeys constant velocity, so $|v| \equiv c_2$. Combined, we have

$$r^2\dot{\phi} = r(r\dot{\phi}) = r|v|\sin\alpha = rc_2 \sin\alpha = c_1,$$

$$\Rightarrow r\sin\alpha = \frac{c_1}{c_2} = C,$$

where C stays constant over the entire geodesic.

This relation shows that:

1. the motion takes place in the region $|\sin\alpha| \leq 1$ with initial α_0 and r_0, $r\sin\alpha \equiv r_0 \sin\alpha_0$, so

$$\sin\alpha = \frac{r_0 \sin\alpha_0}{r} \Rightarrow \left|\frac{r_0 \sin\alpha_0}{r}\right| \leq 1 \Rightarrow r \geq r_0 |\sin\alpha_0|.$$

2. Inclination of the orbit increases as radius increases, $\sin\alpha = \frac{c}{r}$. It becomes horizontal when r reaches the smallest value $r = r_0|\sin\alpha_0|$, the orbit is reflected and return to region with larger r.

3. Since $r \geq r_0|\sin\alpha_0| > 0$, $\sin\alpha = \frac{r_0 \sin\alpha_0}{r}$ never changes sign from the beginning. This also says that if a geodesic starts with a positive/negative v_ϕ component, *then it will always keep that component.*

In general, geodesics on a convex surface can be divided into 3 classes: meridians (C = 0 $\Rightarrow \alpha \equiv 0$), closed curves, and geodesics dense in a ring ($r \geq C$).

4.3 Numerically computed geodesics on surfaces of revolutions

To numerically express the inclination angle α, we shall consider the local linear space spanned by the meridian basis vector $\hat{m}$ and the rotational basis vector $\hat{\phi}$. Then for a normalized velocity vector $\hat{v}$, its components are given by $\sin\alpha\hat{\phi} + \cos\alpha\hat{m}$. $\hat{m}$ can be explicitly given by $(\frac{\partial r}{\partial z}(\cos\phi\hat{i} + \sin\phi\hat{j}) + \hat{z})/||...||$. Then at each time step t_i, we shall find v_ϕ by taking

$$v_\phi = v\sin\alpha = 1 \cdot \frac{r_0 \sin\alpha_0}{r} = \frac{r_0 \sin\alpha_0}{r}.$$

So ϕ gets incremented by

$$\Delta\phi = \Delta t\dot{\phi} = \Delta t \frac{v_\phi}{r} = \frac{r_0 \sin\alpha_0}{r^2}\Delta t.$$

Now change in z in each time step is given by

$$v_z = \cos\alpha\hat{m} \cdot \hat{z} = \cos\alpha \cdot 1/\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2} = \pm\sqrt{1 - \left(\frac{r_0 \sin\alpha_0}{r}\right)^2}/\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2},$$

$$\Delta z = \pm\Delta t\sqrt{1 - \left(\frac{r_0 \sin\alpha_0}{r}\right)^2}/\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}.$$

The determination of the sign of Δz should follow **locality**, where if it was previously larger than 0, then it should be kept as larger than 0, and vice versa. In the case where it is too close to 0 to predict the next sign, use the first order derivative of v_s to predict if the next v_z should be larger or smaller than 0.

Notice that when making the actual implementation, to avoid the problem where v_z goes to 0 at an exponential rate as r approach $r_0|\sin\alpha_0|$, we take the increment of z up to a second order Taylor expansion, namely: $\Delta z = v_z dt + \frac{1}{2}\dot{v}_z dt^2$. This prevents the geodesics from falsely converging to the radius limit and makes the trace more accurate.

