Path Filling Points

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Contents

1 3D parametric linear spline

A 3D parametric linear spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 1 shows the polynomials $\hat{\mathbf{p}}^{(n)}(t)$ with parameter t close to the n-th position.

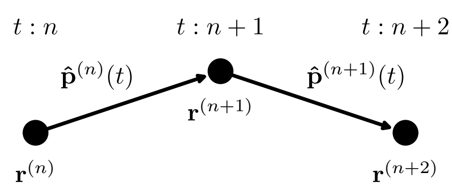

Figure 1: Polynomials in the n-th position of linear spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\ 0 \leq n \leq N-1$, we can generate a linear spline with $N-2$ polynomials $\hat{\mathbf{p}}^{(n)}(t)$, $\forall\ 0 \leq n \leq N-2$, according the Fig. 1. So that, it is fulfilled that

$$\hat{\mathbf{p}}^{(n)}(t) = \begin{bmatrix} \hat{p}_x^{(n)}(t) & \hat{p}_y^{(n)}(t) & \hat{p}_z^{(n)}(t), \end{bmatrix}^T \tag{1}$$

where

$$\hat{p}_x^{(n)}(t) = \hat{a}_x^{(n)} + \hat{b}_x^{(n)}(t-n), \tag{2}$$

$$\hat{p}_y^{(n)}(t) = \hat{a}_y^{(n)} + \hat{b}_y^{(n)}(t-n), \tag{3}$$

$$\hat{p}_z^{(n)}(t) = \hat{a}_z^{(n)} + \hat{b}_z^{(n)}(t-n). \tag{4}$$

1.1 Matricial form of polynomials $\hat{\mathbf{p}}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\hat{\mathbf{w}}^{(n)} = \left[\begin{array}{cc:cc:cc} \hat{a}_x^{(n)} & \hat{b}_x^{(n)} & \hat{a}_y^{(n)} & \hat{b}_y^{(n)} & \hat{a}_z^{(n)} & \hat{b}_z^{(n)} \end{array}\right]^T, \tag{5}$$

$$\hat{\mathbf{A}}^{(n)}(t) = \left[\begin{array}{cc:cc:cc} 1 & (t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (t-n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) \end{array}\right], \tag{6}$$

$$\hat{\mathbf{p}}^{(n)}(t) = \hat{\mathbf{A}}^{(n)}(t)\hat{\mathbf{w}}^{(n)}. \tag{7}$$

Additionally, we can define

$$\hat{\mathbf{w}} \equiv \begin{bmatrix} \hat{\mathbf{w}}^{(0)} \\ \hat{\mathbf{w}}^{(1)} \\ \vdots \\ \hat{\mathbf{w}}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{8}$$

2 Boundary conditions in 3D parametric linear spline

2.1 Conditions in points

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-2$, so that

$$\hat{\mathbf{p}}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{9}$$

$$\hat{\mathbf{p}}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{10}$$

2.1.1 Matricial form of the conditions in points

Using the Eq. 7 in the Eqs. 9 and 10, we obtain for $0 \leq n \leq N-2$

$$\hat{\mathbf{A}}^{(n)}(n)\hat{\mathbf{w}}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{11}$$

$$\hat{\mathbf{A}}^{(N-2)}(N-1)\hat{\mathbf{w}}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{12}$$

where, using the Eq. 6, we know that

$$\hat{\mathbf{A}}^{(n)}(n+1) = \left[\begin{array}{cc:cc:cc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,0)} \in \mathbb{R}^{3\times 6}, \tag{13}$$

$$\hat{\mathbf{A}}^{(n+1)}(n+1) = \hat{\mathbf{A}}^{(n)}(n) = \left[\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \equiv \hat{\mathbf{Q}}^{(0,1)} \in \mathbb{R}^{3\times 6}, \tag{14}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 11 and 12, $\forall 0 \leq n \leq N-2$. We can write the Eqs. 9 and 10 as

$$\hat{\mathbf{P}}\hat{\mathbf{w}} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{15}$$

Where

$$\hat{\mathbf{P}} \equiv \begin{bmatrix} \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N \times 6(N-1)} \tag{16}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{17}$$

2.2 Boundary conditions in internal point

Following the Fig. 1, we can affirm that for $0 \leq n \leq N-3$, so that

$$\hat{\mathbf{p}}^{(n)}(n+1) - \hat{\mathbf{p}}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{18}$$

