Path Filling Points

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Contents

1 3D parametric linear spline

A 3D parametric linear spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 1 shows the polynomials $\hat{\mathbf{p}}^{(n)}(t)$ with parameter t close to the n-th position.

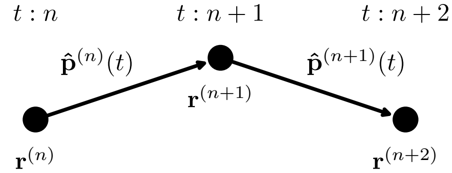

Figure 1: Polynomials in the n-th position of linear spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\ 0 \leq n \leq N-1$, we can generate a linear spline with $N-2$ polynomials $\hat{\mathbf{p}}^{(n)}(t)$, $\forall\ 0 \leq n \leq N-2$, according the Fig. 1. So that, it is fulfilled that

$$\hat{\mathbf{p}}^{(n)}(t) = \begin{bmatrix} \hat{x}^{(n)}(t) & \hat{y}^{(n)}(t) & \hat{z}^{(n)}(t), \end{bmatrix}^T \tag{1}$$

where

$$\hat{x}^{(n)}(t) = \hat{a}_x^{(n)} + \hat{b}_x^{(n)}(t-n), \tag{2}$$

$$\hat{y}^{(n)}(t) = \hat{a}_y^{(n)} + \hat{b}_y^{(n)}(t-n), \tag{3}$$

$$\hat{z}^{(n)}(t) = \hat{a}_z^{(n)} + \hat{b}_z^{(n)}(t-n). \tag{4}$$

1.1 Matricial form of polynomials $\hat{\mathbf{p}}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\hat{\mathbf{w}}^{(n)} = \left[\begin{array}{cc:cc:cc} \hat{a}_x^{(n)} & \hat{b}_x^{(n)} & \hat{a}_y^{(n)} & \hat{b}_y^{(n)} & \hat{a}_z^{(n)} & \hat{b}_z^{(n)} \end{array}\right]^T, \tag{5}$$

$$\hat{\mathbf{A}}^{(n)}(t) = \left[\begin{array}{cc:cc:cc} 1 & (t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (t-n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) \end{array}\right], \tag{6}$$

$$\hat{\mathbf{p}}^{(n)}(t) = \hat{\mathbf{A}}^{(n)}(t)\hat{\mathbf{w}}^{(n)}. \tag{7}$$

Additionally, we can define

$$\hat{\mathbf{w}} \equiv \begin{bmatrix} \hat{\mathbf{w}}^{(0)} \\ \hat{\mathbf{w}}^{(1)} \\ \vdots \\ \hat{\mathbf{w}}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{8}$$

2 Boundary conditions in 3D parametric linear spline

2.1 Conditions in points

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-2$, so that

$$\hat{\mathbf{p}}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{9}$$

$$\hat{\mathbf{p}}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{10}$$

2.1.1 Matricial form of the conditions in points

Using the Eq. 7 in the Eqs. 9 and 10, we obtain for $0 \leq n \leq N-2$

$$\hat{\mathbf{A}}^{(n)}(n)\hat{\mathbf{w}}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{11}$$

$$\hat{\mathbf{A}}^{(N-2)}(N-1)\hat{\mathbf{w}}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{12}$$

where, using the Eq. 6, we know that

$$\hat{\mathbf{A}}^{(n)}(n+1) = \left[\begin{array}{cc:cc:cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,0)} \in \mathbb{R}^{3\times 6}, \tag{13}$$

$$\hat{\mathbf{A}}^{(n+1)}(n+1) = \hat{\mathbf{A}}^{(n)}(n) = \left[\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,1)} \in \mathbb{R}^{3\times 6}, \tag{14}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 11 and 12, $\forall 0 \leq n \leq N-2$. We can write the Eqs. 9 and 10 as

$$\hat{\mathbf{P}}\hat{\mathbf{w}} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{15}$$

Where

$$\hat{\mathbf{P}} \equiv \begin{bmatrix} \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N\times 6(N-1)} \tag{16}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{17}$$

