An Optimal Method for Tuning a Six Hole's Ocarina

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Abstract

In this paper we propose an optimal method for tuning a six hole's ocarina, this optimization will be subordinated by the minimization of the square norm of square difference between the desired and obtained frequency. The results show that it is not possible tuning a John Taylor style six hole's ocarina with a perfect equal temperament scale, but the minimization rule returns a tuning method with an optimal approximation and heterogeneous errors less than 43.697% of a semitone.

1 Introduction

The fingering style of the pendant-type ocarina was developed around 1964 by the Engish mathematician John Taylor [Met+85, pp. 79] [Soc01, pp. 10].

2 Fingering chart in six hole's ocarina

The Fig. 1 represents the hole distribution of the ocarina used in this work, where S_0 represent the sound hole (voicing) and the holes S_1, S_2, S_3, ... and S_6 represent the tone-holes. The fingering chart used in this ocarina model is based in the 4 holes fingering style proposed by John Taylor [Met+85, pp. 79, 146], the Table 1 shows the fingering chart, it indicates the configuration of open or close holes to produce the tones from C until D (a higher octave), where a value $S_i = 1$ represents an open hole and the value $S_i = 0$ a closed hole.

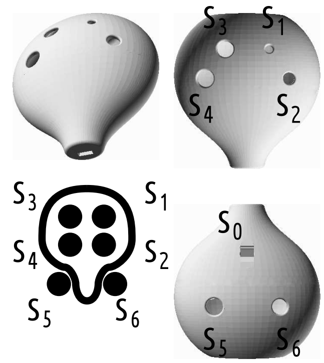

Figure 1: View of six hole's ocarina.

Note	S_1	S_2	S_3	S_4	S_5	S_6
C	0	0	0	0	0	0
$C\#$	0	0	0	0	0	1
D	1	0	0	0	0	0
$D\#$	1	0	0	0	0	1
E	0	1	0	0	0	0
F	1	1	0	0	0	0
$F\#$	0	0	1	0	0	0
G	1	0	1	0	0	0
$G\#$	0	1	1	0	0	0
A	1	1	1	0	0	0
$A\#$	1	0	1	1	0	0
B	0	1	1	1	0	0
C	1	1	1	1	0	0
$C\#$	0	1	1	1	1	0
D	1	1	1	1	1	0

Table 1: Fingering chart

3 Math and music

In this paper we analyze a tuning method with 12 tone equal temperament; so that, the frequency interval between any pair of adjacent notes has the same ratio $\rho = \sqrt[12]{2}$. The Table 2 shows how are distributed the musical notes from tone C until D (a higher octave), where f_0 represent the frequency of C note and the next notes follow the geometric progression $\hat{f}_i = \rho^i f_0$ with i representing the separation degree with C.

$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\hat{f}_3$	$\hat{f}_4$
C	$C\#$	D	$D\#$	E
f_0	$\rho^1 f_0$	$\rho^2 f_0$	$\rho^3 f_0$	$\rho^4 f_0$
$\hat{f}_5$	$\hat{f}_6$	$\hat{f}_7$	$\hat{f}_8$	$\hat{f}_9$
F	$F\#$	G	$G\#$	A
$\rho^5 f_0$	$\rho^6 f_0$	$\rho^7 f_0$	$\rho^8 f_0$	$\rho^9 f_0$
$\hat{f}_{10}$	$\hat{f}_{11}$	$\hat{f}_{12}$	$\hat{f}_{13}$	$\hat{f}_{14}$
$A\#$	B	C	$C\#$	D
$\rho^{10} f_0$	$\rho^{11} f_0$	$\rho^{12} f_0$	$\rho^{13} f_0$	$\rho^{14} f_0$

Table 2: Musical notes

4 Math and ocarina

4.1 Analyzing a whistle

We can think of an ocarina as a whistle with tone-holes in its resonant cavity, so that when we open and close some of these holes we can modify the tone of generated sound. Thus to understand the math of an ocarina first we should to know the science behind the whistles or more formally a Helmholtz resonant cavity [Cor11] (see Fig. 2) excited by a jet-edge-resonator system (see Fig. 3) [Gib13, pp. 3] [NWS53, pp. 138]. In this sense the Eq. 1

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{A_0}{l_0 V}} \tag{1}$$

shows the way of to calculate the frequency f_0 of a theoretic[1] whistle [Gib13, pp. 3] [Kob+09, pp. 5] [Oka+19, pp. 265] in relation with the constants π and c (the velocity of sound), and the variables: V the volume of its resonant cavity, A_0 is its cross-section area and l_0 the effective length of the neck in the resonator.