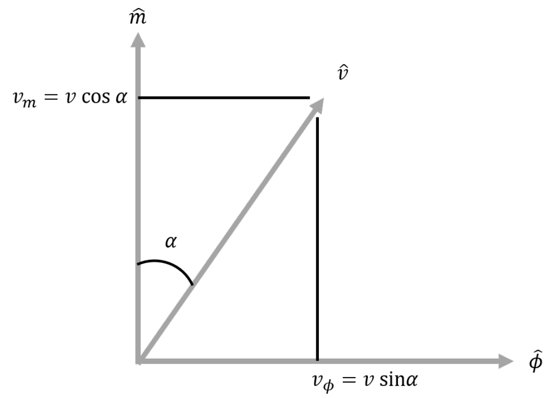

Figure 8: Geometric components of $\hat{v}$

4.3.1 Numerical result on sphere

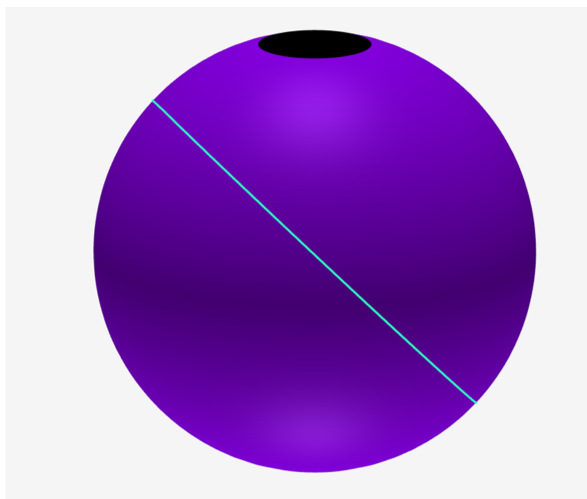

Figure 9: The numerical geodesic on sphere of radius 3 with initial $\alpha_0 = \frac{2}{5}\pi$, $z_0 = -2$, $\phi_0 = 0$ after fixing v_z sign flip

4.3.2 Numerical geodesic on cylinder

Figure 10: A numerical geodesic on cylinder of radius 1 with initial $\alpha_0 = 1$, $z_0 = -5$, $\phi_0 = 0$

4.3.3 Numerical geodesic on surface $r = 1 + \cos(z)$

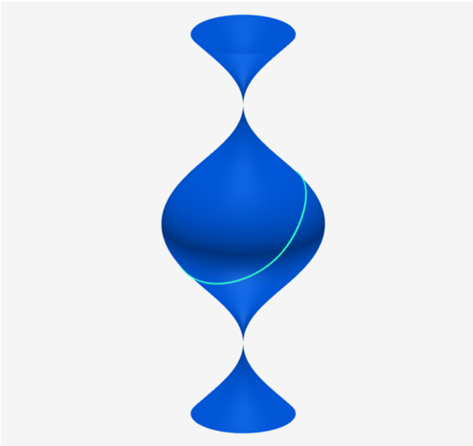

Figure 11: Numerical geodesic on $r = 1 + \cos(z)$ with initial conditions $\alpha_0 = \frac{2\pi}{5}$, $z_0 = 1$, $\phi_0 = 0$

4.3.4 Numerical geodesic on surface $r = 3\left(\frac{z}{2}+1\right)^2 \frac{z}{2}$

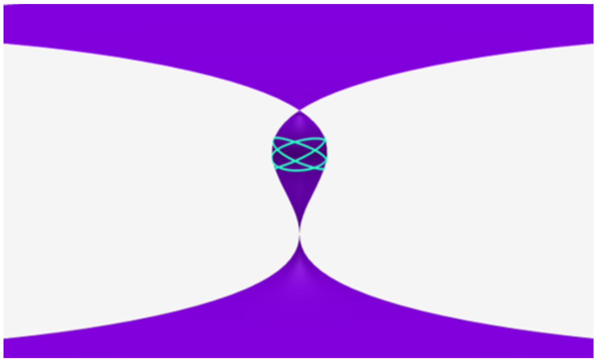

Figure 12: Interesting periodic orbits show up on this surface with initial conditions $\alpha_0 = -\frac{2}{5}$, $z_0 = 0.87$
.

4.3.5 Generic geodesic solver

Our original design of the numerical solver for geodesics on arbitrary revolutionary surfaces is available at **Pendulum Geodesics**. Modify v in the first argument to specify the expression of $r(z)$. Modify the second argument to specify α.

