Université
Paris Cité

January 28$^{\text{th}}$, 2025

Decision Theory
Exam
Duration: 2 hours

All course documents allowed

Exercice 1: checking for defects

A small company producing recyclable components for the automobile industry is wondering about the relevance of establishing a verification procedure to test each component before delivering it to the factories, ensuring the assembly of these recyclable components with other conventional components. Indeed, when it delivers a defective component, it incurs a replacement cost denoted c_r. Furthermore, the verification procedure capable of detecting any defective component results in a unit cost denoted c_v. The cost c_v of the verification procedure includes the cost of possible replacement of the parts detected as defective.

We consider the events D ("*the component is defective*") and $\overline{D}$ ("*the component is not defective*"). Let V denote the action of checking a possibly defective component and $\overline{V}$ the opposite action. In the remainder of the text, we note $p(X)$ the probability that the event X occurs.

1. Based on the criterion of expected cost, express a condition on $p(D)$ as a function of c_r and c_v, indicating in which case it is preferable to carry out or not to carry out a verification. To answer the question, we can rely on a rudimentary decision tree.

2. The values of $p(D)$, c_r and c_v are not known but we can estimate that the probability that a component is defective is at most 10% and that the replacement cost is at most equal to three times the verification cost. Under these hypotheses, indicate whether it is advantageous to establish the verification procedure.

3. We now wish to compare the options V and $\overline{V}$, taking into account a new possibility: it is possible to carry out a preliminary test on the component. If this test indicates a positive result (presumption of defect), the verification procedure will be systematically carried out. If this test indicates a negative result (presumption of absence of defect), the verification procedure will not be carried out. This test, much more basic than the verification procedure, does not generate any cost.

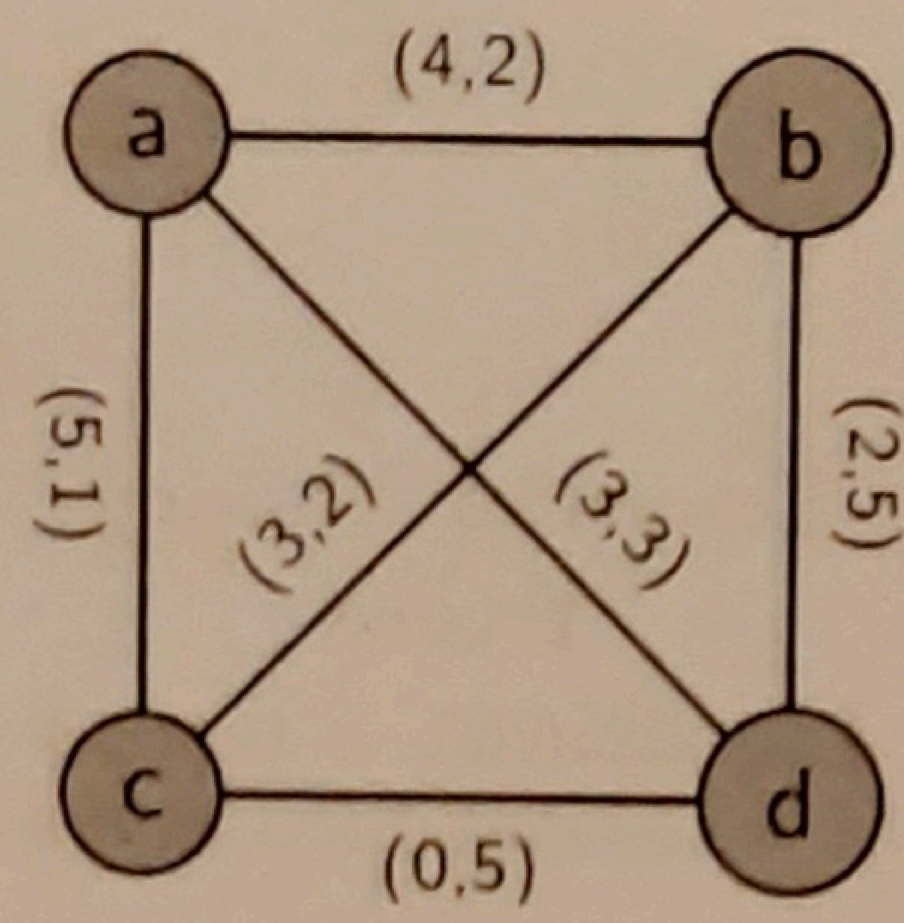
   We therefore consider the following two events: $\oplus$, which means "the test is positive" and $\ominus$, which means the test is negative". We will denote by T the strategy consisting of carrying out the test followed, if necessary, by the verification procedure. Construct the decision tree to describe the choice between the three strategies T, V, and $\overline{V}$ and determine the cost expectations associated with each of the strategies.