[1] A sphere-like resonant cavity and the open area of the neck is much less than the total surface area of the resonant cavity. .

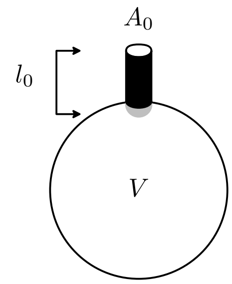

Figure 2: whistle.

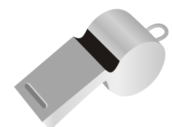

Figure 3: whistle.

4.2 Analyzing an ocarina

The fundamental resonant frequency f of an ocarina, that have at least one circular tone-hole in addition to the sound hole (see Fig. 4), can be represented like a function of the total tone-holes area A.

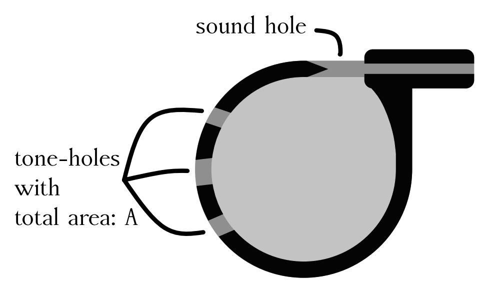

Figure 4: Theoretic ocarina

In the article "La Ocarina de Zanahoria a Carrot Ocarina" [MP10] were analyzed data about of the increment of resonant frequency f in relation to the augment of area A, showing that in that study case the relationships between the variables can be approximated with a line

$$f = c_1 + c_2\ A. \tag{2}$$

where $c_1 = 595.859069$, $c_2 = 4.324273$ and a mean square error $MSE = 8051.1$. The Fig. 5 show the linear curve in relation to the collected data.

By other side, if we follow the equation proposed by Bart Hopkin in its book "Air Columns

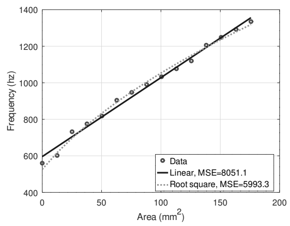

Figure 5: Comparing models.

and Toneholes" [Cab02, pp. 44] [Hop99], we can fit the data of Fig. 5 with a root square function

$$f = d_1\sqrt{d_2 + A}, \tag{3}$$

where $d_1 = 91.215856$, $d_2 = 33.262474$ and a mean square error $MSE = 5993.3$. We must highlight that d_2 represents the area of sound hole.

This last model fit better the data, also the equation proposed by Hopkin keep coherence with the Helmholtz resonator equation (see Eq. 1). In the next section we use the model proposed by Hopkin to obtain a optimal tuning in John Taylor style six hole's ocarina.

4.3 Resonance frequency in ocarinas

If we assume the equation 4 proposed by Bart Hopkin [Cab02, pp. 44] [Hop99],

$$f = \frac{c}{2\pi}\sqrt{\frac{\frac{A_0 S_0}{l_0} + \frac{A_1 S_1}{l_1} + \frac{A_2 S_2}{l_2} + \ldots}{V}} \tag{4}$$

as a good approximation to estimate the fundamental frequency f of globular resonators that have at least one circular pitch hole in addition to the sound hole (a.k.a. an ocarina). in which A_i and l_i are the area an the effective length of the S_i tone-hole, so that the variable S_i has only have two values $\{0, 1\}$ according the Table 1, additionally $S_0 = 1$ and represent the sound hole, being A_0 and l_0 their corresponding variables.

4.3.1 Reordering the resonance equation

We can rearrange the Eq. 4 defining the variable $h_i \propto \frac{A_i}{l_i}$, so that

$$h_i \equiv \left(\frac{c^2}{4\pi^2 V}\right)\left(\frac{A_i}{l_i}\right); \tag{5}$$

This variable change imply that the Eq. 4 can be rewrite as

$$f^2 = \sum_{i=0}^{6} S_i h_i. \tag{6}$$

It is important remember that following the Eqs. 1 and 6, when all the tone-hole's are close we obtain that

$$f_0^2 = h_0. \tag{7}$$

5 Problem Statement

Definition 5.1 ($\hat{\mathbf{z}}$ - Square value of the desired frequencies) *Known the frequency relation between the notes showed in the Table 2, we define the variable $\hat{z}_i \equiv \hat{f}_i^2 = \left(\rho^i f_0\right)^2$, so that the column vector $\hat{\mathbf{z}} = [\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{14}]^T$, that contain the square value of the desired frequencies, can be represented as in the Eq. 8.*

$$\hat{\mathbf{z}} \equiv \mathbf{p} f_0^2 \tag{8}$$

where $\mathbf{p} = [\rho^2,\ \rho^4,\ \rho^6,\ \rho^8,\ \ldots,\ \rho^{28}]^T$.

