Path Filling Points

Fernando Pujaico Rivera

February 22, 2024

Contents

1 3D parametric linear spline

A 3D parametric linear spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 1 shows the polynomials $\hat{\mathbf{p}}^{(n)}(t)$ with parameter t close to the n-th position.

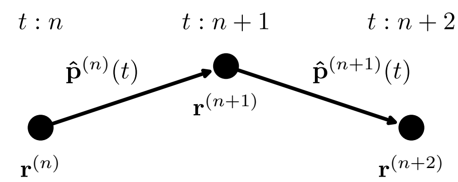

Figure 1: Polynomials in the n-th position of linear spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\ 0 \leq n \leq N-1$, we can generate a linear spline with $N-2$ polynomials $\hat{\mathbf{p}}^{(n)}(t)$, $\forall\ 0 \leq n \leq N-2$, according the Fig. 1. So that, it is fulfilled that

$$\hat{\mathbf{p}}^{(n)}(t) = \begin{bmatrix} \hat{x}^{(n)}(t) & \hat{y}^{(n)}(t) & \hat{z}^{(n)}(t), \end{bmatrix}^T \tag{1}$$

where

$$\hat{x}^{(n)}(t) = \hat{a}_x^{(n)} + \hat{b}_x^{(n)}(t-n), \tag{2}$$

$$\hat{y}^{(n)}(t) = \hat{a}_y^{(n)} + \hat{b}_y^{(n)}(t-n), \tag{3}$$

$$\hat{z}^{(n)}(t) = \hat{a}_z^{(n)} + \hat{b}_z^{(n)}(t-n). \tag{4}$$

1.1 Matricial form of polynomials $\hat{\mathbf{p}}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\hat{\mathbf{w}}^{(n)} = \left[\begin{array}{cc:cc:cc} \hat{a}_x^{(n)} & \hat{b}_x^{(n)} & \hat{a}_y^{(n)} & \hat{b}_y^{(n)} & \hat{a}_z^{(n)} & \hat{b}_z^{(n)} \end{array}\right]^T, \tag{5}$$

$$\hat{\mathbf{A}}^{(n)}(t) = \left[\begin{array}{cc:cc:cc} 1 & (t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (t-n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) \end{array}\right], \tag{6}$$

$$\hat{\mathbf{p}}^{(n)}(t) = \hat{\mathbf{A}}^{(n)}(t)\hat{\mathbf{w}}^{(n)}. \tag{7}$$

Additionally, we can define

$$\hat{\mathbf{w}} \equiv \begin{bmatrix} \hat{\mathbf{w}}^{(0)} \\ \hat{\mathbf{w}}^{(1)} \\ \vdots \\ \hat{\mathbf{w}}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{8}$$

2 Boundary conditions in 3D parametric linear spline

2.1 Conditions in points

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-2$, so that

$$\hat{\mathbf{p}}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{9}$$

$$\hat{\mathbf{p}}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{10}$$

2.1.1 Matricial form of the conditions in points

Using the Eq. 7 in the Eqs. 9 and 10, we obtain for $0 \leq n \leq N-2$

$$\hat{\mathbf{A}}^{(n)}(n)\hat{\mathbf{w}}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{11}$$

$$\hat{\mathbf{A}}^{(N-2)}(N-1)\hat{\mathbf{w}}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{12}$$

where, using the Eq. 6, we know that

$$\hat{\mathbf{A}}^{(n)}(n+1) = \left[\begin{array}{cc:cc:cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,0)} \in \mathbb{R}^{3\times 6}, \tag{13}$$

$$\hat{\mathbf{A}}^{(n+1)}(n+1) = \hat{\mathbf{A}}^{(n)}(n) = \left[\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,1)} \in \mathbb{R}^{3\times 6}, \tag{14}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 11 and 12, $\forall 0 \leq n \leq N-2$. We can write the Eqs. 9 and 10 as

$$\hat{\mathbf{P}}\hat{\mathbf{w}} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{15}$$

Where

$$\hat{\mathbf{P}} \equiv \begin{bmatrix} \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N\times 6(N-1)} \tag{16}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{17}$$

2.2 Boundary conditions in internal point

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-3$, so that

$$\hat{\mathbf{p}}^{(n)}(n+1) - \hat{\mathbf{p}}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{18}$$

2.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eq. 7 in the Eq. 18, we obtain for $0 \leq n \leq N-3$

$$\hat{\mathbf{A}}^{(n)}(n+1)\hat{\mathbf{w}}^{(n)} - \hat{\mathbf{A}}^{n+1}(n+1)\hat{\mathbf{w}}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{19}$$

Grouping in a matrix

$$\begin{bmatrix} \hat{\mathbf{A}}^{(n)}(n+1) & -\hat{\mathbf{A}}^{n+1}(n+1) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3, \tag{20}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 18 and 20, these can be rewritten for $0 \leq n \leq N-3$

$$\begin{bmatrix}\hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)}\end{bmatrix} = \mathbf{0} \in \mathbb{R}^3 \tag{21}$$

Finally, concatenating to all values for $0 \leq n \leq N-3$ in the boundary conditions in internal point equation's, we obtain

$$\hat{\mathbf{Q}} \equiv \begin{bmatrix}\hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)}\end{bmatrix} \in \mathbb{R}^{3(N-2)\times 6(N-1)} \tag{22}$$

$$\hat{\mathbf{Q}}\hat{\mathbf{w}} = \mathbf{0} \in \mathbb{R}^{3(N-2)} \tag{23}$$