2.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eq. 7 in the Eq. 18, we obtain for $0 \leq n \leq N-3$

$$\hat{\mathbf{A}}^{(n)}(n+1)\hat{\mathbf{w}}^{(n)} - \hat{\mathbf{A}}^{n+1}(n+1)\hat{\mathbf{w}}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{19}$$

Grouping in a matrix

$$\begin{bmatrix} \hat{\mathbf{A}}^{(n)}(n+1) & -\hat{\mathbf{A}}^{n+1}(n+1) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3, \tag{20}$$

Thus, using the Eqs. 13 and 14 in the Eqs. 18 and 20, these can be rewritten for $0 \leq n \leq N-3$

$$\begin{bmatrix} \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}^{(n)} \\ \hat{\mathbf{w}}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^3 \tag{21}$$

Finally, concatenating to all values for $0 \leq n \leq N-3$ in the boundary conditions in internal point equation's, we obtain

$$\hat{\mathbf{Q}} \equiv \left[\begin{array}{ccccccc} \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \hat{\mathbf{Q}}^{(0,0)} & -\hat{\mathbf{Q}}^{(0,1)} \end{array}\right] \in \mathbb{R}^{3(N-2) \times 6(N-1)} \tag{22}$$

$$\hat{\mathbf{Q}}\hat{\mathbf{w}} = \mathbf{0} \in \mathbb{R}^{3(N-2)} \tag{23}$$

3 Parameter calculus of 3D parametric linear spline

Following the explained in the Section 2, the equation that should be fulfilled to fit the linear spline in the points can be represented in the next equation

$$\begin{bmatrix} \hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \end{bmatrix} \hat{\mathbf{w}} = \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{24}$$

Thus,

$$\hat{\mathbf{w}} = \begin{bmatrix} \hat{\mathbf{P}} \\ \hat{\mathbf{Q}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6(N-1)} \tag{25}$$

4 3D parametric cubic spline

A 3D parametric cubic spline is a parametric function defined piecewise by parametric polynomials in $\mathbb{R}^3$ space. The Fig. 2 shows the polynomials $\mathbf{p}^{(n)}(t)$ with parameter t close to the n-th position.

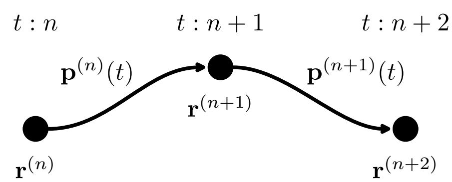

Figure 2: Polynomials in the n-th position of cubic spline

Given a set of N points $\mathbf{r}^{(n)} \in \mathbb{R}^3$, $\forall\ 0 \leq n \leq N-1$, we can generate a cubic spline with $N-2$ polynomials $\mathbf{p}^{(n)}(t)$, $\forall\ 0 \leq n \leq N-2$, according the Fig. 2. So that, it is fulfilled that

$$\mathbf{p}^{(n)}(t) = \begin{bmatrix} p_x^{(n)}(t) & p_y^{(n)}(t) & p_z^{(n)}(t), \end{bmatrix}^T \tag{26}$$

where

$$p_x^{(n)}(t) = a_x^{(n)} + b_x^{(n)}(t-n) + c_x^{(n)}(t-n)^2 + d_x^{(n)}(t-n)^3, \tag{27}$$
$$p_y^{(n)}(t) = a_y^{(n)} + b_y^{(n)}(t-n) + c_y^{(n)}(t-n)^2 + d_y^{(n)}(t-n)^3, \tag{28}$$
$$p_z^{(n)}(t) = a_z^{(n)} + b_z^{(n)}(t-n) + c_z^{(n)}(t-n)^2 + d_z^{(n)}(t-n)^3. \tag{29}$$