2.2 Boundary conditions in internal point

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-3$, so that

$$\hat{\mathbf{p}}^{(n)}(n+1) - \hat{\mathbf{p}}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{18}$$

2.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eq. 7 in the Eq. 18, we obtain for $0 \leq n \leq N-3$

$$\hat{\mathbf{A}}^{(n)}(n+1)\hat{\mathbf{w}}^{(n)} - \hat{\mathbf{A}}^{n+1}(n+1)\hat{\mathbf{w}}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{19}$$

Grouping in a matrix

$$\begin{bmatrix} \hat{\mathbf{A}}^{(n)}(n+1) & -\hat{\mathbf{A}}^{n+1}(n+1) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3, \tag{20}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 18 and 20, these can be rewritten for $0 \leq n \leq N-3$

$$\begin{bmatrix} \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3 \tag{21}$$

Finally, concatenating to all values for $0 \leq n \leq N-3$ in the boundary conditions in internal point equation's, we obtain

$$\hat{\mathbf{Q}} \equiv \left[\begin{array}{ccccccc} \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} \end{array}\right] \in \mathbb{R}^{3(N-2)\times 6(N-1)} \tag{22}$$

$$\hat{\mathbf{Q}}\hat{\mathbf{w}} = \mathbf{0} \in \mathbb{R}^{3(N-2)} \tag{23}$$

3 Parameter calculus of 3D parametric linear spline

Following the explained in the Section 2, the equation that should be fulfilled to fit the linear spline in the points can be represented in the next equation

$$\begin{bmatrix} \hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \end{bmatrix} \hat{\mathbf{w}} = \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{24}$$

Thus,

$$\hat{\mathbf{w}} = \begin{bmatrix} \hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{25}$$

4 3D parametric cubic spline

A 3D parametric cubic spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 2 shows the polynomials $\mathbf{p}^{(n)}(t)$ with parameter t close to the n-th position.

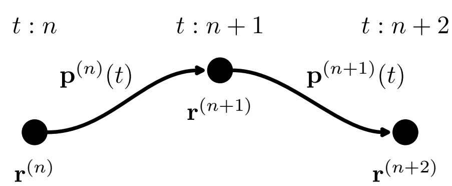

Figure 2: Polynomials in the n-th position of cubic spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\ 0 \leq n \leq N-1$, we can generate a cubic spline with $N-2$ polynomials $\mathbf{p}^{(n)}(t)$, $\forall\ 0 \leq n \leq N-2$, according the Fig. 2. So that, it is fulfilled that

$$\mathbf{p}^{(n)}(t) = \begin{bmatrix} x^{(n)}(t) & y^{(n)}(t) & z^{(n)}(t) \end{bmatrix}^T, \tag{26}$$

where

$$x^{(n)}(t) = a_x^{(n)} + b_x^{(n)}(t-n) + c_x^{(n)}(t-n)^2 + d_x^{(n)}(t-n)^3, \tag{27}$$

$$y^{(n)}(t) = a_y^{(n)} + b_y^{(n)}(t-n) + c_y^{(n)}(t-n)^2 + d_y^{(n)}(t-n)^3, \tag{28}$$

$$z^{(n)}(t) = a_z^{(n)} + b_z^{(n)}(t-n) + c_z^{(n)}(t-n)^2 + d_z^{(n)}(t-n)^3. \tag{29}$$

4.1 Matricial form of polynomials $\mathbf{p}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\mathbf{w}^{(n)} = \left[\begin{array}{cccc:cccc:cccc} a_x^{(n)} & b_x^{(n)} & c_x^{(n)} & d_x^{(n)} & a_y^{(n)} & b_y^{(n)} & c_y^{(n)} & d_y^{(n)} & a_z^{(n)} & b_z^{(n)} & c_z^{(n)} & d_z^{(n)} \end{array}\right]^T, \tag{30}$$

$$\mathbf{A}^{(n)}(t) = \left[\begin{array}{cccc:cccc:cccc} 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 \end{array}\right], \tag{31}$$

$$\mathbf{p}^{(n)}(t) = \mathbf{A}^{(n)}(t)\mathbf{w}^{(n)}. \tag{32}$$

Additionally, we can define

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{w}^{(0)} \\ \mathbf{w}^{(1)} \\ \vdots \\ \mathbf{w}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{12(N-1)} \tag{33}$$

