# Relation Between the Mass-Spring System and the Dynamic Speckle

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**Abstract**

In this article will be studied the relation between the mass-spring system and the dynamic speckle.

*Keywords:* Biospeckle laser, Biospeckle signal, Dynamic speckle,

## 1. Introduction

The biospeckle laser analysis has presented as a versatile tool in the analysis of biological activity.

## 2. System description

The Fig. 1.a) represents the signal $z$ with samples $z(n)$, obtained in a pixel of a dynamic speckle analysis, where $E[z]$ indicates the mean value of $z$. Be

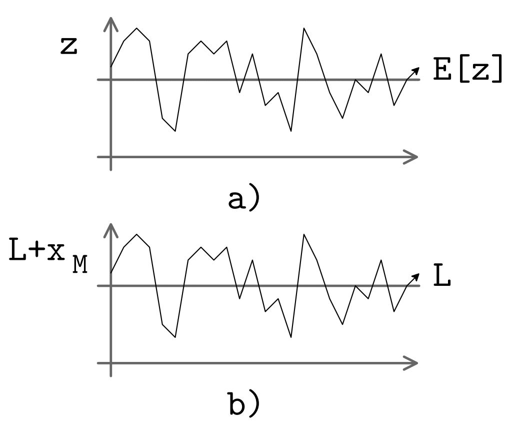


Figure 1: Data acquisition system setup of the coffee seed.

other side the Fig. 1.b) represents the signal $x_M$ with samples $x_M(n)$, obtained in a mass-spring system of $M$ elements, where each mass is separated of another by a distance of $L/M$, like can be seen in the Fig. 2. Thus, in this system the

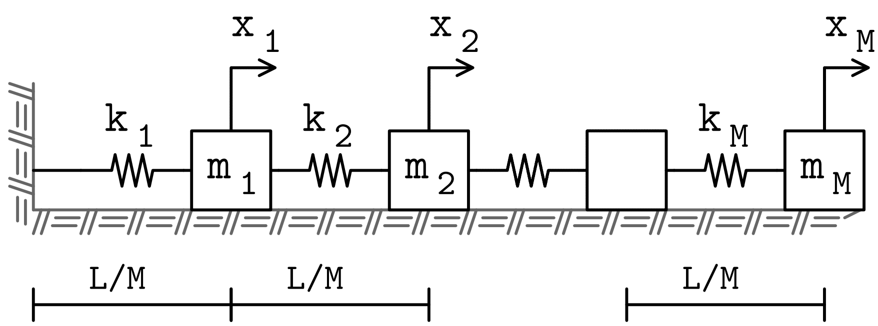


Figure 2: Data acquisition system setup of the coffee seed.

mass are denoted as $m_i$, the springs as $k_i$ and the displacements of each mass by $x_i$, for all $1 \leq i \leq M$.

The objective of this work it is to solve the next inverse problem: Known $y(n)$,

$$y(n) = z(n) - E[z] \tag{1}$$

and assuming $M$ elements with $m_i = m = 1/L$; what values of $k_i$ generate a signal $x_M$ that minimize $E$, where

$$E = \frac{1}{2}\sum_{n}(y(n) - x_M(n))^2 \tag{2}$$

## 3. Mass-spring system

Assuming a mass spring system like seen in the Fig. 2 with $m_i = m$ we can to get the system of Eq. (3).

$$m\ddot{\mathbb{X}} = -\mathbf{P}\mathbb{X}, \tag{3}$$

where

$$\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} \tag{4}$$

and

$$\mathbf{P}(\mathbf{K}) \equiv \mathbf{P} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \dots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{N-1}+x_N & -x_N \\ 0 & 0 & 0 & \dots & -x_N & x_N \end{pmatrix}, \tag{5}$$

so that $\mathbf{P}$ is a function of $\mathbf{K} = (k_1\ k_2\ k_3\ \dots\ k_{M-1}\ k_M)^T$.