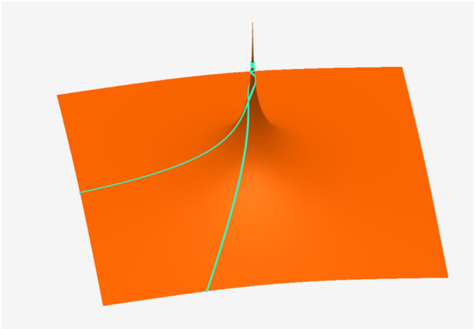

Figure 13: Geodesic on exponential revolutionary surface

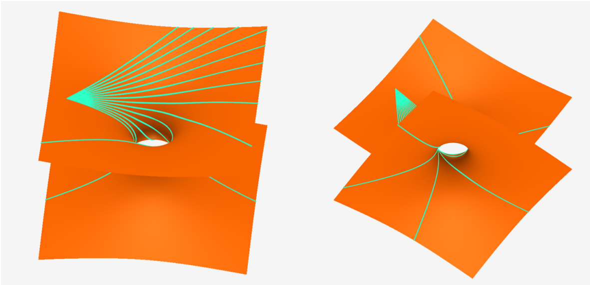

Figure 14: Geodesic near a "wormhole"

4.4 Practical Application

4.4.1 Flight Path Planning

To plan the path for a flight connecting two cities, we could treat the earth as a sphere. As we know that the geodesic on a sphere is the great circles, the path and shortest distance could be easily described and computed.

Suppose the latitudes and colatitudes (90 degrees minus the latitude) of city 1 and city 2 are θ_1, θ_2, ϕ_1, and ϕ_2. Then, the vector representing the position of two cities relative to the center of the earth could be expressed as follows:

$$\overrightarrow{OP} = (R\sin\phi_1\cos\theta_1, R\sin\phi_1\sin\theta_1, R\cos\phi_1)$$
$$\overrightarrow{OQ} = (R\sin\phi_2\cos\theta_2, R\sin\phi_2\sin\theta_2, R\cos\phi_2)$$

where R is the radius of the earth. Using the trigonometry formula

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} = R^2\cos\alpha$$

we could get the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Lastly, the distance between the two cities, s, is given by

$$s = R\alpha$$

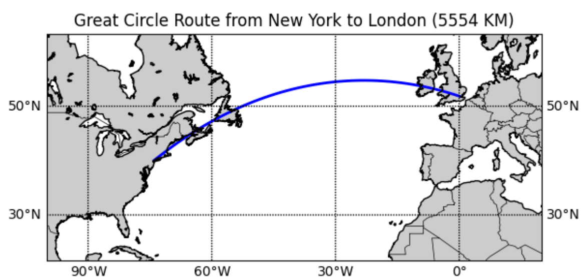

Figure 15: Great Circle Path between London and New York

A Codes

Code implemented for numerical geodesics calculation (Pendulum [1])

```
computeGeodesic(){
    let positions: number[] = [];
    let phi = 0;
    let z = this.z0;
    let alpha0 = this.alpha0;
    let r0 = this.dataInterface(phi, z)[0];
    let C = r0*Math.sin(alpha0);
    let pdz = this.pdz(phi, z);
    let oldVz = Math.cos(alpha0)/Math.sqrt(1+pdz*pdz);
    let dVz = 0;
    let dt = 0.01
    for(let t = 0; t<=15; t+=dt){
        let r = this.dataInterface(phi, z)[0];
        positions.push(Math.cos(phi)*r,Math.sin(phi)*r, z);
        pdz = this.pdz(phi, z);;
        phi = phi+dt*C/r/r;
        let cosl = Math.sqrt(1-(C/r)*(C/r));
        let vz = cosl/Math.sqrt(1+pdz*pdz);
        if(oldVz<0)
            vz = -vz;

        z+=dt*vz;
        //Test for false inversion
        r = this.dataInterface(phi, z)[0];
        pdz = this.pdz(phi, z);
        if(1-(C/r)*(C/r)<0){//Invert z increment at boundaries
            vz=-vz;
            z+=2*dt*vz;
        }
        z+=1/2*dt*dt*dVz;
        dVz = (vz-oldVz)/dt;
        oldVz = vz;
    }
    this.geodesic.geometry.setPositions(positions);
}
```

References

[1] Yuelong Li. Pendulum calculator.

[2] John M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Springer New York, NY, 1997.

[3] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer New York, NY, 1978.

[4] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*. Westview Press, 2000.

[5] Jay Villanueva. Geodesics on surfaces by variational calculus.