Exercise 2 : bi-criteria spanning tree

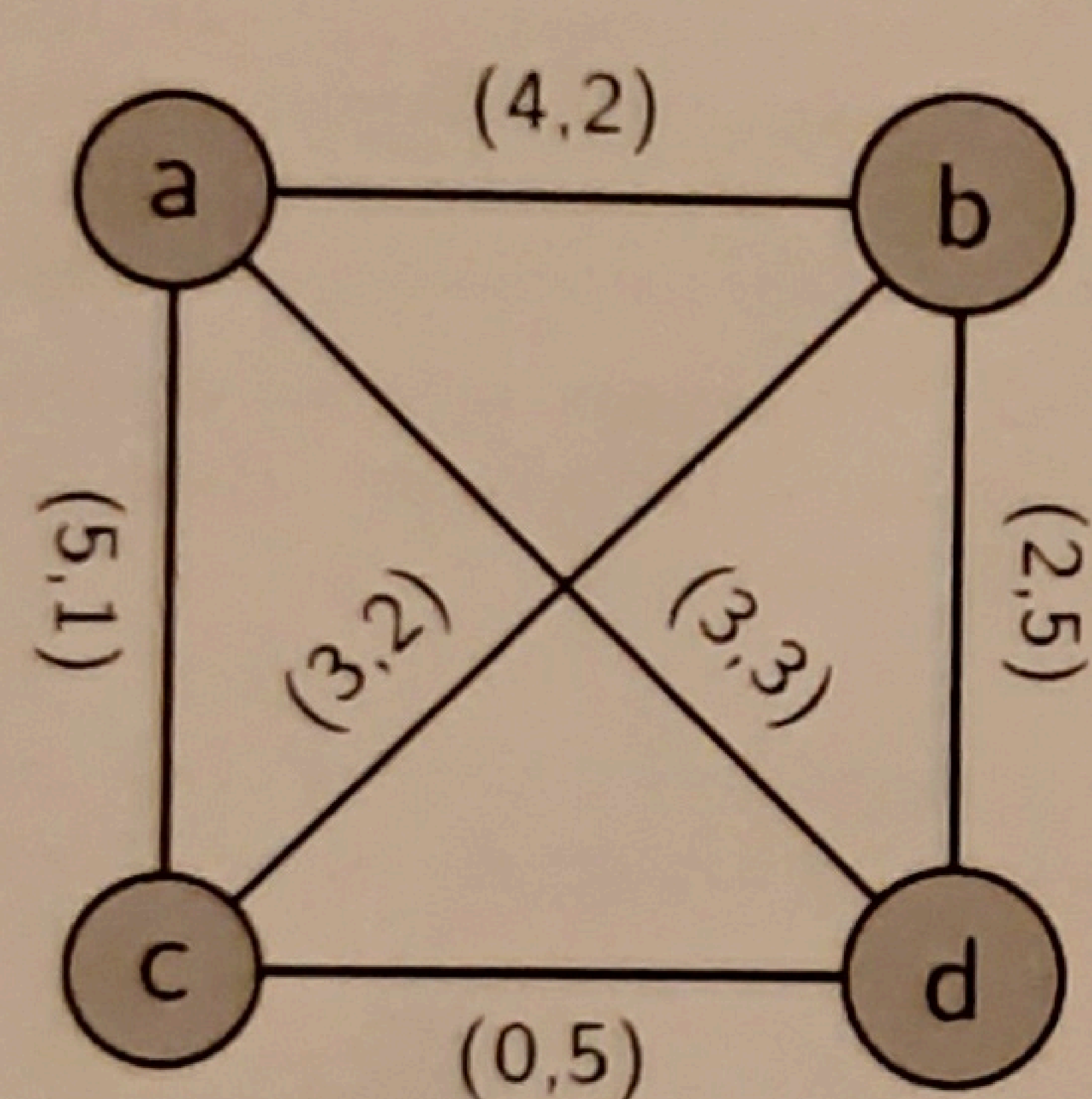
We consider the bi-criteria minimum weight spanning tree problem below. What is the value, on both criteria, of the tree composed of the following edges ((a,c),(b,d),(c,d))?