4.1 Matricial form of polynomials $\mathbf{p}^{(n)}(t)$

For all values $0 \leq n \leq N-2$, we know that

$$\mathbf{w}^{(n)} = \left[\begin{array}{cccc|cccc|cccc} a_x^{(n)} & b_x^{(n)} & c_x^{(n)} & d_x^{(n)} & a_y^{(n)} & b_y^{(n)} & c_y^{(n)} & d_y^{(n)} & a_z^{(n)} & b_z^{(n)} & c_z^{(n)} & d_z^{(n)} \end{array} \right]^T, \tag{30}$$

$$\mathbf{A}^{(n)}(t) = \left[\begin{array}{cccc|cccc|cccc} 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (t-n) & (t-n)^2 & (t-n)^3 \end{array} \right], \tag{31}$$

$$\mathbf{p}^{(n)}(t) = \mathbf{A}^{(n)}(t)\mathbf{w}^{(n)}. \tag{32}$$

Additionally, we can define

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{w}^{(0)} \\ \mathbf{w}^{(1)} \\ \vdots \\ \mathbf{w}^{(N-2)} \end{bmatrix} \in \mathbb{R}^{12(N-1)} \tag{33}$$

4.2 Additional useful derivation of $\mathbf{p}^{(n)}(t)$

$$\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2(t-n) & 3(t-n)^2 \end{array}\right], \tag{34}$$

$$\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6(t-n) \end{array}\right]. \tag{35}$$

If we define $\frac{\partial \mathbf{A}^{(n)}(t)}{\partial t} \equiv D_t \mathbf{A}^{(n)}(t)$ and $\frac{\partial^2 \mathbf{A}^{(n)}(t)}{\partial t^2} \equiv D_t^2 \mathbf{A}^{(n)}(t)$, then

$$\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t} = D_t \mathbf{A}^{(n)}(t)\mathbf{w}^{(n)} \qquad \wedge \qquad \frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2} = D_t^2 \mathbf{A}^{(n)}(t)\mathbf{w}^{(n)}. \tag{36}$$

5 Boundary conditions in 3D parametric cubic spline

5.1 Conditions in points

Following the Fig. 2, we can affirm that for $0 \le n \le N-2$, so that

$$\mathbf{p}^{(n)}(n) = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{37}$$

$$\mathbf{p}^{(N-2)}(N-1) = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{38}$$

5.1.1 Matricial form of the conditions in points

Using the Eq. 32 in the Eqs. 37 and 38, we obtain for $0 \le n \le N-2$

$$\mathbf{A}^{(n)}(n)\mathbf{w}^{(n)} = \mathbf{r}^{(n)} \in \mathbb{R}^3, \tag{39}$$

$$\mathbf{A}^{(N-2)}(N-1)\mathbf{w}^{(N-2)} = \mathbf{r}^{(N-1)} \in \mathbb{R}^3, \tag{40}$$

where, using the Eq. 31, we know that

$$\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \equiv \mathbf{Q}^{(0,0)} \in \mathbb{R}^{3\times 12}, \tag{41}$$

$$\mathbf{A}^{(n+1)}(n+1) = \mathbf{A}^{(n)}(n) = \left[\begin{array}{cccc:cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(0,1)} \in \mathbb{R}^{3\times 12}, \tag{42}$$

Thus, using the Eqs. 41 and 42 in the Eqs. 39 and 40, $\forall 0 \le n \le N-2$; We can write the Eqs. 37 and 38 as

$$\mathbf{P}\mathbf{w} = \mathbf{r} \in \mathbb{R}^{3N}. \tag{43}$$

Where

$$\mathbf{P} \equiv \begin{bmatrix} \mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(0,1)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{(0,0)} \end{bmatrix} \in \mathbb{R}^{3N\times 12(N-1)} \tag{44}$$

and

$$\mathbf{r} \equiv \begin{bmatrix} \mathbf{r}^{(0)} \\ \mathbf{r}^{(1)} \\ \vdots \\ \mathbf{r}^{(N-1)} \end{bmatrix} \in \mathbb{R}^{3N} \tag{45}$$

5.2 Boundary conditions in internal point

Following the Fig. 2, we can affirm that for $0 \leq n \leq N-3$, so that

$$\mathbf{p}^{(n)}(n+1) - \mathbf{p}^{n+1}(n+1) = \mathbf{0} \in \mathbb{R}^3, \tag{46}$$

$$\left.\frac{\partial \mathbf{p}^{(n)}(t)}{\partial t}\right|_{t=n+1} - \left.\frac{\partial \mathbf{p}^{n+1}(t)}{\partial t}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3, \tag{47}$$

$$\left.\frac{\partial^2 \mathbf{p}^{(n)}(t)}{\partial t^2}\right|_{t=n+1} - \left.\frac{\partial^2 \mathbf{p}^{n+1}(t)}{\partial t^2}\right|_{t=n+1} = \mathbf{0} \in \mathbb{R}^3. \tag{48}$$

