Honors Precalculus: Yet Another Survival Guide

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Contents

0 Introduction

Welcome to the Honors Precalculus Survival Guide! This is a document containing a collection of concepts, formulas, and topics that will be covered in the M420 Honors Preacalculus course (at least in the 2023-24 curriculum). Each chapter, or "Unit," will run through all the ideas discussed, and provide a relatively brief and easy to understand explanation for some ideas, as well as calculations and derivations of formulas paired with written logic and reasoning. There will also be "bonus" chapters on subjects that won't really be tested on, but are cool and helpful to know.

I decided to write the Precalc Guide because there was a lot of demand for a guide for this class specifically, and because of the notorious difficulty for this class. I'll just say upfront: Honors Precalc isn't hard. That is, if you pay good attention and try to understand the "why" rather than the "what," everything falls into place. But I still wanted to make this to hopefully break the "harder" parts of this class into easier pieces and alleviate some stress behind this class.

In the end though, this was heavily inspired by Michael Y. '24, the author of the original Lakeside math course guide titled "Multivariable Calculus: A Survival Guide." Without his contributions to Lakeside math, this would not have been possible. I highly suggest checking his document out for all your Multi needs.

Because I am "by no means an expert at [pre]calculus," to quote Michael, there will inevitably be errors and confusing parts. If anything doesn't add up (get it?), feel free to let me know.

In closing, I wish you all good luck on your journey through Precalculus. The document is titled such because it covers Honors Precalculus topics, but anyone is free to use this as a reference. However, Honors or not, Precalculus is not as terrifying as it seems. Like I always say, "As long as you study, you'll be fine." Prioritize sleep, pay attention, and enjoy the wild ride. It's a fun class that will lead you to life-changing discoveries as long as you just stay curious.

1 Sequences and Series

§1.1 Definitions

Definition 1.1.1. A *sequence* is a list of things, usually numbers. Each of the things in a sequence are called *terms*.

Example 1.1.2
The sequence of the first 10 positive even integers is

$$\{2, 4, \dots, 18, 20\}.$$

When we want to talk about sequences, we want to be able to label each of the terms in the sequence. Even though we can just refer to the first thing in a sequence as "the first thing in a sequence", it's very unwieldy to write

$$\text{the first thing in the sequence above} = 2$$

every time we want to talk about the first thing in a sequence.

Instead, mathematicians use subscripts to refer to these elements. For example, we have that

$$a_1 = 2.$$

Here, the letter a is just the name of our sequence, and the number 1 means that it is the first thing in our sequence. We can treat a_1 as any other variable.

When we introduce a sequence, we can say that our sequence is a_n, $\{a\}$, $\{a_n\}$, or even just a. For example, we can alternatively say that

Example 1.1.3
Let a_n be the sequence of the first 10 positive even integers, or

$$a_n = \{2, 4, \dots, 18, 20\}.$$

There also exist *infinite* sequences. What this means is that the sequence has no end, compared to a *finite* sequence which does have an end. Our example above is a finite sequence, since we can clearly see that it ends. The following sequence, on the other hand, does not end:

Example 1.1.4

An example of an infinite sequence is the sequence of all positive even integers, or

$$a_n = \{2, 4, 6, \dots\}.$$

Notice that here, we cannot identify a concrete "last term". Even though the last term listed out is 6, the ellipsis signify that there is more; in fact, infinitely more. Unlike with the earlier example where we used ellipsis because we're too lazy to list out all of the terms, here we do so because it is impossible to actually list out all of the terms.

A key thing to note is that sequences don't necessarily have any pattern. For example,

$$a_n = \{1, 4, 3, 4, 12, 10, 7, -3, \pi\}$$

is a sequence. However, since these sequences are very random, we can't really identify special properties about them, so we won't discuss them very much.

On the other hand, a series is the sum of the things in a sequence. We may rigorously define it as follows:

Definition 1.1.5. The *series* S of a sequence a_n is defined as

$$S = a_1 + a_2 + \dots$$

Example 1.1.6

Earlier, we had the example of the sequence $a_n = \{2, 4, \dots, 18, 20\}$. Then, the series of this sequence is

$$S = 2 + 4 + 6 + \dots + 18 + 20,$$

which calculates to be 110.

