# EXAM (2h)
January 2025

## Exercise 1: Fair allocation of indivisible goods (6 pts)

We are given a set $O = \{o_1, \ldots, o_m\}$ of $m$ indivisible goods, and a set $N = \{1, \ldots, n\}$ of $n$ agents. Each agent $i \in N$ has preferences over the possible subsets of goods she gets, which are expressed by an additive utility function $u_i : 2^O \to \mathbb{R}^+$. The goal is to find a fair allocation of the indivisible goods to the agents $A = (A_1, \ldots, A_n)$, which is a partition of the set of goods, i.e., $A_i \subseteq O$ for every $i \in N$, $\bigcup_{i \in N} A_i = O$, and $A_i \cap A_j = \emptyset$, for every $i, j \in N$ with $i \neq j$, where $A_i$ denotes the bundle of goods allocated to agent $i \in N$ in $A$.

1. Recall three classic fairness criteria used in the context of fair allocation of indivisible goods under additive utilities, as described above, and give their connections (which criterion is stronger than the others, etc.) [no need to prove these implications]. **(1 pt)**
2. Consider the following instance of fair allocation of indivisible goods under additive utilities where the agents' utilities over single goods are given below:

| | $o_1$ | $o_2$ | $o_3$ | $o_4$ | $o_5$ | $o_6$ |
|---|---|---|---|---|---|---|
| agent 1 | 10 | 2 | 8 | 6 | 9 | 4 |
| agent 2 | 9 | 7 | 2 | 6 | 5 | 1 |
| agent 3 | 9 | 4 | 3 | 5 | 8 | 1 |

   For each of the three criteria you previously mentioned, give an allocation in this instance which is fair w.r.t. this criterion or state that no fair allocation exists. Justify. **(2 pts)**
3. An allocation $A$ is a competitive equilibrium with equal incomes (CEEI) if there exists a price vector $p \in [0, 1]^m$ such that $A_i \in \arg\max_{O' \subseteq O}\{u_i(O') : \sum_{o \in O'} p_o \leq 1\}$, for every agent $i \in N$. Locate the CEEI criterion in the hierarchy of fairness criteria, i.e., what are the connections between the CEEI criterion and the other fairness criteria you have previously mentioned? Prove your statements. **(2 pts)**
4. In the instance described in question 2, give an allocation that satisfies the CEEI criterion or state that no such fair allocation exists. Justify. **(1 pt)**

## Exercise 2: Stability in Euclidean roommate matchings (8 pts)

In a roommate instance, we are given a set $N = \{1, \ldots, n\}$ of $n$ agents with $n$ even. Each agent $i \in N$ has preferences over the other agents which are expressed by a linear order $\succ_i$ over $N \setminus \{i\}$.

The goal is to find a perfect matching over $N$.

A roommate instance $(N, \{\succ_i\}_{i \in N})$ is said to be 1-Euclidean if there exists an embedding of the agents in the real line $E : N \to \mathbb{R}^+$ such that for each triple of agents $i, j, k \in N$ with $j \neq i \neq k$, we have $j \succ_i k$ iff $|E(j) - E(i)| < |E(k) - E(i)|$.

For all considered instances $(N, \{\succ_i\}_{i \in N})$, we assume, w.l.o.g., that the agents are ordered w.r.t. the axis $1 < 2 < \cdots < n$, i.e., if there exists an embedding $E : N \to \mathbb{R}^+$ such that the instance is 1-Euclidean w.r.t. $E$ then we have $E(1) < E(2) < \cdots < E(n)$.

1. Are the two following preference profiles 1-Euclidean? Justify. **(2 pts)**

(a)

$$\begin{array}{lccccccccc}
1: & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
2: & 3 & \succ & 1 & \succ & 4 & \succ & 5 & \succ & 6 \\
3: & 4 & \succ & 2 & \succ & 1 & \succ & 5 & \succ & 6 \\
4: & 5 & \succ & 3 & \succ & 6 & \succ & 2 & \succ & 1 \\
5: & 6 & \succ & 4 & \succ & 3 & \succ & 2 & \succ & 1 \\
6: & 5 & \succ & 4 & \succ & 3 & \succ & 2 & \succ & 1
\end{array}$$

(b)

$$\begin{array}{lccccccccc}
1: & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
2: & 1 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
3: & 4 & \succ & 5 & \succ & 2 & \succ & 6 & \succ & 1 \\
4: & 5 & \succ & 3 & \succ & 6 & \succ & 2 & \succ & 1 \\
5: & 6 & \succ & 4 & \succ & 3 & \succ & 2 & \succ & 1 \\
6: & 5 & \succ & 4 & \succ & 3 & \succ & 2 & \succ & 1
\end{array}$$

2. Given the assumption that a plausible embedding $E$ is necessarily such that $E(1) < E(2) < \cdots < E(n)$, give a linear program to efficiently check that a roommate preference profile is 1-Euclidean. **(2 pts)**
3. There always exists a unique blocking-pair stable matching in a 1-Euclidean roommate instance. Describe a polynomial-time algorithm to construct such a blocking-pair stable matching, assuming the instance is 1-Euclidean and show that the constructed matching is indeed blocking-pair stable. **(2 pts)**
4. Prove that the unique blocking-pair stable matching is also swap-stable, assuming the 1-Euclidean restriction. **(2 pts)**

## Exercise 3: Pareto-optimality in additively separable hedonic games (6pts)

In a hedonic game, we are given a set $N = \{1, \dots, n\}$ of $n$ agents, and the goal is to partition the set of agents into disjoint coalitions, i.e., a solution is a partition $\pi = (C_1, \dots, C_k)$ where $C_i \subseteq N$ for each $i \in \{1, \dots, k\}$, $\bigcup_{i=1}^{k} C_i = N$, and $C_i \cap C_j = \emptyset$, for each $1 \leq i < j \leq k$. Let $\pi(i)$ denote the coalition to which agent $i$ belongs in partition $\pi$. In an additively separable hedonic game (ASGH), each agent $i \in N$ is associated with a valuation function $v_i : N \setminus \{i\} \to \mathbb{R}$ and evaluates each partition $\pi$ as follows: $u_i(\pi) = \sum_{j \in \pi(i) \setminus \{i\}} v_i(j)$.

1. Assume that the preferences are such that $v_i(j) = 0$ implies $v_j(i) = 0$, for each pair of agents $i, j \in N$. Give a polynomial-time algorithm to compute a Pareto-optimal partition in such an ASHG. **(2 pts)**
2. Is a Pareto-optimal partition necessarily individually rational? Prove your claim. **(2 pts)**
3. Is a Pareto-optimal partition necessarily contractually individually stable? Prove your claim. **(2 pts)**