4.2 Additional useful derivation of $\mathbf{p}^{(n)}(t)$

$$\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 \end{array}\right], \tag{34}$$

$$\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) \end{array}\right]. \tag{35}$$

If we define $\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} \equiv D_t \mathbf{A}^{(n)}(t)$ and $\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} \equiv D_t^2 \mathbf{A}^{(n)}(t)$, then

$$\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} = D_t \mathbf{A}^{(n)}(t) \mathbf{w}^{(n)} \tag{36}$$

$$= \begin{bmatrix} b_x^{(n)} + 2(t-n)c_x^{(n)} + 3(t-n)^2 d_x^{(n)} \\ b_y^{(n)} + 2(t-n)c_y^{(n)} + 3(t-n)^2 d_y^{(n)} \\ b_z^{(n)} + 2(t-n)c_z^{(n)} + 3(t-n)^2 d_z^{(n)} \end{bmatrix} \tag{37}$$

$$\frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2} = D_t^2 \mathbf{A}^{(n)}(t) \mathbf{w}^{(n)} \tag{38}$$

$$= \begin{bmatrix} 2c_x^{(n)} + 6(t-n)d_x^{(n)} \\ 2c_y^{(n)} + 6(t-n)d_y^{(n)} \\ 2c_z^{(n)} + 6(t-n)d_z^{(n)} \end{bmatrix} \tag{39}$$

5 Boundary conditions in 3D parametric cubic spline

5.1 Conditions in points

Following the Fig. 2, we can affirm that for $0 \leq n \leq N-2$, so that

$$\mathbf{p}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{40}$$

$$\mathbf{p}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{41}$$

5.1.1 Matricial form of the conditions in points

Using the Eq. 32 in the Eqs. 40 and 41, we obtain for $0 \leq n \leq N-2$

$$\mathbf{A}^{(n)}(n) \mathbf{w}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{42}$$

$$\mathbf{A}^{(N-2)}(N-1) \mathbf{w}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{43}$$

where, using the Eq. 31, we know that

$$\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \equiv \mathbf{Q}^{(0,0)} \in \mathbb{R}^{3\times 12}, \tag{44}$$

$$\mathbf{A}^{(n+1)}(n+1) = \mathbf{A}^{(n)}(n) = \left[\begin{array}{cccc:cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(0,1)} \in \mathbb{R}^{3\times 12}, \tag{45}$$

Thus, using the Eqs. 44 and 45 in the Eqs. 42 and 43, $\forall 0 \leq n \leq N-2$; We can write the Eqs. 40 and 41 as

$$\mathbf{P}\mathbf{w} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{46}$$

Where

$$\mathbf{P} \equiv \begin{bmatrix} \mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N \times 12(N-1)} \tag{47}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{48}$$

5.2 Boundary conditions in internal point

Following the Fig. 2, we can affirm that for $0 \leq n \leq N-3$, so that

$$\mathbf{p}^{(n)}(n+1) - \mathbf{p}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{49}$$

$$\left.\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t}\right|_{t=n+1} - \left.\frac{\partial \mathbf{p}^{n+1}(t)}{\partial t}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3, \tag{50}$$

$$\left.\frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2}\right|_{t=n+1} - \left.\frac{\partial^2 \mathbf{p}^{n+1}(t)}{\partial t^2}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3. \tag{51}$$

5.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eqs. 32, 34, 35, 36 and 38 in the Eqs. 49, 50 and 51, we obtain for $0 \leq n \leq N-3$

$$\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - \mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{52}$$

$$D_t\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t\mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{53}$$

$$D_t^2\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t^2\mathbf{A}^{(n+1)}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3. \tag{54}$$