### *3.1. Exact solution*

Knowing the system shown in the Eq. (3), we can solve It using the Eq. (6),

$$\mathbb{X}(t) = \mathbf{V}\left(\mathbf{D}_1 cos(\mathbf{w}t) + \mathbf{D}_2 sin(\mathbf{w}t)\right), \tag{6}$$

or Eq. (7)

$$\mathbb{X}(t) = \mathbf{V}\left(cos(\mathbf{W}t)\mathbf{d}_1 + sin(\mathbf{W}t)\mathbf{d}_2\right), \tag{7}$$

where, $\mathbf{V} = (e_1, e_2, \dots, e_M)$ and $\mathbf{w} = \left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_M}\right)^T$ are a matrix and a column vector conform using the eigenvectors $e_i$ and eigenvalues $\lambda_i$ of $\mathbf{P}/m$, being $\mathbf{W}$ a diagonal matrix conform with the elements of vector $\mathbf{w}$. By other side, $\mathbf{D}_1$ and $\mathbf{D}_2$ are two any constant diagonal matrices conform by the elements of column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ respectively. Thus, we now that $\dot{\mathbb{X}}(t)$ and $\ddot{\mathbb{X}}(t)$ are defined by the Eqs. (8) and (9) respectively.

$$\dot{\mathbb{X}}(t) = \mathbf{V}\left(-\mathbf{D}_1\mathbf{W} sin(\mathbf{w}t) + \mathbf{D}_2\mathbf{W} cos(\mathbf{w}t)\right), \tag{8}$$

$$\ddot{\mathbb{X}}(t) = -\mathbf{V}\left(\mathbf{D}_1\mathbf{W}^2 cos(\mathbf{w}t) + \mathbf{D}_2\mathbf{W}^2 sin(\mathbf{w}t)\right), \tag{9}$$

thus, It is fulfill that $\mathbf{W}^2 = (VD_1)^{-1}(P/m)(VD_1) = (VD_2)^{-1}(P/m)(D_2)$.

### *3.1.1. Constant values from two points*

Now, to get the constant values in the column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, we can use the Eq. (10)

$$\begin{pmatrix} \mathbf{V}cos(\mathbf{W}t_1) & \mathbf{V}sin(\mathbf{W}t_1) \\ \mathbf{V}cos(\mathbf{W}t_2) & \mathbf{V}sin(\mathbf{W}t_2) \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{X}(t_1) \\ \mathbb{X}(t_2) \end{pmatrix} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} \tag{10}$$

### *3.1.2. Constant values from position and velocity of a point*

Now, to get the constant values in the column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, we can use the Eq. (11)

$$\begin{pmatrix} \mathbf{V}cos(\mathbf{W}t_1) & \mathbf{V}sin(\mathbf{W}t_1) \\ -\mathbf{V}\mathbf{W}sin(\mathbf{W}t_1) & \mathbf{V}\mathbf{W}cos(\mathbf{W}t_1) \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{X}(t_1) \\ \dot{\mathbb{X}}(t_1) \end{pmatrix} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} \tag{11}$$

### *3.2. Finite differences: Knowing two consecutive samples*

Applying finite differences we known that $\mathbb{X} \equiv \mathbb{X}(n)$ and $\ddot{\mathbb{X}} \equiv (\mathbb{X}(n+1)$ $-2\mathbb{X}(n) + \mathbb{X}(n-1))/\tau^2$, so that the Eq. (3) can be rewrite as

$$\mathbb{X}(n) = \left(2\mathbf{I} - \mathbf{P}\frac{\tau^2}{m}\right)\mathbb{X}(n-1) - \mathbb{X}(n-2), \tag{12}$$

now deriving $\mathbb{X}(n)$ by the vector $\mathbf{K} = (k_1\ k_2\ k_3\ \ldots\ k_{M-1}\ k_M)$, so that $\mathbb{J}(n) \equiv \frac{\partial \mathbb{X}(n)}{\partial \mathbf{K}}$, we get the Eq. (13)

$$\mathbb{J}(n) = -\frac{\tau^2}{m}\bigcup_i \left[\frac{\partial \mathbf{P}}{\partial k_i}\mathbb{X}(n-1)\right] + \left(2\mathbf{I} - \mathbf{P}\frac{\tau^2}{m}\right)\mathbb{J}(n-1) - \mathbb{J}(n-2), \tag{13}$$

where

$$\frac{\partial \mathbf{P}}{\partial k_i} = \begin{array}{cc} \begin{array}{c} \quad\quad\quad i-th \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} & \begin{array}{c} \\ \\ i-th \\ \\ \\ \\ \end{array} \end{array}, \tag{14}$$