Sometimes, we would like to only add up some of the elements of a sequence. We can define it as such:

Definition 1.1.7 (Partial Sums)**.** For a sequence a_n, the *nth partial sum* is

$$a_1 + a_2 + \dots + a_n.$$

In other words, this is the sum of the first n terms.

Exercise 1.1.8. Find the 6th partial sum of the Fibonacci sequence

$$a_n = \{0, 1, 1, 2, 3, 5, 8, \dots\}.$$

Later on in the section, we will explore special types of sequences which have special ways to calculate their partial sums.

§1.2 Explicit and Recursive Definitions

There are many ways to define the terms of a sequence. For example, we can *explicitly* define the terms of a sequence. Here, we explicitly tell the reader what the nth term exactly is.

Example 1.2.1

The sequence of positive even integers a_n is defined as

$$a_n = 2n.$$

This is equivalent to saying ”the nth term of our sequence can be written as $2n$”.

Another way to define the terms of a sequence is *recursively*. This is done by telling the reader how to compute each term of the sequence based on the previous terms of the sequence.

Example 1.2.2

The sequence of positive even integers a_n is defined with $a_1 = 2$, and

$$a_n = a_{n-1} + 2.$$

This is equivalent to saying ”you can get the next term of the sequence by adding 2 to the previous one”.

Initially, recursive definitions may seem incredibly useless, however examples like the Fibonacci sequence show that they make for much nicer presentations of sequences.

Example 1.2.3

The famous Fibonacci sequence F_n is defined with $F_0 = 0$, $F_1 = 1$, and for all other terms, that

$$F_n = F_{n-1} + F_{n-2}.$$

This is equivalent to saying ”the nth term of the Fibonacci sequence is the sum of the terms before it”.

> **Remark 1.2.4.** Although there exist explicit definitions of the Fibonacci sequence, they are quite ugly and hard to work with, and it is best not to venture off in that direction until later.

The most important thing to remember for recursive definitions is **ALWAYS DEFINE THE FIRST TERMS OF A RECURSIVE SEQUENCE**. It is impossible for the reader to calculate the rest of the terms of your sequence if you don’t explicitly define the first terms. Otherwise, it’s like trying to knock dominoes down: even if the dominoes are set up to knock each subsequent one down, if you don’t knock down the first one, the others will not fall.

Exercise 1.2.5. Find explicit and recursive definitions for the sequence

$$\{1, 2, 4, 8, 16, 32, 64, \dots\}$$

Exercise 1.2.6. Find a recursive definition for the sequence

$$\{-2, 3, 1, 4, 5, 9, 14, \dots\}$$

§1.3 Arithmetic Sequences

One sequence you will commonly see is the *arithmetic sequence*. We will give a couple of examples before we formally define it.

Example 1.3.1

- The arithmetic sequence with first term 1 and common difference 1 is
$$\{1, 2, 3, 4, 5, \dots\}.$$
- The arithmetic sequence with first term 2 and common difference 2 is
$$\{2, 4, 6, 8, 10, \dots\}.$$
- The arithmetic sequence with first term 102 and common difference 2 is
$$\{102, 104, 106, 108, 110, \dots\}.$$
- The arithmetic sequence with first term -8 and common difference 3 is
$$\{-8, -5, -2, 1, 4, \dots\}.$$
- The arithmetic sequence with first term 4 and common difference -3 is
$$\{4, 1, -2, -5, 4-8, \dots\}.$$

As you can see, an arithmetic sequence is really just the idea of "counting up by a number". For example, the arithmetic sequence with first term -8 and common difference 3 is just the same thing as counting up from -8 by 3s. We will rigorously define this as such:

Definition 1.3.2. An arithmetic sequence a_n is a sequence defined by a starting term a and common difference d, with the sequence being

$$a_n = \{a, a+d, a+2d, a+3d, \dots\}.$$

Notice that the nth term of this sequence is explicitly defined by

$$\begin{aligned} a_1 &= a \\ a_2 &= a+d \\ a_3 &= a+2d \end{aligned}$$

$$\vdots$$
$$a_n = a + (n-1)d.$$

The reason it's not

$$a_n = a + nd$$

is because our first term is a_1, which does not have any d term.

Exercise 1.3.3. Biff is counting up by 5s, starting from 2. What is the 10th number he says? What about the 100th number? What about the nth number?

When given an arithmetic sequence, we may easily calculate its common difference by subtracting any two adjacent terms of an arithmetic sequence.

