Relation Between the Mass-Spring System and the Dynamic Speckle

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1 Introduction

The biospeckle laser analysis has presented as a versatile tool in the analysis of biological activity.

2 System description

The signal y with samples $y(n)$, represent the adquisition of the signal x_M with samples $x_M(n)$, obtained in a mass-spring system of M elements, where each mass is separated of another by a distance of L/M, like can be seen in the Fig. 1.

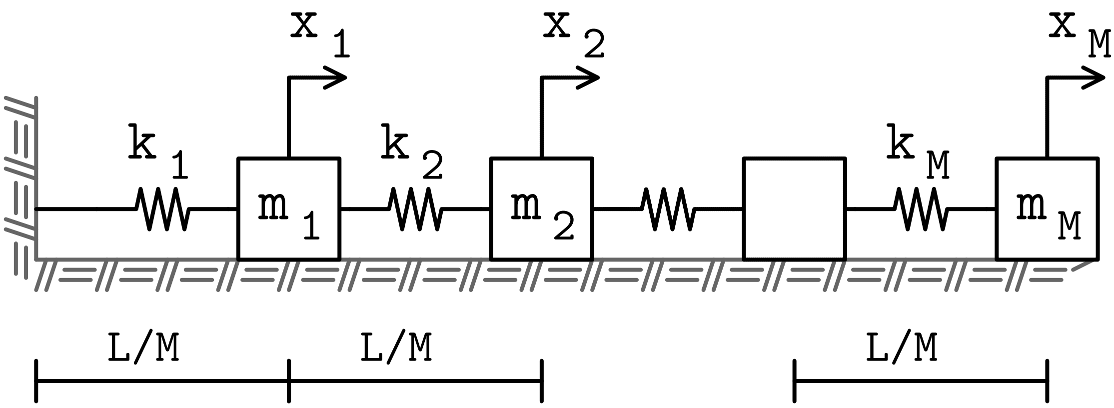

Figure 1: Data acquisition system setup of the coffee seed.

Thus, in this system the mass are denoted as m_i, the springs as k_i and the displacements of each mass by x_i, for all $1 \leq i \leq M$.

The objective of this work it is to solve the next inverse problem: Known $y(n)$, and assuming M elements with $m_i = m = 1/L$; what values of k_i generate a signal x_M that minimize E, where

$$E = \frac{1}{2} \sum_n (y(n) - x_M(n))^2 \tag{1}$$

3 Mass-spring system

Assuming a mass spring system like seen in the Fig. 1 with $m_i = m$ we can to get the system of Eq. (2).

$$m\ddot{\mathbf{x}} = -\mathbf{P}\mathbf{x}, \tag{2}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} \tag{3}$$

and

$$\mathbf{P}(\mathbf{k}) \equiv \mathbf{P} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \dots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{N-1}+x_N & -x_N \\ 0 & 0 & 0 & \dots & -x_N & x_N \end{pmatrix}, \tag{4}$$

so that $\mathbf{P}$ is a function of $\mathbf{k} = (k_1\ k_2\ k_3\ \dots\ k_{M-1}\ k_M)^T$.

3.1 Exact solution

Knowing the system shown in the Eq. (2), we can solve It using the Eq. (5),

$$\mathbf{x}(t) = \mathbf{V}\left(\mathbf{D}_1 cos(\mathbf{w}t) + \mathbf{D}_2 sin(\mathbf{w}t)\right), \tag{5}$$

or Eq. (6)

$$\mathbf{x}(t) = \mathbf{V}\left(cos(\mathbf{W}t)\mathbf{d}_1 + sin(\mathbf{W}t)\mathbf{d}_2\right), \tag{6}$$

where, $\mathbf{V} = (\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_M)$ and $\mathbf{w} = \left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_M}\right)^T$ are a matrix and a column vector conform using the eigenvectors $\mathbf{e}_i$ and eigenvalues λ_i of $\mathbf{P}/m$, being $\mathbf{W}$ a diagonal matrix conform with the elements of vector $\mathbf{w}$. By other side, $\mathbf{D}_1$ and $\mathbf{D}_2$ are two any constant diagonal matrices conform by the elements of column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ respectively. Thus, we now that $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$ are defined by the Eqs. (7) and (8) respectively.

$$\dot{\mathbf{x}}(t) = \mathbf{V}\left(-\mathbf{D}_1\mathbf{W} sin(\mathbf{w}t) + \mathbf{D}_2\mathbf{W} cos(\mathbf{w}t)\right), \tag{7}$$

$$\ddot{\mathbf{x}}(t) = -\mathbf{V}\left(\mathbf{D}_1\mathbf{W}^2 cos(\mathbf{w}t) + \mathbf{D}_2\mathbf{W}^2 sin(\mathbf{w}t)\right), \tag{8}$$

thus, It is fulfill that $\mathbf{W}^2 = (\mathbf{V}\mathbf{D}_1)^{-1}(\mathbf{P}/m)(\mathbf{V}\mathbf{D}_1) = (\mathbf{V}\mathbf{D}_2)^{-1}(\mathbf{P}/m)(\mathbf{V}\mathbf{D}_2)$.