Definition 5.2 ($\mathbf{z(h)}$ - Square value of the generated frequencies) *If we rearrange the Eqs. 6 and 7 for all notes in Table 1, we obtain the next equation*

$$\mathbf{z}(\mathbf{h}) = \mathbf{q}h_0 + \mathbf{A}\mathbf{h} \tag{9}$$

where

$$\mathbf{q} = \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&1&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&1&0&0&0\\ 0&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&0&1&1&0&0\\ 0&1&1&1&0&0\\ 1&1&1&1&0&0\\ 0&1&1&1&1&0\\ 1&1&1&1&1&0 \end{bmatrix}, \tag{10}$$

$\mathbf{h} = [h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6]^T$ *and* $\mathbf{z}(\mathbf{h}) = [z_1(\mathbf{h}),\ z_2(\mathbf{h}),\ \ldots,\ z_{14}(\mathbf{h})]^T$. *Being $z_i(\mathbf{h})$ the* *square value of $i-th$ frequency generate by the ocarina given a* $\mathbf{h}$ *relation of areas and effective lengths in its tone-holes.*

5.1 Minimization rule and optimal value

Known and calculated the desired and generated frequencies seen in the Eqs. 8 and 9, we propose the next minimization rule

$$e^2(\mathbf{h}) = ||\hat{\mathbf{z}} - \mathbf{z}(\mathbf{h})||^2, \tag{11}$$

where the unknown variable is the vector $\mathbf{h}$. This problem can be fitted as a case of multiple linear regression that can be solved using the Tikhonov regularization [Puj20, pp. 79, 80]. Thus, following this method, we obtain the vector $\mathbf{h}^*$ that minimize the function $e^2(\mathbf{h})$,

$$\mathbf{h}^* = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\left(\mathbf{p}-\mathbf{q}\right)h_0, \tag{12}$$

if we use this results joint the Eqs. 7 and 9, we obtain that

$$\mathbf{z}^* = \left(\mathbf{q} + \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\left(\mathbf{p}-\mathbf{q}\right)\right)f_0^2, \tag{13}$$

being $\mathbf{z}^*$ the square value of the generated frequencies that minimize the function $e^2(\mathbf{h})$.

6 Resultados

Definition 6.1 (E_1 - relative tuning error) *The first type of error used in this work is*

$$E_1 = 100\,\frac{f_b - f_a}{f_a}\ \%, \tag{14}$$

that measures the difference error of f_b in relation to the target frequency (f_a) or relative tuning error.

To obtain the biggest accepted relative tuning error, It is necessary to know the fact that between any two consecutive notes, with frequencies f_a and ρf_a, exist a proportion of $\rho \approx 1.059463$ (a semitone), this mean that the next consecutive note is ever a $+5.9463\%$ that the last note. Thus, if we set as maximum error the midway position in geometric proportion between notes, them the biggest accepted positive relative tuning error to avoid a tone recognition mistake, it is equal to $+2.9302\% \equiv 100(\sqrt{\rho}f_a - f_a)/f_a\ \%$. By other side the biggest accepted negative relative tuning error to avoid a tone recognition mistake, it is equal to $-2.8468\% \equiv 100(f_a/\sqrt{\rho} - f_a)/f_a\ \%$.

Definition 6.2 (E_2 - semitone error) *The logarithmic difference error between two frequencies f_a and f_b in relation to a semitone (ρ) is defined as*

$$E_2 = 100\,\frac{ln(f_b) - ln(f_a)}{ln(\rho)}\ \%, \tag{15}$$

called here as semitone error.

To obtain the biggest accepted semitone error, we use a similar criterion as seen in the Def. 6.1, so that if we set as maximum error the midway position in geometric proportion between notes (ex: C and C#), them the biggest accepted positive tuning error to avoid a tone recognition mistake, it is equal to $+50\% \equiv 100\,\frac{ln(\sqrt{\rho}f_a)-ln(f_a)}{ln(\rho)}\ \%$. By other side the biggest accepted negative semitone error to avoid a tone recognition mistake, it is equal to $-50\% \equiv 100\,\frac{ln(f_a/\sqrt{\rho})-ln(f_a)}{ln(\rho)}\ \%$.