Grouping in a matrix

$$\begin{bmatrix} \mathbf{A}^{(n)}(n+1) & -\mathbf{A}^{(n+1)}(n+1) \\ D_t\mathbf{A}^{(n)}(n+1) & -D_t\mathbf{A}^{(n+1)}(n+1) \\ D_t^2\mathbf{A}^{(n)}(n+1) & -D_t^2\mathbf{A}^{(n+1)}(n+1) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(n)} \\ \mathbf{w}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^9, \tag{55}$$

using the Eqs. 34 and 35, we know that

$$D_t\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \equiv \mathbf{Q}^{(1,0)} \in \mathbb{R}^{3\times 12}, \tag{56}$$

$$D_t\mathbf{A}^{(n+1)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(1,1)} \in \mathbb{R}^{3\times 12}, \tag{57}$$

$$D_t^2\mathbf{A}^{(n)}(n+1)=\left[\begin{array}{cccc|cccc|cccc}0&0&2&6&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&6&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&2&6\end{array}\right]\equiv\mathbf{Q}^{(2,0)}\in\mathbb{R}^{3\times 12}, \tag{58}$$

$$D_t^2\mathbf{A}^{(n+1)}(n+1)=\left[\begin{array}{cccc|cccc|cccc}0&0&2&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&2&0\end{array}\right]\equiv\mathbf{Q}^{(2,1)}\in\mathbb{R}^{3\times 12}. \tag{59}$$

Thus, using the Eqs. 44, 45, 56, 57, 58 and 59 in the Eqs. 49, 50 and 51, these can be rewritten for $0\le n\le N-3$

$$\begin{bmatrix}\mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)}\\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)}\\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)}\end{bmatrix}\begin{bmatrix}\mathbf{w}^{(n)}\\ \mathbf{w}^{(n+1)}\end{bmatrix}=\mathbf{0}\in\mathbb{R}^9 \tag{60}$$

Finally, concatenating to all values for $0\le n\le N-3$ in the boundary conditions in internal point equation's, we obtain

$$\mathbf{Q}\equiv\left[\begin{array}{cccccccc}\mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)}\end{array}\right]\in\mathbb{R}^{9(N-2)\times 12(N-1)} \tag{61}$$

$$\mathbf{Q}\mathbf{w}=\mathbf{0}\in\mathbb{R}^{9(N-2)} \tag{62}$$

6 Cost function in 3D parametric cubic spline

6.1 Cost function of fitting the cubic spline in the points $\mathbf{r}^{(n)}$

Following the explained in the Section 5, the equation that should be fulfilled to fit the cubic spline in the points can be represented in the next equation

$$\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w}=\begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix}\in\mathbb{R}^{12(N-1)-6} \tag{63}$$

Defining the cost function $E_1(\mathbf{w})$ of fitting of cubic spline in the points $\mathbf{r}^{(n)}$, $\forall 0\le n\le N-1$

$$E_1(\mathbf{w})=\left\|\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w}-\begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix}\right\|_{\mathbf{D}}^2 \tag{64}$$

where $\mathbf{D}\in\mathbb{R}^{(12N-18)\times(12N-18)}$ is a diagonal matrix with the weight of each line equation.

Applying the derivative in relation of vector $\mathbf{w}$ [1, pp. 11] in the cost function $E_1(\mathbf{w})$ of Eq. 64, we obtain

$$\frac{\partial E_1(\mathbf{w})}{\partial\mathbf{w}}=2\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}^T\mathbf{D}\left(\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}\mathbf{w}-\begin{bmatrix}\mathbf{r}\\ \mathbf{0}\end{bmatrix}\right) \tag{65}$$

7 Parameter calculus of 3D parametric cubic spline

Using the Eq. 65 and the gradient descent technique, we obtain

$$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_i - \alpha \frac{\partial E_1(\mathbf{w})}{\partial \mathbf{w}} \tag{66}$$

$$\mathbf{w}_{i+1} \leftarrow \mathbf{w}_i - 2\alpha \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}^T \mathbf{D} \left(\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \mathbf{w} - \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \right) \tag{67}$$

where α is a learning hyper-parameter and $\mathbf{w}_0$ will be calculated following the calculus of 3D parametric linear spline seen in the Section 3. The iteration follow until a maximum number od iteration i or until reach a minimum defined error $E_1(\mathbf{w}_i)$.