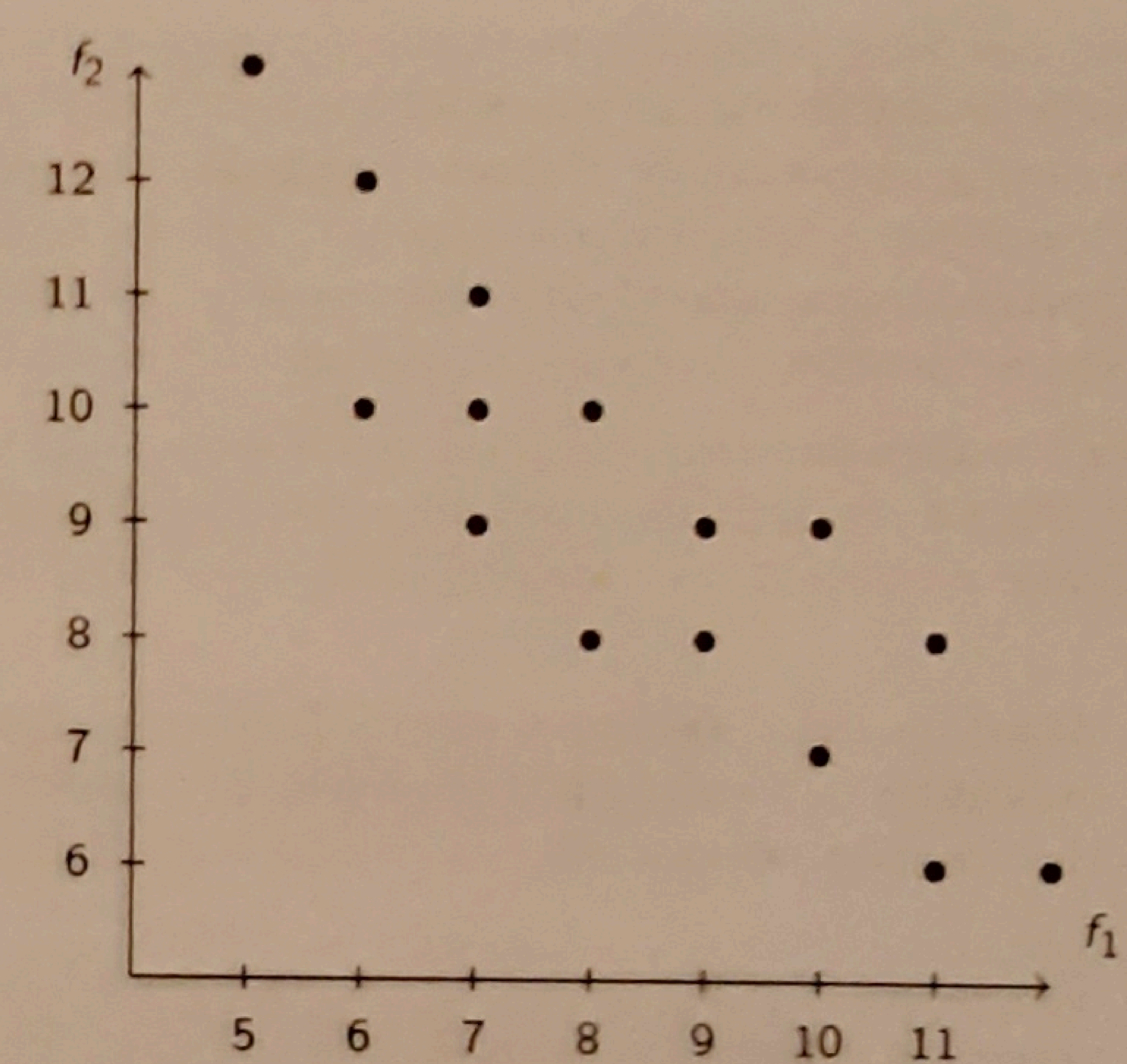
What are the spanning trees with value 8 on the second criterion (check the correct answer/s, several answers are possible)?