*3.3. Finite differences: Knowing one sample and velocity*

In this case, It is necessary define $\dot{\mathbb{X}} = \mathbb{V}$, so that the Eq. (3) can be rewrite as

$$\begin{pmatrix} \dot{\mathbb{X}} \\ \dot{\mathbb{V}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{M\times M} \\ -\mathbf{P}/m & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbb{X} \\ \mathbb{V} \end{pmatrix}, \tag{15}$$

so that we got

$$\dot{\mathbb{U}} = A\mathbb{U}, \tag{16}$$

where $\mathbb{U} = (\mathbb{X}; \mathbb{V})$ and $A = (\mathbf{0}, \mathbb{I}_{M\times M}; -\mathbf{P}/m, \mathbf{0})$.

Applying finite differences we known that $\mathbb{U} \equiv \mathbb{U}(n)$ and $\dot{\mathbb{U}} \equiv (\mathbb{U}(n) - \mathbb{U}(n-1))/\tau_2$, so that the Eq. (16) can be rewrite as

$$\mathbb{U}(n) = (\mathbf{I} - \mathbf{A}\tau_2)^{-1}\, \mathbb{U}(n-1), \tag{17}$$

Problems with finite differences: To got a good aproximation it is necessary to choose $\tau_2 \gg \tau$ (where $\tau$ is the value used in the section 3.2). Experimentaly was see that $\tau_2 \geq \tau^2$.

Now deriving $\mathbb{U}(n)$ by the vector $\mathbf{K} = (k_1\ k_2\ k_3\ \dots\ k_{M-1}\ k_M)$, so that $\mathbb{Q}(n) \equiv \frac{\partial \mathbb{U}(n)}{\partial \mathbf{K}}$, we get the Eq. (18)

$$\mathbb{Q}(n) = \tau_2 (\mathbf{I} - \mathbf{A}\tau_2)^{-1} \bigcup_i \left[ \frac{\partial \mathbf{A}}{\partial k_i} \mathbb{U}(n) \right], + (\mathbf{I} - \mathbf{A}\tau_2)^{-1}\, \mathbb{Q}(n-1), \tag{18}$$

where

$$\frac{\partial \mathbf{A}}{\partial k_i} = \begin{pmatrix} \mathbf{0} & \mathbf{0}_{M\times M} \\ -\frac{1}{m}\frac{\partial \mathbf{P}}{\partial k_i} & \mathbf{0} \end{pmatrix},$$

(19)

## 4. Minimization problem

The minimization problem seen in the Eq. (2) can be rewrite as

$$E(\mathbf{K}) = \frac{1}{2}\sum_{n}\big(y(n) - \mathbf{B}^T\mathbb{X}(n,\mathbf{K})\big)^2 \tag{20}$$

where $\mathbf{B} = (0\ 0\ 0\ \ldots\ 0\ 1)^T$, $y(n)$ are known values and $\mathbb{X}(n)$ that is a function of $\mathbf{K} = (k_1\ k_2\ k_3\ \ldots\ k_{M-1}\ k_M)^T$.

Now, knowing that a minimum of $E(\mathbf{K})$ in $\mathbf{K}$ is found when $\frac{\partial E(\mathbf{K})}{\partial k_i} = 0$ for all integer $1 \leq i \leq M$; we calculate the Eq. (21).

$$\frac{\partial E(\mathbf{K})}{\partial k_i} = \sum_{n}\left(\mathbf{B}^T\frac{\partial \mathbb{X}(n,\mathbf{K})}{\partial k_i}\right)^T\big(\mathbf{B}^T\mathbb{X}(n,\mathbf{K}) - y(n)\big), \tag{21}$$

Now reordering the Eq. (21) using a vectorial differentiation by $\mathbf{K}$, we get the Eq. (22).

$$\frac{\partial E(\mathbf{K})}{\partial \mathbf{K}} = \sum_{n}\big(\mathbf{B}^T\mathbb{J}(n,\mathbf{K})\big)^T\big(\mathbf{B}^T\mathbb{X}(n,\mathbf{K}) - y(n)\big), \tag{22}$$

where $\mathbb{J}(n) = \frac{\partial \mathbb{X}(n)}{\partial \mathbf{K}}$.