8 Curvature of a 3D parametric function

The curvature $\mathcal{K}_{(n)}(t)$ of a 3D parametric function, $\mathbf{p}^{(n)}(t) \in \mathbb{R}^3$ with parameter t, can be calculated [2, pp. 21] using the next equation,

$$\mathcal{K}_{(n)}(t) = \frac{\left\| \frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} \times \frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2} \right\|}{\left\| \frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} \right\|^3} \equiv \frac{\left\| \mathbf{p}^{(n)\prime}(t) \times \mathbf{p}^{(n)\prime\prime}(t) \right\|}{\left\| \mathbf{p}^{(n)\prime}(t) \right\|^3} \tag{68}$$

if we calculate the square curvature $\mathcal{K}^2_{(n)}(t)$ to the cases when $t \equiv n$ and $t \equiv n+1$, then we obtain

$$\mathcal{K}^2_{(n)}(n) = \frac{\left\| \left(\mathbf{Q}^{(1,1)} \mathbf{w}^{(n)} \right) \times \left(\mathbf{Q}^{(2,1)} \mathbf{w}^{(n)} \right) \right\|^2}{\left\| \mathbf{Q}^{(1,1)} \mathbf{w}^{(n)} \right\|^6} \tag{69}$$

$$\mathcal{K}^2_{(n)}(n+1) = \frac{\left\| \left(\mathbf{Q}^{(1,0)} \mathbf{w}^{(n)} \right) \times \left(\mathbf{Q}^{(2,0)} \mathbf{w}^{(n)} \right) \right\|^2}{\left\| \mathbf{Q}^{(1,0)} \mathbf{w}^{(n)} \right\|^6} \tag{70}$$

or rewritten in a simplified form to minimize the indices

$$\mathcal{K}^2(\bar{\mathbf{w}}) = \frac{\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2}{\left\| \bar{\mathbf{b}} \right\|^6}. \tag{71}$$

Knowing that $\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2$ represents the square area of parallelepid form by the vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{c}}$, then $\left\| \bar{\mathbf{b}} \times \bar{\mathbf{c}} \right\|^2 = \left(\left\| \bar{\mathbf{b}} \right\| sin(\theta) \left\| \bar{\mathbf{c}} \right\| \right)^2 = \left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 cos(\theta)^2$. Thus, the Eq. 71 can be written as

$$\mathcal{K}^2(\bar{\mathbf{w}}) = \frac{\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2}{\left\| \bar{\mathbf{b}} \right\|^6} \tag{72}$$

derivating in function of vector $\mathbf{w}$

$$\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\frac{\partial \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2 \right)}{\partial \bar{\mathbf{w}}} \left\| \bar{\mathbf{b}} \right\|^6 - \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2 \right) \frac{\partial \left\| \bar{\mathbf{b}} \right\|^6}{\partial \bar{\mathbf{w}}}}{\left\| \bar{\mathbf{b}} \right\|^{12}} \tag{73}$$

$$\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left(\frac{\partial \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 \right)}{\partial \bar{\mathbf{w}}} - \frac{\partial \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2}{\partial \bar{\mathbf{w}}} \right) \left\| \bar{\mathbf{b}} \right\|^6 - \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2 \right) \frac{\partial \left\| \bar{\mathbf{b}} \right\|^6}{\partial \bar{\mathbf{w}}}}{\left\| \bar{\mathbf{b}} \right\|^{12}} \tag{74}$$