Extending that,

$$d = \frac{A_m - A_n}{m - n}$$

since there are $m - n$ differences between A_m and A_n.

Exercise 1.3.4. Eho wrote the following arithmetic sequence on the board, but accidentally smudged some of the numbers, which have become black squares. What were the numbers Eho originally wrote?

$$23, \blacksquare, \blacksquare, 47, \blacksquare, \blacksquare$$

§1.3.1 The Arithmetic Series Formula

We propose the following:

> **Lemma 1.3.5** (Sum of Arithmetic Series)
> The sum of $\{a, a+d, a+2d, \dots, a+d(n-1)\}$ is
> $$\frac{n(2a + d(n-1))}{2}$$

Now let's derive the arithmetic series formula. Here's a cool story that my parents have been teaching me since I was a child. It has been translated for your purposes.

> **Remark 1.3.6** (Storytime)**.** There once was a famous mathematician called Gauss. One day, his teacher gave his entire class a tricky problem: find the sum of the first 100 integers. The whole class got down and started adding, frustrated. However, Gauss opted to avoid the grueling summation and decided to match each integer with its partner on the other side of the set. That is, he paired 1 and 100, 2 and 99, etc. Quickly, he found 50 pairs of 101, leading him to find 5050 while the rest of the class was still stupidly chipping away at the problem.

We can apply that thinking and write our proof.

Proof.

$$S = a + (a+d) + (a+2d) + \cdots + (a + d(n-2)) + (a + d(n-1))$$

$2S = (2a+d(n-1))+(2a+d(n-1))+(2a+d(n-1))+\cdots+(2a+d(n-1))+(2a+d(n-1))$

We see $2a + d(n-1)$ appears n times, so we get

$$S = \frac{n(2a + d(n-1))}{2}$$

$\square$

Another way to think of this formula is

$$\frac{n(First + Last)}{2}$$

This reason why should be pretty trivial. Lastly, we will finish this section off with a saying in Chinese that always helped me remember the formula

> shou xiang jia mo xiang cheng yu xiang shu chu yu er

It simply means "First plus Last times occurences divided by two".

Problem 1.3.7. Find $19 + 25 + 31 + 37 + \cdots + 79 + 85 + 91 + 97$.

Problem 1.3.8. For an arithmetic sequence $S_n = \{a, a+d, a+2d, \ldots, a+(n-2)d, a+(n-1)d\}$, find a formula for $S_k + S_{k+1} + S_{k+2} + \cdots + S_{l-1} + S_l$ for arbitrary values k, l such that $k \leq l$.

§1.4 Geometric Sequences

The second type of special sequence is a geometric sequence. This is where any two consecutive terms have the same common ratio. This leads us to our general geometric series.

Definition 1.4.1. The general geometric sequence G_n for two arbitrary constants a, r is

$$G_n = \{a, ar, ar^2, \ldots, ar^{n-1}\}.$$

We denote r as the common ratio. The explicit definition of that sequence is

$$G_n = ar^n.$$

The recursive definition of that sequence is

$$G_n = G_{n-1} \cdot r$$

$$G_n = a.$$

The common ratio can be calculated by dividing two consecutive terms.

Similar to an arithemtic series,

$$\frac{\frac{G_k}{G_n}}{k-n}$$

Similarily, the thinking goes that there are $k - n$ ratios between G_k and G_n.

§1.4.1 The Geometric Series Formula

We propose the following:

Lemma 1.4.2
The sum of $\{a, ar, ar^2, \ldots, ar^{n-1}\}$ is

$$\frac{a(r^n - 1)}{r - 1}$$

Sadly, there is no story to derive this formula. But there is a really interesting connection between this and polynomials. It just so happens that

$$\frac{x^n - 1}{n - 1} = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$$

due to polynomial division (you can try this yourself with synthetic division). This is one way to derive the geometric series formula. Or you can do it the "official" way.