6.1 Analyzing the optimal $\mathbf{h}^*$

In the Eq. 12 was calculated $\mathbf{h}^* = [h_1^*,\ h_2^*,\ ...,\ h_6^*]^T$, the optimal vector that minimize the function $e(\mathbf{h})$. The Table 3 shows each element of this vector in relation to h_0, being that $h_i^*/h_0 \equiv \left(\frac{l_0}{A_0}\right)\left(\frac{A_i}{l_i}\right)$.

h_1^*/h_0	h_2^*/h_0	h_3^*/h_0	h_4^*/h_0	h_5^*/h_0	h_6^*/h_0
29.3%	58.1%	96.1%	103.5%	104.1%	12.2%

Table 3: Percentage ratio between h_i^* and h_0.

In this table we observe that if we consider l_i a constant in all ocarina then $h_i^*/h_0 \propto A_i$ and the percentages presented in the Table 3 describes the ratio of proportions between areas A_i. So that below this conditions the optimal area A_2 is about twice A_1, the optimal area A_3 is about $3.27A_1$ and so on.

6.2 Analyzing the optimal frequencies

The Tabla 4 shows a comparison between the desired and optimal generated frequencies for each note in a six hole's ocarina. The first column

Note	Desired (hz)	Optimal (hz)	E_1	E_2
C	1.0000 f_0	1.0000 f_0	0.000 %	0.000 %
$C\#$	1.0595 f_0	1.0591 f_0	-0.030 %	-0.524 %
D	1.1225 f_0	1.1371 f_0	1.308 %	22.506 %
$D\#$	1.1892 f_0	1.1895 f_0	0.024 %	0.415 %
E	1.2599 f_0	1.2574 f_0	-0.204 %	-3.528 %
F	1.3348 f_0	1.3690 f_0	2.556 %	43.697 %
$F\#$	1.4142 f_0	1.4005 f_0	-0.973 %	-16.926 %
G	1.4983 f_0	1.5015 f_0	0.210 %	3.639 %
$G\#$	1.5874 f_0	1.5944 f_0	0.443 %	7.652 %
A	1.6818 f_0	1.6838 f_0	0.122 %	2.108 %
$A\#$	1.7818 f_0	1.8138 f_0	1.796 %	30.819 %
B	1.8877 f_0	1.8915 f_0	0.198 %	3.421 %
C	2.0000 f_0	1.9674 f_0	-1.628 %	-28.417 %
$C\#$	2.1189 f_0	2.1490 f_0	1.419 %	24.401 %
D	2.2449 f_0	2.2161 f_0	-1.282 %	-22.333 %

Table 4: Errors in optimal tuning

shows all notes used in the optimization calculation. The second column has the desired frequencies (equal temperament scale), this column is equivalent to the root square value of $\hat{\mathbf{z}}$. The third column has the optimal generated frequencies, this column is equivalent to the root square value of $\mathbf{z}^*$. The fourth and fifth columns have the errors described in the Definitions 6.1 and 6.1, respectively.

The worst error E_1 is equal to $+2.556\%$ and correspond to the note F, similarly this note present the worst error E_2 with a detour of 43.697% of a semitone. The next worst error E_1 is equal to $+1.796\%$ and correspond to the note $A\#$, having an error E_2 with a detour of 30.819% of a semitone.

6.3 Proposed tuning method

From the data of the Table 4, we can infer an ocarina construction method that fulfill an optimal tuning following the minimization rule of the Eq. 11. In this sense if we choose of this table a set of notes, in which each current note use a new tone hole in relation to the last tuned note, we can obtain a tuning method or path to reach an optimal tuning.

With this considerations, the Table 5 shows one tuning method, where is considered a C note equal to $523.2511hz$.

Note	Desired (hz)	Optimal (hz)	E_2
C	523.2511	523.2511	0.000 %
$C\#$	554.3653	554.1976	-0.524 %
D	587.3295	595.0147	22.506 %
E	659.2551	657.9130	-3.528 %
G	783.9909	785.6404	3.639 %
B	987.7666	989.7206	3.421 %
D	1174.6591	1159.6030	-22.333 %

Table 5: Proposed tuning method.

7 Conclusion

The conclusion goes here.

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