3.1.1 Constant values from two points

Now, to get the constant values in the column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, we can use the Eq. (9)

$$\begin{pmatrix} \mathbf{V} cos(\mathbf{W}t_1) & \mathbf{V} sin(\mathbf{W}t_1) \\ \mathbf{V} cos(\mathbf{W}t_2) & \mathbf{V} sin(\mathbf{W}t_2) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{x}(t_1) \\ \mathbf{x}(t_2) \end{pmatrix} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} \tag{9}$$

3.1.2 Constant values from position and velocity of a point

Now, to get the constant values in the column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, we can use the Eq. (10)

$$\begin{pmatrix} \mathbf{V} cos(\mathbf{W}t_1) & \mathbf{V} sin(\mathbf{W}t_1) \\ -\mathbf{V}\mathbf{W} sin(\mathbf{W}t_1) & \mathbf{V}\mathbf{W} cos(\mathbf{W}t_1) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{x}(t_1) \\ \dot{\mathbf{x}}(t_1) \end{pmatrix} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} \tag{10}$$

3.2 Finite differences: Knowing two consecutive samples

Applying finite differences we known that $\mathbf{x} \equiv \mathbf{x}(n)$ and $\ddot{\mathbf{x}} \equiv (\mathbf{x}(n+1) - 2\mathbf{x}(n) + \mathbf{x}(n-1))/\tau^2$, so that the Eq. (2) can be rewrite as

$$\mathbf{x}(n) = \left(2\mathbf{I} - \mathbf{P}\frac{\tau^2}{m}\right)\mathbf{x}(n-1) - \mathbf{x}(n-2), \tag{11}$$

now deriving $\mathbf{x}(n)$ by the vector $\mathbf{k} = (k_1\ k_2\ k_3\ \ldots\ k_{M-1}\ k_M)$, so that $\mathbf{J}(n) \equiv \frac{\partial \mathbf{x}(n)}{\partial \mathbf{k}}$, we get the Eq. (12)

$$\mathbf{J}(n) = -\frac{\tau^2}{m}\bigcup_i \left[\frac{\partial \mathbf{P}}{\partial k_i}\mathbf{x}(n-1)\right] + \left(2\mathbf{I} - \mathbf{P}\frac{\tau^2}{m}\right)\mathbf{J}(n-1) - \mathbf{J}(n-2), \tag{12}$$

where

$$\frac{\partial \mathbf{P}}{\partial k_i} = \begin{array}{cc} \begin{array}{ccccccc} & & & i-th & & & \end{array} & \\ \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & \ldots & 1 & -1 & \ldots & 0 & 0 \\ 0 & \ldots & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} & i-th \end{array}, \tag{13}$$

3.3 Finite differences: Knowing one sample and velocity

In this case, It is necessary define $\dot{\mathbf{x}} = \mathbf{v}$, so that the Eq. (2) can be rewrite as

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{M\times M} \\ -\mathbf{P}/m & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}, \tag{14}$$

so that we got

$$\dot{\mathbf{u}} = A\mathbf{u}, \tag{15}$$

where $\mathbf{u} = (\mathbf{x}; \mathbf{v})$ and $A = (\mathbf{0}, \mathbf{I}_{M\times M}; -\mathbf{P}/m, \mathbf{0})$.

Applying finite differences we known that $\mathbf{u} \equiv \mathbf{u}(n)$ and $\dot{\mathbf{u}} \equiv (\mathbf{u}(n) - \mathbf{u}(n-1))/\tau_2$, so that the Eq. (15) can be rewrite as

$$\mathbf{u}(n) = (\mathbf{I} - \mathbf{A}\tau_2)^{-1}\mathbf{u}(n-1), \tag{16}$$

Problems with finite differences: To got a good aproximation it is necessary to choose $\tau_2 \gg \tau$ (where τ is the value used in the section 3.2). Experimentaly was see that $\tau_2 \geq \tau^2$.

Now deriving $\mathbf{u}(n)$ by the vector $\mathbf{k} = (k_1\ k_2\ k_3\ \ldots\ k_{M-1}\ k_M)$, so that $\mathbf{Q}(n) \equiv \frac{\partial \mathbf{u}(n)}{\partial \mathbf{k}}$, we get the Eq. (17)

$$\mathbf{Q}(n) = \tau_2(\mathbf{I} - \mathbf{A}\tau_2)^{-1}\bigcup_i \left[\frac{\partial \mathbf{A}}{\partial k_i}\mathbf{u}(n)\right], + (\mathbf{I} - \mathbf{A}\tau_2)^{-1}\mathbf{Q}(n-1), \tag{17}$$

where

$$\frac{\partial \mathbf{A}}{\partial k_i} = \begin{pmatrix} \mathbf{0} & \mathbf{0}_{M\times M} \\ -\frac{1}{m}\frac{\partial \mathbf{P}}{\partial k_i} & \mathbf{0} \end{pmatrix},$$

(18)

4 Minimization problem

The minimization problem seen in the Eq. (1) can be rewrite as

$$E(\mathbf{k}) = \frac{1}{2} \sum_n \big(y(n) - \mathbf{b}^T \mathbf{x}(n, \mathbf{k}) \big)^2 \tag{19}$$

where $\mathbf{b} = (0\ 0\ 0\ \ldots\ 0\ 1)^T$, $y(n)$ are known values and $\mathbf{x}(n)$ that is a function of $\mathbf{k} = (k_1\ k_2\ k_3\ \ldots\ k_{M-1}\ k_M)^T$.