Proof.

$$S = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$S \cdot r = ar + ar^2 + ar^3 + \cdots + ar^n$$
$$Sr - S = ar^n - a$$
$$S(r-1) = a(r^n - 1)$$
$$S = \frac{a(r^n - 1)}{r - 1}$$

□

Problem 1.4.3. Compute

$$\frac{5}{12} + \frac{5}{3} + \frac{20}{3} + \cdots + \frac{800}{3}$$

§1.5 Infinite Geometric Series

One funny (and really cool) thing we can do with geometric series that we can't do with arithmetic series is infinite geometric series. First, we need to define convergence and divergence. Convergence is when the sum of a series starts to approach, or **converge** onto a point as more terms are added. Divergence is when the sum of a series doesn't approach anything, and in fact moves farther away from 0 on the number line. Geometric series, however, have the potential for values to approach 0 as n grows larger if the common ratio is less than 1. Therefore, the series has the ability to converge onto a value. A famous problem is the dichotomy problem, which details the story of someone trying to get from point A to point B. Each turn, they move half the remaining distance to point B. Will they ever reach point B? It turns out, sorta.

§1.5.1 Infinite Geometric Series Formula

We really want to find a value for our dichotomy problem, but to do so, we have to eliminate the infinite terms. Luckily, since it's infinite, we can multiply our series by 2.

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$2S = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$
$$2S - S = S = 1.$$

Now we can apply this thinking and obtain our infinite geometric series formula.

Lemma 1.5.1

The sum of $\{a + ar + ar^2 + ar^3 + \dots\}$ is

$$\frac{a}{1-r}$$

if and only if $|r| < 1$.

Proof.

$$S = a + ar + ar^2 + ar^3 + \dots$$
$$S \cdot r = ar + ar^2 + ar^3 + ar^4 + \dots$$
$$S - S \cdot r = a$$
$$S = \frac{a}{1-r}$$

Obviously, if $r \geq 1$, the terms can never converge, so the restriction that $|r| < 1$ applies. □

If we decide to violate this restriction, we get really interesting values. This is the study of infinite partial sums. Though it technically isn't defined, we can apply this to some series with common ratio ≥ 1 and get some pretty strange answers. Though it isn't an infinite geometric series, the Ramanujan Summation tells us

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}.$$

This is obviously false. Or is it? Srinivas Ramanjuan developed a very interesting derivation, which is explained in the subappendices.

Problem 1.5.2. Compute

$$\frac{1}{3} + \frac{3}{6} + \frac{9}{12} + \frac{27}{24} + \dots$$

Problem 1.5.3 (Challenge). Compute

$$\frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \dots$$

§1.6 Sigma Notation

As a mathematician, I love efficiency. One thing that cosntanlty annoys me is writing out a summation, like $a + ar + ar^2 + ar^3 + ar^4 + \dots$. Don't you wish there could be a quicker way of writing this? Well, luckily there is. Welcome to the wonderful world of sigma! Sigma notation is an efficient way to write out summations or products (which is technically called product notation, but we're putting them in the same section).

$$\sum_{k=a}^{n} f(k)$$

This is sigma. If you know coding, think of it as a for loop. k is the index of this summation, a is the start, and n is the stop. That is, it will sum $f(k)$ for every value of k from a to n, inclusive. k will only increase by 1 each time.

Example 1.6.1 (Sum of All Natural Numbers)

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \dots$$

The start is 1, so we start with 1. Then we add 2. Then 3. This continues for eternity, since you can have the stop be infinity. I suppose in that case, the summation just doesn't stop.

Product notation is very similar. Just replace $\sum$ with $\prod$, and instead of summing terms, multiply terms.

Example 1.6.2 (n!)

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n = n!$$

You are allowed to nest summations, such as

$$\sum_{i=1}^{n} \sum_{j=1}^{i} j = (1 + 2 + 3 + \dots) + (2 + 3 + 4 + \dots) + \dots$$

However, something you are not allowed to do is have the index move backwards. This is not standard:

$$\sum_{k=10}^{5} k.$$

Problem 1.6.3. Write $(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \dots$ using sigma and product notation.

§1.7 Interest

- explain APR/Yield - explain how sequences+series tie into it - problem

§1.A Bonus: Sums of n^k powers

- formulas, how to derive next

§1.B Explicit Formulas for Linear Recurrences

§1.C The Ramanujan Summation

2 Freaky Functions

- function notation - what is a Function - what is not a function - functions from tables - problem

§2.1 Graphs of Functions

- graphs of functions (lin, quad, general polynomial, everything else) - verti line test, why it works - end behavior - problem

§2.2 Domain and Range

- what is d and r - how it shows up on a graph - interpreting a graph - problem

§2.3 Inverse Functions

- explain inverse functions - change of variable - how it shows up on a graph - problem