3 Parameter calculus of 3D parametric linear spline

Following the explained in the Section 2, the equation that should be fulfilled to fit the linear spline in the points can be represented in the next equation

$$\begin{bmatrix}\hat{\mathbf{P}} \\ \hat{\mathbf{Q}}\end{bmatrix}\hat{\mathbf{w}} = \begin{bmatrix}\mathbf{r} \\ \mathbf{0}\end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{24}$$

Thus,

$$\hat{\mathbf{w}} = \begin{bmatrix}\hat{\mathbf{P}} \\ \hat{\mathbf{Q}}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{r} \\ \mathbf{0}\end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{25}$$

4 3D parametric cubic spline

A 3D parametric cubic spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 2 shows the polynomials $\mathbf{p}^{(n)}(t)$ with parameter t close to the n-th position.

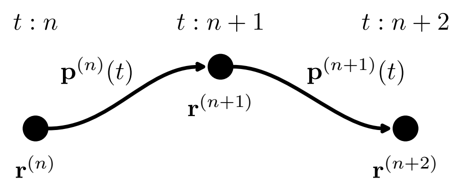

Figure 2: Polynomials in the n-th position of cubic spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\, 0 \leq n \leq N-1$, we can generate a cubic spline with $N-2$ polynomials $\mathbf{p}^{(n)}(t)$, $\forall\, 0 \leq n \leq N-2$, according the Fig. 2. So that, it is fulfilled that

$$\mathbf{p}^{(n)}(t) = \begin{bmatrix}x^{(n)}(t) & y^{(n)}(t) & z^{(n)}(t)\end{bmatrix}^T, \tag{26}$$

where

$$x^{(n)}(t) = a_x^{(n)} + b_x^{(n)}(t-n) + c_x^{(n)}(t-n)^2 + d_x^{(n)}(t-n)^3, \tag{27}$$

$$y^{(n)}(t) = a_y^{(n)} + b_y^{(n)}(t-n) + c_y^{(n)}(t-n)^2 + d_y^{(n)}(t-n)^3, \tag{28}$$

$$z^{(n)}(t) = a_z^{(n)} + b_z^{(n)}(t-n) + c_z^{(n)}(t-n)^2 + d_z^{(n)}(t-n)^3. \tag{29}$$

4.1 Matricial form of polynomials $\mathbf{p}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\mathbf{w}^{(n)} = \left[\begin{array}{cccc:cccc:cccc} a_x^{(n)} & b_x^{(n)} & c_x^{(n)} & d_x^{(n)} & a_y^{(n)} & b_y^{(n)} & c_y^{(n)} & d_y^{(n)} & a_z^{(n)} & b_z^{(n)} & c_z^{(n)} & d_z^{(n)} \end{array}\right]^T, \tag{30}$$

$$\mathbf{A}^{(n)}(t) = \left[\begin{array}{cccc:cccc:cccc} 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 \end{array}\right], \tag{31}$$

$$\mathbf{p}^{(n)}(t) = \mathbf{A}^{(n)}(t)\mathbf{w}^{(n)}. \tag{32}$$

Additionally, we can define

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{w}^{(0)} \\ \mathbf{w}^{(1)} \\ \vdots \\ \mathbf{w}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{12(N-1)} \tag{33}$$

4.2 Additional useful derivation of $\mathbf{p}^{(n)}(t)$

$$\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 \end{array}\right], \tag{34}$$

$$\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) \end{array}\right]. \tag{35}$$

If we define $\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} \equiv D_t\mathbf{A}^{(n)}(t)$ and $\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} \equiv D_t^2\mathbf{A}^{(n)}(t)$, then

$$\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} = D_t\mathbf{A}^{(n)}(t)\mathbf{w}^{(n)} \tag{36}$$

$$= \begin{bmatrix} b_x^{(n)} + 2(t-n)c_x^{(n)} + 3(t-n)^2 d_x^{(n)} \\ b_y^{(n)} + 2(t-n)c_y^{(n)} + 3(t-n)^2 d_y^{(n)} \\ b_z^{(n)} + 2(t-n)c_z^{(n)} + 3(t-n)^2 d_z^{(n)} \end{bmatrix} \tag{37}$$

$$\frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2} = D_t^2\mathbf{A}^{(n)}(t)\mathbf{w}^{(n)} \tag{38}$$

$$= \begin{bmatrix} 2c_x^{(n)} + 6(t-n)d_x^{(n)} \\ 2c_y^{(n)} + 6(t-n)d_y^{(n)} \\ 2c_z^{(n)} + 6(t-n)d_z^{(n)} \end{bmatrix} \tag{39}$$

5 Boundary conditions in 3D parametric cubic spline

5.1 Conditions in points

Following the Fig. 2, we can affirm that for $0 \leq n \leq N-2$, so that

$$\mathbf{p}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{40}$$

$$\mathbf{p}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{41}$$

5.1.1 Matricial form of the conditions in points

Using the Eq. 32 in the Eqs. 40 and 41, we obtain for $0 \le n \le N-2$

$$\mathbf{A}^{(n)}(n)\mathbf{w}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{42}$$

$$\mathbf{A}^{(N-2)}(N-1)\mathbf{w}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{43}$$

where, using the Eq. 31, we know that

$$\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \equiv \mathbf{Q}^{(0,0)} \in \mathbb{R}^{3\times 12}, \tag{44}$$

$$\mathbf{A}^{(n+1)}(n+1) = \mathbf{A}^{(n)}(n) = \left[\begin{array}{cccc:cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(0,1)} \in \mathbb{R}^{3\times 12}, \tag{45}$$