Solving [1, pp. 11] the Eq. 74 by parts

$$
\begin{aligned}
\frac{\partial \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 \right)}{\partial \bar{\mathbf{w}}} &= \frac{\partial \left\| \bar{\mathbf{b}} \right\|^2}{\partial \bar{\mathbf{w}}} \left\| \bar{\mathbf{c}} \right\|^2 + \left\| \bar{\mathbf{b}} \right\|^2 \frac{\partial \left\| \bar{\mathbf{c}} \right\|^2}{\partial \bar{\mathbf{w}}} && (75) \\
&= \frac{\partial \left(\bar{\mathbf{w}}^T \mathbf{Q}^{(1)T} \mathbf{Q}^{(1)} \bar{\mathbf{w}} \right)}{\partial \bar{\mathbf{w}}} \left\| \bar{\mathbf{c}} \right\|^2 + \left\| \bar{\mathbf{b}} \right\|^2 \frac{\partial \left(\bar{\mathbf{w}}^T \mathbf{Q}^{(2)T} \mathbf{Q}^{(2)} \bar{\mathbf{w}} \right)}{\partial \bar{\mathbf{w}}} && (76) \\
&= 2\mathbf{Q}^{(1)T} \mathbf{Q}^{(1)} \bar{\mathbf{w}} \left\| \bar{\mathbf{c}} \right\|^2 + 2 \left\| \bar{\mathbf{b}} \right\|^2 \mathbf{Q}^{(2)T} \mathbf{Q}^{(2)} \bar{\mathbf{w}} && (77) \\
&= 2 \left\| \bar{\mathbf{c}} \right\|^2 \mathbf{Q}^{(1)T} \bar{\mathbf{b}} + 2 \left\| \bar{\mathbf{b}} \right\|^2 \mathbf{Q}^{(2)T} \bar{\mathbf{c}} && (78)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2}{\partial \bar{\mathbf{w}}} &= 2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \frac{\partial \left(\bar{\mathbf{w}}^T \mathbf{Q}^{(1)T} \mathbf{Q}^{(2)} \bar{\mathbf{w}} \right)}{\partial \bar{\mathbf{w}}} && (79) \\
&= 2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \left(\mathbf{Q}^{(1)T} \mathbf{Q}^{(2)} + \mathbf{Q}^{(2)T} \mathbf{Q}^{(1)} \right) \bar{\mathbf{w}} && (80) \\
&= 2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \left(\mathbf{Q}^{(1)T} \bar{\mathbf{c}} + \mathbf{Q}^{(2)T} \bar{\mathbf{b}} \right) && (81)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \left\| \bar{\mathbf{b}} \right\|^6}{\partial \bar{\mathbf{w}}} &= \frac{\partial \left(\left\| \bar{\mathbf{b}} \right\|^2 \right)^3}{\partial \bar{\mathbf{w}}} && (82) \\
&= 3 \left(\left\| \bar{\mathbf{b}} \right\|^2 \right)^2 \frac{\partial \left\| \bar{\mathbf{b}} \right\|^2}{\partial \bar{\mathbf{w}}} && (83) \\
&= 3 \left\| \bar{\mathbf{b}} \right\|^4 \frac{\partial \bar{\mathbf{b}}^T \bar{\mathbf{b}}}{\partial \bar{\mathbf{w}}} && (84) \\
&= 3 \left\| \bar{\mathbf{b}} \right\|^4 \frac{\partial \bar{\mathbf{w}}^T \mathbf{Q}^{(1)T} \mathbf{Q}^{(1)} \bar{\mathbf{w}}}{\partial \bar{\mathbf{w}}} && (85) \\
&= 6 \left\| \bar{\mathbf{b}} \right\|^4 \mathbf{Q}^{(1)T} \mathbf{Q}^{(1)} \bar{\mathbf{w}} && (86) \\
&= 6 \left\| \bar{\mathbf{b}} \right\|^4 \mathbf{Q}^{(1)T} \bar{\mathbf{b}} && (87)
\end{aligned}
$$

We obtain

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left[2 \left\| \bar{\mathbf{c}} \right\|^2 \mathbf{Q}^{(1)T} \bar{\mathbf{b}} + 2 \left\| \bar{\mathbf{b}} \right\|^2 \mathbf{Q}^{(2)T} \bar{\mathbf{c}} - 2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \left(\mathbf{Q}^{(1)T} \bar{\mathbf{c}} + \mathbf{Q}^{(2)T} \bar{\mathbf{b}} \right) \right] \left\| \bar{\mathbf{b}} \right\|^6 - \left(\left\| \bar{\mathbf{b}} \right\|^2 \left\| \bar{\mathbf{c}} \right\|^2 - \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right)^2 \right) 6 \left\| \bar{\mathbf{b}} \right\|^4 \mathbf{Q}^{(1)T} \bar{\mathbf{b}}}{\left\| \bar{\mathbf{b}} \right\|^{12}} \tag{88}
$$