§2.4 Function Composition

- explain how function composotion works - show it on a graph - problem

§2.5 Transformations

- explain what is a transformation - 4 types and why they work - composition of transformations - problem

3 The Derivative

- why are these useful(NOT LONG)

§3.1 Limits

- explain what a limit is - how to find limits - problem

§3.2 Average Rate of Change

- explain change, average rate of change - manual derivative calculation - problem

§3.3 Definition of a Derivative

- what is a Derivative, how to find - derivative on graphs - problem

§3.4 Derivative Rules

-

§3.4.1 Sum Rule

- derive sum rule - problem

§3.4.2 Power Rule

- derive power Rule

§3.4.3 Quotient Rule

- quotient rule - explain why its inferior

- problem

§3.4.4 Chain Rule

- derive chain Rule - problem

- some common derivatives (polys, 1/x, sqrt(x)) - THEYRE ALL JUST EXPONENTS

§3.5 Derivatives and Graphs

- ??

§3.5.1 Graphing the Derivative

- how to graph a derivative based on a graph - problem

§3.5.2 Minima and Maxima

- explain what those are - how to find - problem

§3.5.3 Inflection Poitns

- explain what they are

§3.5.4 Polynomial Construction

- building a polynomial given the Derivative - derivative - problem

§3.5.5 Inverse Functions

- derivative of inverse function behavior - pesky formula - problem

4 Exponential and Logarithmic Functions

- introduction to what an exponential and logarithmic function is - explain log - how they look on a graph - problem

§4.1 Exponential Laws

- quick runthrough of all laws, why they work - include log laws too - problem

§4.2 e

- include image of flame grilled e - explain how to find

§4.2.1 ln

- correct pronounciation

- why it is so useful (deri, rael world apps) - problem??

§4.3 Exponential Growth

- explain exponential grwoth+decay - real world examples - curve fitting?? - problem

§4.4 Derivatives of Logarithmic Functions

- derive derivative of e^x - derive derivative of ln(x) - derive ferviative of a^x - derive derivative of $log_a b$ - problem

§4.A Logistic Curves

- logistic curves what they are - Properties - uses - problem

5 Tricky Trigonometry

- explain that this mostly coveres derivatives

§5.1 Radians

- explain what a radian is - estimating radians - problem

§5.2 Intro to Trigonometric Functions

- introductoin to trigonometry Functions (what they are, how to get)

§5.2.1 Properties of Trigonometric Functions

- Amplitude - Vertical shift - Frequency/Period - Phase

§5.2.2 Estimation

- how to estimate trig values

§5.2.3 Trigonometric Equations

- solving them - problem

§5.3 Trigonometric Identities

- derive sin2x+cos2x=1 - derive sinpi/2-x=cosx

§5.4 The Unit Circle

- unit circle what it is - trig functions on a unit cricle - same value diff parity - laws on circle - problem

§5.4.1 Special Values of Trigonometric Functions

- pi, 0, pi/6, pi/3, etc - how they appear on a unit circle - problem

§5.5 Trigonometric Functions and their Derivatives

§5.5.1 sin(x)

- derive derivative of sin(x)

§5.5.2 cos(x)

- derive derivative of cos(x)

§5.5.3 tan(x)

- derive derivative of tan(x)

§5.6 More Trigonometric Functions

- sec, csc, cot - their derivative - problem

§5.7 Inverse Trigonometric Functions

- arcsin arccos arctan - their Derivative - problem

§5.8 Taylor Series

- what it is, hwo to get - basic taylor series

§5.A Angle Addition

- derive angle Addition - problem

6 Unit N: Topic 1

Did you know that I will be talking about animals?

§6.1 Subtopic

Did you know that fish have 0 legs?

Theorem 6.1.1 (The Fish)
Fish have 0 legs

Proof. Because they can't walk, dummy □

What else can fish do?

Lemma 6.1.2
Fish also can't fly

That is for our purposes, true.

> **Remark 6.1.3.** Flying fish can fly, but they are a conspiracy

§6.2 Subtopic 2

Did you know that chickens have two legs?

Subsubtopic 1

Did you know that those chickens who have two legs also have beaks?

§6.A Bonus: Special Topic that isn't on tests

Did you know that horses have four legs? That's crazy.

Example 6.A.1 (Three-legged Horse)
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A Afterword

B Special Thanks

Michael Y. '24 for being a gigachad.

C About the Author

D Works Cited

- cite Works - michaels doc