Thus, using the Eqs. 44 and 45 in the Eqs. 42 and 43, $\forall 0 \le n \le N-2$; We can write the Eqs. 40 and 41 as

$$\mathbf{P}\mathbf{w} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{46}$$

Where

$$\mathbf{P} \equiv \begin{bmatrix} \mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N\times 12(N-1)} \tag{47}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{48}$$

5.2 Boundary conditions in internal point

Following the Fig. 2, we can affirm that for $0 \le n \le N-3$, so that

$$\mathbf{p}^{(n)}(n+1) - \mathbf{p}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{49}$$

$$\left.\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t}\right|_{t=n+1} - \left.\frac{\partial \mathbf{p}^{n+1}(t)}{\partial t}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3, \tag{50}$$

$$\left.\frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2}\right|_{t=n+1} - \left.\frac{\partial^2 \mathbf{p}^{n+1}(t)}{\partial t^2}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3. \tag{51}$$

5.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eqs. 32, 34, 35, 36 and 38 in the Eqs. 49, 50 and 51, we obtain for $0 \leq n \leq N-3$

$$\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - \mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{52}$$

$$D_t\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t\mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{53}$$

$$D_t^2\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t^2\mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3. \tag{54}$$

Grouping in a matrix

$$\begin{bmatrix} \mathbf{A}^{(n)}(n+1) & -\mathbf{A}^{(n+1)}(n+1) \\ D_t\mathbf{A}^{(n)}(n+1) & -D_t\mathbf{A}^{(n+1)}(n+1) \\ D_t^2\mathbf{A}^{(n)}(n+1) & -D_t^2\mathbf{A}^{(n+1)}(n+1) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(n)} \\ \mathbf{w}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^9, \tag{55}$$

using the Eqs. 34 and 35, we know that

$$D_t\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \equiv \mathbf{Q}^{(1,0)} \in \mathbb{R}^{3\times 12}, \tag{56}$$

$$D_t\mathbf{A}^{(n+1)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(1,1)} \in \mathbb{R}^{3\times 12}, \tag{57}$$

$$D_t^2\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{array}\right] \equiv \mathbf{Q}^{(2,0)} \in \mathbb{R}^{3\times 12}, \tag{58}$$

$$D_t^2\mathbf{A}^{(n+1)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right] \equiv \mathbf{Q}^{(2,1)} \in \mathbb{R}^{3\times 12}. \tag{59}$$

Thus, using the Eqs. 44, 45, 56, 57, 58 and 59 in the Eqs. 49, 50 and 51, these can be rewritten for $0 \leq n \leq N-3$

$$\begin{bmatrix} \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} \\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} \\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(n)} \\ \mathbf{w}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^9 \tag{60}$$

Finally, concatenating to all values for $0 \leq n \leq N-3$ in the boundary conditions in internal point equation's, we obtain

$$\mathbf{Q} \equiv \left[\begin{array}{ccccccccc} \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} \end{array}\right] \in \mathbb{R}^{9(N-2)\times 12(N-1)} \tag{61}$$

$$\mathbf{Qw} = \mathbf{0} \in \mathbb{R}^{9(N-2)} \tag{62}$$

6 Curvature of a 3D parametric cubic spline

The curvature $\mathcal{K}_{(n)}(t)$ of a 3D parametric function, $\mathbf{p}^{(n)}(t) \in \mathbb{R}^3$ with parameter t, can be calculated [2, pp. 21] using the next equation,

$$\mathcal{K}_{(n)}(t) = \frac{\left\| \frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} \times \frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2} \right\|}{\left\| \frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} \right\|^3} \equiv \frac{\left\| \mathbf{p}^{(n)\prime}(t) \times \mathbf{p}^{(n)\prime\prime}(t) \right\|}{\left\| \mathbf{p}^{(n)\prime}(t) \right\|^3} \tag{63}$$

if we calculate the square curvature $\mathcal{K}^2_{(n)}(t)$ to the cases when $t \equiv n$ and $t \equiv n+1$, then we obtain

$$\mathcal{K}^2_{(n)}(n) = \frac{\left\| \left(\mathbf{Q}^{(1,1)}\mathbf{w}^{(n)}\right) \times \left(\mathbf{Q}^{(2,1)}\mathbf{w}^{(n)}\right) \right\|^2}{\left\| \mathbf{Q}^{(1,1)}\mathbf{w}^{(n)} \right\|^6} \tag{64}$$

$$\mathcal{K}^2_{(n)}(n+1) = \frac{\left\| \left(\mathbf{Q}^{(1,0)}\mathbf{w}^{(n)}\right) \times \left(\mathbf{Q}^{(2,0)}\mathbf{w}^{(n)}\right) \right\|^2}{\left\| \mathbf{Q}^{(1,0)}\mathbf{w}^{(n)} \right\|^6} \tag{65}$$

or rewritten in a simplified form to minimize the indices

$$\mathcal{K}^2(\bar{\mathbf{w}}) = \frac{\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2}{\left\| \bar{\mathbf{b}} \right\|^6}. \tag{66}$$