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{\left[2 \left\| \bar{\mathbf{c}} \right\|^2 \mathbf{Q}^{(1)T} \bar{\mathbf{b}} + 2 \left\| \bar{\mathbf{b}} \right\|^2 \mathbf{Q}^{(2)T} \bar{\mathbf{c}} - 2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \left(\mathbf{Q}^{(1)T} \bar{\mathbf{c}} + \mathbf{Q}^{(2)T} \bar{\mathbf{b}} \right) \right] \left\| \bar{\mathbf{b}} \right\|^6 - \mathcal{K}^2(\bar{\mathbf{w}}) 6 \left\| \bar{\mathbf{b}} \right\|^{10} \mathbf{Q}^{(1)T} \bar{\mathbf{b}}}{\left\| \bar{\mathbf{b}} \right\|^{12}} \tag{89}
$$

$$
\frac{\partial \mathcal{K}^2(\bar{\mathbf{w}})}{\partial \bar{\mathbf{w}}} = \frac{2 \left\| \bar{\mathbf{c}} \right\|^2 \mathbf{Q}^{(1)T} \bar{\mathbf{b}} + 2 \left\| \bar{\mathbf{b}} \right\|^2 \mathbf{Q}^{(2)T} \bar{\mathbf{c}}}{\left\| \bar{\mathbf{b}} \right\|^6} - \frac{2 \left(\bar{\mathbf{b}}^T \bar{\mathbf{c}} \right) \left(\mathbf{Q}^{(1)T} \bar{\mathbf{c}} + \mathbf{Q}^{(2)T} \bar{\mathbf{b}} \right)}{\left\| \bar{\mathbf{b}} \right\|^6} - \frac{6 \mathcal{K}^2(\bar{\mathbf{w}}) \mathbf{Q}^{(1)T} \bar{\mathbf{b}}}{\left\| \bar{\mathbf{b}} \right\|^2} \tag{90}
$$

Thus

$$
\begin{aligned}
\frac{\partial \mathcal{K}^2_{(n)}(n+1)}{\partial \mathbf{w}^{(n)}} = & \, 2\frac{\left\|\mathbf{p}^{(n)''}(n+1)\right\|^2 \mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)'}(n+1) + \left\|\mathbf{p}^{(n)'}(n+1)\right\|^2 \mathbf{Q}^{(2,0)T}\mathbf{p}^{(n)''}(n+1)}{\left\|\mathbf{p}^{(n)'}(n+1)\right\|^6} \\
& - 2\frac{\left(\mathbf{p}^{(n)'}(n+1)^T \mathbf{p}^{(n)''}(n+1)\right)\left(\mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)''}(n+1) + \mathbf{Q}^{(2,0)T}\mathbf{p}^{(n)'}(n+1)\right)}{\left\|\mathbf{p}^{(n)'}(n+1)\right\|^6} \\
& - 6\frac{\mathcal{K}^2_{(n)}(n+1)\mathbf{Q}^{(1,0)T}\mathbf{p}^{(n)'}(n+1)}{\left\|\mathbf{p}^{(n)'}(n+1)\right\|^2}
\end{aligned} \tag{91}
$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{K}^2_{(n)}(n)}{\partial \mathbf{w}^{(n)}} = & \, 2\frac{\left\|\mathbf{p}^{(n)''}(n)\right\|^2 \mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)'}(n) + \left\|\mathbf{p}^{(n)'}(n)\right\|^2 \mathbf{Q}^{(2,1)T}\mathbf{p}^{(n)''}(n)}{\left\|\mathbf{p}^{(n)'}(n)\right\|^6} \\
& - 2\frac{\left(\mathbf{p}^{(n)'}(n)^T \mathbf{p}^{(n)''}(n)\right)\left(\mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)''}(n) + \mathbf{Q}^{(2,1)T}\mathbf{p}^{(n)'}(n)\right)}{\left\|\mathbf{p}^{(n)'}(n)\right\|^6} \\
& - 6\frac{\mathcal{K}^2_{(n)}(n)\mathbf{Q}^{(1,1)T}\mathbf{p}^{(n)'}(n)}{\left\|\mathbf{p}^{(n)'}(n)\right\|^2}
\end{aligned} \tag{92}
$$

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