5.2.1 Matricial form of the boundary conditions in internal point equation's

Using the Eqs. 32, 34, 35 and 36 in the Eqs. 46, 47 and 48, we obtain for $0 \leq n \leq N-3$

$$\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - \mathbf{A}^{n+1}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{49}$$

$$D_t\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t\mathbf{A}^{n+1}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3, \tag{50}$$

$$D_t^2\mathbf{A}^{(n)}(n+1)\mathbf{w}^{(n)} - D_t^2\mathbf{A}^{n+1}(n+1)\mathbf{w}^{(n+1)} = \mathbf{0} \in \mathbb{R}^3. \tag{51}$$

Grouping in a matrix

$$\begin{bmatrix} \mathbf{A}^{(n)}(n+1) & -\mathbf{A}^{n+1}(n+1) \\ D_t\mathbf{A}^{(n)}(n+1) & -D_t\mathbf{A}^{n+1}(n+1) \\ D_t^2\mathbf{A}^{(n)}(n+1) & -D_t^2\mathbf{A}^{n+1}(n+1) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(n)} \\ \mathbf{w}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^9, \tag{52}$$

using the Eqs. 34 and 35, we know that

$$D_t\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \equiv \mathbf{Q}^{(1,0)} \in \mathbb{R}^{3\times 12}, \tag{53}$$

$$D_t\mathbf{A}^{n+1}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \equiv \mathbf{Q}^{(1,1)} \in \mathbb{R}^{3\times 12}, \tag{54}$$

$$D_t^2\mathbf{A}^{(n)}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{array}\right] \equiv \mathbf{Q}^{(2,0)} \in \mathbb{R}^{3\times 12}, \tag{55}$$

$$D_t^2\mathbf{A}^{n+1}(n+1) = \left[\begin{array}{cccc:cccc:cccc} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right] \equiv \mathbf{Q}^{(2,1)} \in \mathbb{R}^{3\times 12}. \tag{56}$$

Thus, using the Eqs. 41, 42, 53, 54, 55 and 56 in the Eqs. 46, 47 and 48, these can be rewritten for $0 \leq n \leq N-3$

$$\begin{bmatrix} \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} \\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} \\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(n)} \\ \mathbf{w}^{(n+1)} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^{9} \tag{57}$$

Finally, concatenating to all values for $0 \leq n \leq N-3$ in the boundary conditions in internal point equation's, we obtain

$$\mathbf{Q} \equiv \left[\begin{array}{cccccccc} \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(0,0)} & -\mathbf{Q}^{(0,1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(1,0)} & -\mathbf{Q}^{(1,1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}^{(2,0)} & -\mathbf{Q}^{(2,1)} \end{array}\right] \in \mathbb{R}^{9(N-2)\times 12(N-1)} \tag{58}$$

$$\mathbf{Q}\mathbf{w} = \mathbf{0} \in \mathbb{R}^{9(N-2)} \tag{59}$$

6 Cost function in 3D parametric cubic spline

6.1 Cost function of fitting the cubic spline in the points $\mathbf{r}^{(n)}$

Following the explained in the Section 5, the equation that should be fulfilled to fit the cubic spline in the points can be represented in the next equation

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \mathbf{w} = \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{12(N-1)-6} \tag{60}$$

Defining the cost function $E_1(\mathbf{w})$ of fitting of cubic spline in the points $\mathbf{r}^{(n)}$, $\forall 0 \leq n \leq N-1$

$$E_1(\mathbf{w}) = \left\| \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \mathbf{w} - \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \right\|_{\mathbf{D}}^{2} \tag{61}$$

where $\mathbf{D} \in \mathbb{R}^{(12N-18)\times(12N-18)}$ is a diagonal matrix with the weight of each line equation.

Applying the derivative in relation of vector $\mathbf{w}$ [1, pp. 11] in the cost function $E_1(\mathbf{w})$ of Eq. 61, we obtain

$$\frac{\partial E_1(\mathbf{w})}{\partial \mathbf{w}} = 2 \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}^{T} \mathbf{D} \left(\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \mathbf{w} - \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix} \right) \tag{62}$$

References

[1] Kaare Brandt Petersen, Michael Syskind Pedersen, et al. "The matrix cookbook". In: *Technical University of Denmark* 7.15 (2008), p. 510.