Knowing that $\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2$ represents the square area of parallelepid form by the vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{c}}$, then $\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2 = \left(\left\|\bar{\mathbf{b}}\right\| sin(\theta) \left\|\bar{\mathbf{c}}\right\|\right)^2 = \left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 - \left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 cos(\theta)^2$. Thus, the Eq. 66 can be written as

$$\mathcal{K}^2(\bar{\mathbf{w}}) = \frac{\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 - \left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2}{\left\|\bar{\mathbf{b}}\right\|^6} \tag{67}$$

derivating in function of vector $\mathbf{w}$

$$\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\frac{\partial\left(\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 - \left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2\right)}{\partial \bar{\mathbf{w}}} \left\|\bar{\mathbf{b}}\right\|^6 - \left(\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 - \left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2\right) \frac{\partial\left\|\bar{\mathbf{b}}\right\|^6}{\partial \bar{\mathbf{w}}}}{\left\|\bar{\mathbf{b}}\right\|^{12}} \tag{68}$$

$$\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left(\frac{\partial\left(\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2\right)}{\partial \bar{\mathbf{w}}} - \frac{\partial\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2}{\partial \bar{\mathbf{w}}}\right) \left\|\bar{\mathbf{b}}\right\|^6 - \left(\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2 - \left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2\right) \frac{\partial\left\|\bar{\mathbf{b}}\right\|^6}{\partial \bar{\mathbf{w}}}}{\left\|\bar{\mathbf{b}}\right\|^{12}} \tag{69}$$

Solving [1, pp. 11] the Eq. 69 by parts

$$\begin{aligned}
\frac{\partial\left(\left\|\bar{\mathbf{b}}\right\|^2 \left\|\bar{\mathbf{c}}\right\|^2\right)}{\partial \bar{\mathbf{w}}} &= \frac{\partial\left\|\bar{\mathbf{b}}\right\|^2}{\partial \bar{\mathbf{w}}} \left\|\bar{\mathbf{c}}\right\|^2 + \left\|\bar{\mathbf{b}}\right\|^2 \frac{\partial\left\|\bar{\mathbf{c}}\right\|^2}{\partial \bar{\mathbf{w}}} &(70)\\
&= \frac{\partial\left(\bar{\mathbf{w}}^T\mathbf{Q}^{(1)T}\mathbf{Q}^{(1)}\bar{\mathbf{w}}\right)}{\partial \bar{\mathbf{w}}} \left\|\bar{\mathbf{c}}\right\|^2 + \left\|\bar{\mathbf{b}}\right\|^2 \frac{\partial\left(\bar{\mathbf{w}}^T\mathbf{Q}^{(2)T}\mathbf{Q}^{(2)}\bar{\mathbf{w}}\right)}{\partial \bar{\mathbf{w}}} &(71)\\
&= 2\mathbf{Q}^{(1)T}\mathbf{Q}^{(1)}\bar{\mathbf{w}} \left\|\bar{\mathbf{c}}\right\|^2 + 2\left\|\bar{\mathbf{b}}\right\|^2 \mathbf{Q}^{(2)T}\mathbf{Q}^{(2)}\bar{\mathbf{w}} &(72)\\
&= 2\left\|\bar{\mathbf{c}}\right\|^2 \mathbf{Q}^{(1)T}\bar{\mathbf{b}} + 2\left\|\bar{\mathbf{b}}\right\|^2 \mathbf{Q}^{(2)T}\bar{\mathbf{c}} &(73)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2}{\partial \bar{\mathbf{w}}} &= 2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right) \frac{\partial\left(\bar{\mathbf{w}}^T\mathbf{Q}^{(1)T}\mathbf{Q}^{(2)}\bar{\mathbf{w}}\right)}{\partial \bar{\mathbf{w}}} &(74)\\
&= 2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right) \left(\mathbf{Q}^{(1)T}\mathbf{Q}^{(2)} + \mathbf{Q}^{(2)T}\mathbf{Q}^{(1)}\right)\bar{\mathbf{w}} &(75)\\
&= 2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right) \left(\mathbf{Q}^{(1)T}\bar{\mathbf{c}} + \mathbf{Q}^{(2)T}\bar{\mathbf{b}}\right) &(76)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \left\|\bar{\mathbf{b}}\right\|^6}{\partial \bar{\mathbf{w}}} &= \frac{\partial \left(\left\|\bar{\mathbf{b}}\right\|^2\right)^3}{\partial \bar{\mathbf{w}}} && (77)\\
&= 3\left(\left\|\bar{\mathbf{b}}\right\|^2\right)^2 \frac{\partial \left\|\bar{\mathbf{b}}\right\|^2}{\partial \bar{\mathbf{w}}} && (78)\\
&= 3\left\|\bar{\mathbf{b}}\right\|^4 \frac{\partial \bar{\mathbf{b}}^T\bar{\mathbf{b}}}{\partial \bar{\mathbf{w}}} && (79)\\
&= 3\left\|\bar{\mathbf{b}}\right\|^4 \frac{\partial \bar{\mathbf{w}}^T\mathbf{Q}^{(1)T}\mathbf{Q}^{(1)}\bar{\mathbf{w}}}{\partial \bar{\mathbf{w}}} && (80)\\
&= 6\left\|\bar{\mathbf{b}}\right\|^4 \mathbf{Q}^{(1)T}\mathbf{Q}^{(1)}\bar{\mathbf{w}} && (81)\\
&= 6\left\|\bar{\mathbf{b}}\right\|^4 \mathbf{Q}^{(1)T}\bar{\mathbf{b}} && (82)
\end{aligned}
$$