Now, knowing that a minimum of $E(\mathbf{k})$ in $\mathbf{k}$ is found when $\frac{\partial E(\mathbf{k})}{\partial k_i} = 0$ for all integer $1 \leq i \leq M$; we calculate the Eq. (20).

$$\frac{\partial E(\mathbf{k})}{\partial k_i} = \sum_n \left(\mathbf{b}^T \frac{\partial \mathbf{x}(n, \mathbf{k})}{\partial k_i} \right)^T \big(\mathbf{b}^T \mathbf{x}(n, \mathbf{k}) - y(n) \big), \tag{20}$$

Now reordering the Eq. (20) using a vectorial differentiation by $\mathbf{k}$, we get the Eq. (21).

$$\frac{\partial E(\mathbf{k})}{\partial \mathbf{k}} = \sum_n \big(\mathbf{b}^T \mathbf{J}(n, \mathbf{k}) \big)^T \big(\mathbf{b}^T \mathbf{x}(n, \mathbf{k}) - y(n) \big), \tag{21}$$

where $\mathbf{J}(n) = \frac{\partial \mathbf{x}(n)}{\partial \mathbf{k}}$.

4.1 Landweber iterative method

The Landweber iteration method propose that the minimization of a nonlinear function $E(\mathbf{k})$ can be found using the gradient descent method, so that

$$\mathbf{k}_j \leftarrow \mathbf{k}_{j-1} - \alpha \frac{\partial E(\mathbf{k}_{j-1})}{\partial \mathbf{k}} \tag{22}$$

where $0 < \alpha < 2/||\frac{\partial E(\mathbf{k})}{\partial \mathbf{k}}||^2$ and $\|\cdot\|$ is the spectral norm. Thus, following the Landweber iteration method and using the Eq. (21) in our minimization problem, It can be solved using the Eq. (23).

$$\mathbf{k}_j \leftarrow \mathbf{k}_{j-1} - \alpha \sum_n \big(\mathbf{b}^T \mathbf{J}(n, \mathbf{k}_{j-1}) \big)^T \big(\mathbf{b}^T \mathbf{x}(n, \mathbf{k}_{j-1}) - y(n) \big) \tag{23}$$

4.2 Tikhonov iterative method

If we assume that the problem of to get $\mathbf{k}$ will be solved iteratively, we can rewrite the Eq. (21) as if was evaluated by $\mathbf{x}_j(n)$ and $\mathbf{J}_{j-1}(n)$, as in the Eq. (24).

$$\sum_n \left\{ \big(\mathbf{b}^T \mathbf{J}_{j-1}(n) \big)^T \big(\mathbf{b}^T \mathbf{x}_j(n) - y(n) \big) \right\} = \mathbf{0}. \tag{24}$$

Where $\mathbf{J}_{j-1}(n) = \mathbf{J}(n, \mathbf{k}_{j-1})$ and $\mathbf{x}_j(n) = \mathbf{x}(n, \mathbf{k}_{j-1})$.

Knowing by the Taylor theorem that $\mathbf{x}_j(n) \approx \mathbf{x}_{j-1}(n) + \mathbf{J}_{j-1}(n)\,(\mathbf{k}_j - \mathbf{k}_{j-1})$

$$\mathbf{k}_j = \mathbf{k}_{j-1} + \left(\sum_n \big(\mathbf{b}^T \mathbf{J}_{j-1}(n) \big)^T \big(\mathbf{b}^T \mathbf{J}_{j-1}(n) \big) \right)^{-1} \sum_n \big(\mathbf{b}^T \mathbf{J}_{j-1}(n) \big)^T \big(y(n) - \mathbf{b}^T \mathbf{x}_{j-1}(n) \big). \tag{25}$$

joint with the Eqs. (11) and (12) we got:

$$\mathbf{x}(n,\mathbf{k}) = \left(2\mathbf{I} - \mathbf{P}(\mathbf{k})\frac{\tau^2}{m}\right)\mathbf{x}(n-1,\mathbf{k}) - \mathbf{x}(n-2,\mathbf{k}), \tag{26}$$

$$\mathbf{J}(n,\mathbf{k}) = -\frac{\tau_2^2}{m}\bigcup_i \left[\frac{\partial\left(\mathbf{P}\right)}{\partial k_i}\mathbf{x}(n-1,\mathbf{k})\right] + \left(2\mathbf{I} - \mathbf{P}(\mathbf{k})\frac{\tau_2^2}{m}\right)\mathbf{J}(n-1,\mathbf{k}) - \mathbf{J}(n-2,\mathbf{k}), \tag{27}$$

5 Numerical results

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6 Conclusion

In this work were presented

7 Acknowledgment

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8 Bibliography

References