### *4.1. Landweber iterative method*

The Landweber iteration method propose that the minimization of a non-linear function $E(\mathbf{K})$ can be found using the gradient descent method, so that

$$\mathbf{K}_j \leftarrow \mathbf{K}_{j-1} - \alpha\frac{\partial E(\mathbf{K}_{j-1})}{\partial \mathbf{K}} \tag{23}$$

where $0 < \alpha < 2/||\frac{\partial E(\mathbf{K})}{\partial \mathbf{K}}||^2$ and $\|\cdot\|$ is the spectral norm. Thus, following the Landweber iteration method and using the Eq. (22) in our minimization problem, It can be solved using the Eq. (24).

$$\mathbf{K}_j \leftarrow \mathbf{K}_{j-1} - \alpha\sum_{n}\big(\mathbf{B}^T\mathbb{J}(n,\mathbf{K}_{j-1})\big)^T\big(\mathbf{B}^T\mathbb{X}(n,\mathbf{K}_{j-1}) - y(n)\big) \tag{24}$$

### 4.2. *Tikhonov iterative method*

If we assume that the problem of to get $\mathbf{K}$ will be solved iteratively, we can rewrite the Eq. (22) as if was evaluated by $\mathbb{X}_j(n)$ and $\mathbb{J}_{j-1}(n)$, as in the Eq. (25).

$$\sum_n \left\{ \left(\mathbf{B}^T \mathbb{J}_{j-1}(n)\right)^T \left(\mathbf{B}^T \mathbb{X}_j(n) - y(n)\right) \right\} = \mathbf{0}. \tag{25}$$

Where $\mathbb{J}_{j-1}(n) = \mathbb{J}(n, \mathbf{K}_{j-1})$ and $\mathbb{X}_j(n) = \mathbb{X}(n, \mathbf{K}_{j-1})$.

Knowing by the Taylor theorem that $\mathbb{X}_j(n) \approx \mathbb{X}_{j-1}(n) + \mathbb{J}_{j-1}(n)\left(\mathbf{K}_j - \mathbf{K}_{j-1}\right)$

$$\mathbf{K}_j = \mathbf{K}_{j-1} + \left( \sum_n \left(\mathbf{B}^T \mathbb{J}_{j-1}(n)\right)^T \left(\mathbf{B}^T \mathbb{J}_{j-1}(n)\right) \right)^{-1} \sum_n \left(\mathbf{B}^T \mathbb{J}_{j-1}(n)\right)^T \left(y(n) - \mathbf{B}^T \mathbb{X}_{j-1}(n)\right). \tag{26}$$

joint with the Eqs. (12) and (13) we got:

$$\mathbb{X}(n, \mathbf{K}) = \left(2\mathbf{I} - \mathbf{P}(\mathbf{K}) \frac{\tau^2}{m}\right) \mathbb{X}(n-1, \mathbf{K}) - \mathbb{X}(n-2, \mathbf{K}), \tag{27}$$

$$\mathbb{J}(n, \mathbf{K}) = -\frac{\tau_2^2}{m} \bigcup_i \left[ \frac{\partial\left(\mathbf{P}\right)}{\partial k_i} \mathbb{X}(n-1, \mathbf{K}) \right] + \left(2\mathbf{I} - \mathbf{P}(\mathbf{K}) \frac{\tau_2^2}{m}\right) \mathbb{J}(n-1, \mathbf{K}) - \mathbb{J}(n-2, \mathbf{K}), \tag{28}$$

## 5. Numerical results

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## 6. Conclusion

In this work were presented

## 7. Acknowledgment

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## 8. Bibliography