We obtain

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left[2\left\|\bar{\mathbf{c}}\right\|^2 \mathbf{Q}^{(1)T}\bar{\mathbf{b}} + 2\left\|\bar{\mathbf{b}}\right\|^2 \mathbf{Q}^{(2)T}\bar{\mathbf{c}} - 2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)\left(\mathbf{Q}^{(1)T}\bar{\mathbf{c}} + \mathbf{Q}^{(2)T}\bar{\mathbf{b}}\right)\right]\left\|\bar{\mathbf{b}}\right\|^6 - \left(\left\|\bar{\mathbf{b}}\right\|^2\left\|\bar{\mathbf{c}}\right\|^2 - \left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)^2\right)6\left\|\bar{\mathbf{b}}\right\|^4 \mathbf{Q}^{(1)T}\bar{\mathbf{b}}}{\left\|\bar{\mathbf{b}}\right\|^{12}} \tag{83}
$$

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left[2\left\|\bar{\mathbf{c}}\right\|^2 \mathbf{Q}^{(1)T}\bar{\mathbf{b}} + 2\left\|\bar{\mathbf{b}}\right\|^2 \mathbf{Q}^{(2)T}\bar{\mathbf{c}} - 2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)\left(\mathbf{Q}^{(1)T}\bar{\mathbf{c}} + \mathbf{Q}^{(2)T}\bar{\mathbf{b}}\right)\right]\left\|\bar{\mathbf{b}}\right\|^6 - \mathcal{K}^2(\bar{\mathbf{w}})6\left\|\bar{\mathbf{b}}\right\|^{10} \mathbf{Q}^{(1)T}\bar{\mathbf{b}}}{\left\|\bar{\mathbf{b}}\right\|^{12}} \tag{84}
$$

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{2\left\|\bar{\mathbf{c}}\right\|^2 \mathbf{Q}^{(1)T}\bar{\mathbf{b}} + 2\left\|\bar{\mathbf{b}}\right\|^2 \mathbf{Q}^{(2)T}\bar{\mathbf{c}}}{\left\|\bar{\mathbf{b}}\right\|^6} - \frac{2\left(\bar{\mathbf{b}}^T\bar{\mathbf{c}}\right)\left(\mathbf{Q}^{(1)T}\bar{\mathbf{c}} + \mathbf{Q}^{(2)T}\bar{\mathbf{b}}\right)}{\left\|\bar{\mathbf{b}}\right\|^6} - \frac{6\mathcal{K}^2(\bar{\mathbf{w}})\mathbf{Q}^{(1)T}\bar{\mathbf{b}}}{\left\|\bar{\mathbf{b}}\right\|^2} \tag{85}
$$

Thus

$$
\begin{aligned}
\frac{\partial \mathcal{K}^2_{(n)}(n+1)}{\partial \mathbf{w}^{(n)}} = {} & 2\frac{\left\|\mathbf{p}^{(n)\prime\prime}(n+1)\right\|^2 \mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)\prime}(n+1) + \left\|\mathbf{p}^{(n)\prime}(n+1)\right\|^2 \mathbf{Q}^{(2,0)T}\mathbf{p}^{(n)\prime\prime}(n+1)}{\left\|\mathbf{p}^{(n)\prime}(n+1)\right\|^6}\\
& - 2\frac{\left(\mathbf{p}^{(n)\prime}(n+1)^T\mathbf{p}^{(n)\prime\prime}(n+1)\right)\left(\mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)\prime\prime}(n+1) + \mathbf{Q}^{(2,0)T}\mathbf{p}^{(n)\prime}(n+1)\right)}{\left\|\mathbf{p}^{(n)\prime}(n+1)\right\|^6}\\
& - 6\frac{\mathcal{K}^2_{(n)}(n+1)\mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)\prime}(n+1)}{\left\|\mathbf{p}^{(n)\prime}(n+1)\right\|^2}
\end{aligned} \tag{86}
$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{K}^2_{(n)}(n)}{\partial \mathbf{w}^{(n)}} = {} & 2\frac{\left\|\mathbf{p}^{(n)\prime\prime}(n)\right\|^2 \mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)\prime}(n) + \left\|\mathbf{p}^{(n)\prime}(n)\right\|^2 \mathbf{Q}^{(2,1)T}\mathbf{p}^{(n)\prime\prime}(n)}{\left\|\mathbf{p}^{(n)\prime}(n)\right\|^6}\\
& - 2\frac{\left(\mathbf{p}^{(n)\prime}(n)^T\mathbf{p}^{(n)\prime\prime}(n)\right)\left(\mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)\prime\prime}(n) + \mathbf{Q}^{(2,1)T}\mathbf{p}^{(n)\prime}(n)\right)}{\left\|\mathbf{p}^{(n)\prime}(n)\right\|^6}\\
& - 6\frac{\mathcal{K}^2_{(n)}(n)\mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)\prime}(n)}{\left\|\mathbf{p}^{(n)\prime}(n)\right\|^2}
\end{aligned} \tag{87}
$$