- ((a,c),(b,c),(b,d))
- ((a,b),(a,c),(c,d))
- ((b,c),(c,d),(b,a))
- ((a,b),(a,c),(b,c))
- ((a,b),(b,c),(b,d))
- none



On this spanning tree instance in which the two criteria are minimized, we represent here the set of feasible solutions in the bicriteria space





- number of non-dominated solutions ....
- number of weakly non-dominated solutions ....
- number of supported non-dominated solutions ....
- number of unsupported non-dominated solutions ....

Exercise 3: student admission

Consider a situation in which a committee for a higher education program has to decide about the admission of students on the basis of their evaluations in 4 courses: mathematics (M), physics (P), literature (L) and history (H). Evaluations on all courses range in the [0,20] interval.

To be accepted ($\mathcal{A}$) in the program, the committee considers that a student should obtain at least 12 on a "*majority*" of courses, otherwise, the student is refused ($\mathcal{R}$). From the committee point of view, all courses (criteria) do not have the same importance. To define the required majority of courses, the committee attaches a weight $w_j \geq 0$ to each course such that they sum to 1; a subset of courses $C \subseteq \{M, P, L, H\}$ is considered as a majority if $\sum_{j \in C} w_j \geq \lambda$, where $\lambda \in [0, 1]$ is a required majority level.

i. Define the weights w_j, and λ when a student is admitted when (s)he obtains evaluation above 12 on 3 courses out of 4.

ii. If there were only 3 courses, how would you interpret the following values for w_j, and λ.

- $w = (0.49, 0.49, 0.02)$ with $\lambda = 0.5$?
- $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with $\lambda = 0.5$?

What can you conclude from this observation?

iii. The committee considers that the strength of a coalition of courses varies as a function of the courses belonging to the coalition, and states that in order to be accepted a student should be evaluated above 12/20 on a scientific course (math or physics), and on a non-scientific course (literature or history). Prove that it is not possible to represent such an admission rule with an additive definition of majority. What can you conclude?

iv. We now consider a general case with n criteria in which the "sufficient sets" of criteria, are not represented by additive weights anymore. We denote $\mathcal{N} = \{1, 2, ..., n\}$ the set of criteria, and $\mathcal{P}(\mathcal{N})$ the set of subsets of $\mathcal{N}$. We consider (Maj, Min) a bi-partition of $\mathcal{P}(\mathcal{N})$: $\mathcal{P}(\mathcal{N}) = Maj \cup Min$ and $Maj \cap Min = \emptyset$, where Maj (Min, resp.) can be interpreted as the subsets of criteria in $\mathcal{N}$ which forms a *majority* (a *minority*, resp.). For such an interpretation, we pose:

(i) $\mathcal{N} \in Maj$ and $\emptyset \in Min$,

(ii) $\forall A, B \in \mathcal{P}(\mathcal{N})$ such that $A \subset B$, $A \in Maj \Rightarrow B \in Maj$ and $B \in Min \Rightarrow A \in Min$

Give a clear interpretation of the two conditions (i) and (ii). How can you check whether a given a bi-partition is additively representable by a weight vector w and a threshold λ ?

v. A capacity is a set function $\mu : 2^{\mathcal{N}} \to [0, 1]$ such that:

(i) $\mu(A) \leq \mu(B)$, for all $A \subseteq B \subseteq \mathcal{N}$

(ii) $\mu(\emptyset) = 0$ and $\mu(\mathcal{N}) = 1$.

define a capacity μ from which you can derive the bi-partition implementing the admission rule defined in question iii.

vi. The following table shows:

- the number $n(|\mathcal{N}|)$ of bi-partitions of $\mathcal{P}(\mathcal{N})$ verifying (i) $\mathcal{N} \in Maj$ and $\emptyset \in Min$, and (ii) $\forall A, B \in \mathcal{P}(\mathcal{N})$ such that $A \subset B$, $A \in Maj \Rightarrow B \in Maj$ and $B \in Min \Rightarrow A \in Min$,
- the number n^{add} of additively representable such bi-partitions, and
- the proportion $p^{add}(|\mathcal{N}|)$ of additively representable such bi-partitions,

as a function of $|\mathcal{N}|$. What can be learned from this table?

$\lvert\mathcal{N}\rvert$	3	4	5	6
$n(\lvert\mathcal{N}\rvert)$	20	168	7581	7828354
$n^{add}(\lvert\mathcal{N}\rvert)$	20	150	3287	244158
$p^{add}(\lvert\mathcal{N}\rvert)$	100%	89.1%	43.4%	3.1%