7 Cost function in 3D parametric cubic spline

7.1 Cost function of fitting the cubic spline in the points $\mathbf{r}^{(n)}$

Following the explained in the Section 5, the equation that should be fulfilled to fit the cubic spline in the points can be represented in the next equation

$$\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w} = \begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix} \in \mathbb{R}^{12(N-1)-6} \tag{88}$$

Defining the cost function $E_1(\mathbf{w})$ of fitting of cubic spline in the points $\mathbf{r}^{(n)}$, $\forall 0 \leq n \leq N-1$

$$E_1(\mathbf{w}) = \left\| \begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w} - \begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix} \right\|_{\mathbf{D}}^2 \tag{89}$$

where $\mathbf{D} \in \mathbb{R}^{(12N-18)\times(12N-18)}$ is a diagonal matrix with the weight of each line equation and $\|\mathbf{a}\|_{\mathbf{D}}^2 \equiv \mathbf{a}^T\mathbf{D}\mathbf{a}$.

Applying the derivative in relation of vector $\mathbf{w}$ [1, pp. 11] in the cost function $E_1(\mathbf{w})$ of Eq. 89, we obtain

$$\frac{\partial E_1(\mathbf{w})}{\partial \mathbf{w}} = 2\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}^T \mathbf{D}\left(\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w} - \begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix}\right) \tag{90}$$

7.2 Cost function of curvature the cubic spline in the points $\mathbf{r}^{(n)}$

Following the explained in the Section 6, the equation relative to the curvature that should be minimized can be represented in the next equation

$$E_2 = \frac{1}{2(N-1)}\sum_{n=0}^{N-2}\left\{\mathcal{K}_{(n)}^2(n) + \mathcal{K}_{(n)}^2(n+1)\right\}. \tag{91}$$

Derivating in function of $\mathbf{w}$

$$\frac{\partial E_2}{\partial \mathbf{w}^{(n)}} = \frac{1}{2(N-1)}\left\{\frac{\partial \mathcal{K}_{(n)}^2(n)}{\partial \mathbf{w}^{(n)}} + \frac{\partial \mathcal{K}_{(n)}^2(n+1)}{\partial \mathbf{w}^{(n)}}\right\} \tag{92}$$

$$\frac{\partial E_2}{\partial \mathbf{w}} = \begin{bmatrix}\frac{\partial E_2}{\partial \mathbf{w}^{(0)}}\\ \frac{\partial E_2}{\partial \mathbf{w}^{(1)}}\\ \vdots \\ \frac{\partial E_2}{\partial \mathbf{w}^{(N-2)}}\end{bmatrix} = \frac{1}{2(N-1)}\begin{bmatrix}\frac{\partial \mathcal{K}_{(0)}^2(0)}{\partial \mathbf{w}^{(0)}} + \frac{\partial \mathcal{K}_{(0)}^2(1)}{\partial \mathbf{w}^{(0)}}\\ \frac{\partial \mathcal{K}_{(1)}^2(1)}{\partial \mathbf{w}^{(1)}} + \frac{\partial \mathcal{K}_{(1)}^2(2)}{\partial \mathbf{w}^{(1)}}\\ \vdots \\ \frac{\partial \mathcal{K}_{(N-2)}^2(N-2)}{\partial \mathbf{w}^{(N-2)}} + \frac{\partial \mathcal{K}_{(N-2)}^2(N-1)}{\partial \mathbf{w}^{(N-2)}}\end{bmatrix}. \tag{93}$$

7.3 Parameter calculus of 3D parametric cubic spline

Defining the cost function $E(\mathbf{w})$ as

$$E(\mathbf{w}) = E_1(\mathbf{w}) + \beta E_2(\mathbf{w}) \tag{94}$$

Using the gradient descent technique, we obtain

$$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_i - \alpha \left.\frac{\partial\left\{E_1(\mathbf{w}) + \beta E_2(\mathbf{w})\right\}}{\partial \mathbf{w}}\right|_{\mathbf{w}=\mathbf{w}_i}. \tag{95}$$

Applying the Eqs. 90, 93 we obtain

$$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_i - \alpha\left\{\left.\frac{\partial E_1(\mathbf{w})}{\partial \mathbf{w}}\right|_{\mathbf{w}=\mathbf{w}_i} + \beta\left.\frac{\partial E_2(\mathbf{w})}{\partial \mathbf{w}}\right|_{\mathbf{w}=\mathbf{w}_i}\right\} \tag{96}$$

$$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_i - \alpha \left\{ 2 \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}^T \mathbf{D} \left(\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \mathbf{w}_i - \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \right) + \frac{\beta}{2(N-1)} \begin{bmatrix} \frac{\partial \mathcal{K}^2_{(0)}(0)}{\partial \mathbf{w}^{(0)}} + \frac{\partial \mathcal{K}^2_{(0)}(1)}{\partial \mathbf{w}^{(0)}} \\ \frac{\partial \mathcal{K}^2_{(1)}(1)}{\partial \mathbf{w}^{(1)}} + \frac{\partial \mathcal{K}^2_{(1)}(2)}{\partial \mathbf{w}^{(1)}} \\ \vdots \\ \frac{\partial \mathcal{K}^2_{(N-2)}(N-2)}{\partial \mathbf{w}^{(N-2)}} + \frac{\partial \mathcal{K}^2_{(N-2)}(N-1)}{\partial \mathbf{w}^{(N-2)}} \end{bmatrix} \right\} \tag{97}$$

where α and β are learning hyper-parameters and $\mathbf{w}_0$ will be calculated following the calculus of 3D parametric linear spline seen in the Section 3. The iteration follow until a maximum number of iteration i or until reach a minimum defined error $E_1(\mathbf{w}_i)$.

8 Derivative

8.1 Derivative of $\mathbf{f}\left(\mathbf{A}\mathbf{w} - \mathbf{c}\right)$

Given a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$, a vector $\mathbf{w} \in \mathbb{R}^N$, a vector $\mathbf{c} \in \mathbb{R}^M$, a vectorial function $\mathbf{f} : \mathbb{R}^M \rightarrow \mathbb{R}^L$ with variable vectorial and the variable $t \in \mathbb{R}$. Then

$$\frac{\partial \mathbf{f}\left(\mathbf{A}\mathbf{w} - \mathbf{c}\right)}{\partial t} = \begin{bmatrix} \frac{\partial f_1(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} \\ \frac{\partial f_2(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} \\ \vdots \\ \frac{\partial f_l(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} \\ \vdots \frac{\partial f_L(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} \end{bmatrix} \tag{98}$$

Using the chain rule [1, pp. 15] and given a function $f : \mathbb{R}^M \rightarrow \mathbb{R}$ a vector $\mathbf{v} \in \mathbb{R}^M$ and the variable $t \in \mathbb{R}$, we now that

$$\frac{\partial f\left(\mathbf{v}\right)}{\partial t} = \sum_m^M \frac{\partial f\left(\mathbf{v}\right)}{\partial v_m} \frac{\partial v_m}{\partial t} \tag{99}$$

Thus, if we define

$$[\ldots,\ v_m,\ \ldots]^T = \mathbf{v} = \mathbf{A}\mathbf{w} - \mathbf{c}, \tag{100}$$

then

$$\frac{\partial f_l\left(\mathbf{A}\mathbf{w} - \mathbf{c}\right)}{\partial t} = \frac{\partial f_l\left(\mathbf{v}\right)}{\partial t} \tag{101}$$

$$= \sum_m^M \frac{\partial f_l\left(\mathbf{v}\right)}{\partial v_m} \frac{\partial v_m}{\partial t} \tag{102}$$

If we define $\boxed{\mathbf{1}}_m \equiv [\ldots,\ 0,\ 1, 0, \ldots]^T$, a vector with a unique 1 in the position m. Then $v_m = \boxed{\mathbf{1}}_m^T \left(\mathbf{A}\mathbf{w} - \mathbf{c}\right)$, so that the last equation can be written as

$$\frac{\partial f_l(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} = \sum_m^M \frac{\partial f_l(\mathbf{v})}{\partial v_m} \frac{\partial \boxed{\mathbf{1}}_m^T (\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} \tag{103}$$

$$= \sum_m^M \frac{\partial f_l(\mathbf{v})}{\partial v_m} \boxed{\mathbf{1}}_m^T \left(\mathbf{A}\frac{\partial \mathbf{w}}{\partial t}\right) \tag{104}$$

$$= \left\{\sum_m^M \frac{\partial f_l(\mathbf{v})}{\partial v_m} \boxed{\mathbf{1}}_m^T\right\} \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{105}$$

$$= \begin{bmatrix} \frac{\partial f_l(\mathbf{v})}{\partial v_1} & \dots & \frac{\partial f_l(\mathbf{v})}{\partial v_m} & \dots & \frac{\partial f_l(\mathbf{v})}{\partial v_M} \end{bmatrix}^T \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{106}$$

$$= \nabla^T f_l(\mathbf{v})\, \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{107}$$

Appplying in the Eq. 98, we obtain

$$\frac{\partial \mathbf{f}(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} = \begin{bmatrix} \nabla^T f_1(\mathbf{v})\, \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \\ \nabla^T f_2(\mathbf{v})\, \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \\ \vdots \\ \nabla^T f_l(\mathbf{v})\, \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \\ \vdots \nabla^T f_L(\mathbf{v})\, \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \end{bmatrix} \tag{108}$$

$$= \begin{bmatrix} \nabla^T f_1(\mathbf{v}) \\ \nabla^T f_2(\mathbf{v}) \\ \vdots \\ \nabla^T f_l(\mathbf{v}) \\ \vdots \nabla^T f_L(\mathbf{v}) \end{bmatrix} \mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{109}$$

$$\tag{110}$$

$$\frac{\partial \mathbf{f}(\mathbf{A}\mathbf{w}-\mathbf{c})}{\partial t} = \nabla \mathbf{f}(\mathbf{v})\mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{111}$$

where $\nabla \mathbf{f}(\mathbf{v})$ is the jacobian matrix of $\mathbf{f}(\mathbf{v})$.

8.2 Derivative of $\|\mathbf{f}(\mathbf{A}\mathbf{w}-\mathbf{c})\|_{\mathbf{Q}}^2$

Given a diagonal matrix $\mathbf{Q} \in \mathbb{R}^{L\times L}$, a matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$, a vector $\mathbf{w} \in \mathbb{R}^N$, a vector $\mathbf{c} \in \mathbb{R}^M$ and a vectorial function $\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^L$.

$$E = \|\mathbf{f}(\mathbf{A}\mathbf{w}-\mathbf{c})\|_{\mathbf{Q}}^2 \tag{112}$$

$$= \mathbf{f}^T(\mathbf{A}\mathbf{w}-\mathbf{c})\,\mathbf{Q}\mathbf{f}(\mathbf{A}\mathbf{w}-\mathbf{c}) \tag{113}$$

If we define $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{c}$ and $\mathbf{f}$ not depend on the variable t.

$$\frac{\partial E}{\partial t} = \frac{\partial \left\{\mathbf{f}^T \left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\right\}}{\partial t} \tag{114}$$

$$= \left\{\frac{\partial \mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)}{\partial t}\right\}^T \mathbf{Q}\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right) + \mathbf{f}^T\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\frac{\partial\left\{\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\right\}}{\partial t} \tag{115}$$

$$= 2\mathbf{f}^T\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\frac{\partial \mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)}{\partial t} \tag{116}$$

Applying the Eq. 111

$$\frac{\partial E}{\partial t} = 2\mathbf{f}^T\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\nabla\mathbf{f}(\mathbf{v})\mathbf{A}\frac{\partial \mathbf{w}}{\partial t} \tag{117}$$

8.3 Derivative of $\|\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\|_{\mathbf{Q}}^2$ in relation to $\mathbf{w}$

Given the expression $E = \|\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\|_{\mathbf{Q}}^2$, where the matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$, the vector $\mathbf{w} \in \mathbb{R}^N$, the vector $\mathbf{c} \in \mathbb{R}^M$ and the vectorial function $\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^L$. If we define $\mathbf{A}$, $\mathbf{Q}$, $\mathbf{c}$ and $\mathbf{f}$ not depend of vector $\mathbf{w}$. Then

$$\frac{\partial E}{\partial \mathbf{w}} = \begin{bmatrix}\frac{\partial E}{\partial w_1} & \frac{\partial E}{\partial w_1} & \ldots & \frac{\partial E}{\partial w_n} & \ldots & \frac{\partial E}{\partial w_N}\end{bmatrix}^T \tag{118}$$

Using the Eq. 117

$$\frac{\partial E}{\partial \mathbf{w}} = \left\{2\mathbf{f}^T\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\nabla\mathbf{f}(\mathbf{v})\mathbf{A}\begin{bmatrix}\frac{\partial \mathbf{w}}{\partial w_1} & \frac{\partial \mathbf{w}}{\partial w_2} & \ldots & \frac{\partial \mathbf{w}}{\partial w_n} & \ldots & \frac{\partial \mathbf{w}}{\partial w_N}\end{bmatrix}\right\}^T \tag{119}$$

$$= \left\{2\mathbf{f}^T\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right)\mathbf{Q}\nabla\mathbf{f}(\mathbf{v})\mathbf{A}\right\}^T \tag{120}$$

$$= 2\left\{\mathbf{Q}\nabla\mathbf{f}(\mathbf{v})\mathbf{A}\right\}^T\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right) \tag{121}$$

$$= 2\mathbf{A}^T\left\{\mathbf{Q}\nabla\mathbf{f}(\mathbf{v})\right\}^T\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right) \tag{122}$$

$$\frac{\partial E}{\partial \mathbf{w}} = 2\mathbf{A}^T\left\{\nabla\mathbf{f}(\mathbf{v})\right\}^T\mathbf{Q}\mathbf{f}\left(\mathbf{A}\mathbf{w}-\mathbf{c}\right) \tag{123}$$

8.3.1 Derivative for a specific type of function $\mathbf{f}(\mathbf{v})$

Given a vector $\mathbf{v} \in \mathbb{R}^M$ and a vectorial function $\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^M$ with vector parameter $\mathbf{v} \equiv [v_1, v_2, \ldots, v_m, \ldots v_M]^T$. If we define

$$\mathbf{f}(\mathbf{v}) \equiv \begin{bmatrix} f_1(v_1) & f_2(v_2) & \ldots & f_m(v_m) & \ldots & f_M(v_M) & \end{bmatrix}^T, \tag{124}$$

where $f_m : \mathbb{R} \to \mathbb{R}, \forall 1 \leq m \leq M$. Then, we can calculate the jacobian matrix of function $\mathbf{f}(\mathbf{v})$ as

$$\nabla\mathbf{f}(\mathbf{v}) \equiv \begin{bmatrix} \frac{\partial f_1(v_1)}{\partial v_1} & 0 & \ldots & 0 & 0 \\ 0 & \frac{\partial f_2(v_2)}{\partial v_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{\partial f_{M-1}(v_{M-1})}{\partial v_{M-1}} & 0 \\ 0 & 0 & \ldots & 0 & \frac{\partial f_M(v_M)}{\partial v_M} \end{bmatrix}, \tag{125}$$

$$\nabla\mathbf{f}^T(\mathbf{v}) \equiv \nabla\mathbf{f}(\mathbf{v}) \equiv diag\left(\begin{bmatrix}\frac{\partial f_1(v_1)}{\partial v_1} & \frac{\partial f_2(v_2)}{\partial v_2} & \ldots & \frac{\partial f_m(v_m)}{\partial v_m} & \ldots & \frac{\partial f_M(v_M)}{\partial v_M} & \end{bmatrix}\right), \tag{126}$$

References

[1] Kaare Brandt Petersen, Michael Syskind Pedersen, et al. "The matrix cookbook". In: *Technical University of Denmark* 7.15 (2008), p. 510. URL: `https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf`.

[2] Victor A Toponogov. *Differential geometry of curves and surfaces: A Concise Guide*. Springer, 2006.