Chapter 1

Convergence

In this chapter we will deduce convergence rate estimates for the modified finite element method from the previous chapter, (**??**). We will then apply the results to Richards' equation. To get started we state an extended version of Cea's lemma, where one can estimate approximation errors in the linear and bi-linear form:

Lemma 1.0.1 (First Lemma of Strang, page 155 [**?**])**.** *Suppose there exists some* $\alpha > 0$ *such that for all* $h > 0$ *and* $v \in V_h$

$$\alpha \left\| v \right\|_1^2 \leq a_h(v, v)$$

and let a *be continuous in* $V \times V$*. Then there exist some constant* C *independent of* V_h *such that*

$$\left\| u - u_h \right\|_1 \leq C \left\{ \inf_{v \in V_h} \left\{ \left\| u - v \right\|_1 + \sup_{w \in V_h} \frac{|a(v, w) - a_h(v, w)|}{\left\| w \right\|_1} + \sup_{w \in V_h} \frac{|l(w) - l_h(w)|}{\left\| w \right\|_1} \right\} \right\} \tag{1.1}$$

From (**??**) we see that our bi-linear form looks like

$$a_u(u, v) = \left\langle \hat{I}_h u, \hat{I}_h v \right\rangle_{0,\Omega} + \left\langle \boldsymbol{K} \nabla u, \nabla v \right\rangle_{0,\Omega} \tag{1.2}$$

And the linear form:

$$b_h(v) = \left\langle F, \hat{I}_h v \right\rangle_{0,\Omega} + \left\langle g, \hat{I}_{\Gamma_N} v \right\rangle_{0.\Gamma_N} \tag{1.3}$$

To apply the first Lemma of Strang 1.0.1 we first show that a_h is coercive:

$$\begin{aligned}
\|u\|_1^2 &= \|\partial_{x_1} u\|_0^2 + \|\partial_{x_2} u\|_0^2 + \|u\|_0^2 \\
&= \frac{\langle \nabla u, \nabla u \rangle}{\tau K} + \|u\|_0^2 \\
&\leq \frac{\langle \nabla u, \nabla u \rangle}{\tau K} + C_\Omega \langle \nabla u, \nabla u \rangle \\
&\leq (\frac{1}{\tau K} + C_\Omega)(\langle \nabla u, \nabla u \rangle + \left\langle \hat{I}_h u, \hat{I}_h u \right\rangle) \\
&\leq \frac{1}{\alpha} a_h(u,u)
\end{aligned} \tag{1.4}$$

Another important piece that must be in place for a convergence proof is the piecewise interpolation error:

Lemma 1.0.2. *For the previously defined piecewise global interpolator $\hat{I}_h$, definition* **??**, *we have the estimate:*

$$\left\| \hat{I}_h u - u \right\|_0 \leq ch \|u\|_0 \quad \forall u \in H^1 \tag{1.5}$$

Proof.

todo

content... □

We are now ready to state the H^1 error estimate for the modified finite element method and thus the MPFA-L method:

Theorem 1.0.3. *Let u solve* (**??**) *and u_h be the solution resulting from MPFA-L , then there exists a positive constant C such that*

$$\|u_h - u\|_1 \leq ch(\|u\|_2 + \|F\|_0 + \|g\|_{\frac{1}{2},\Gamma_N}) \tag{1.6}$$

Proof. We start by controlling the second term on the right hand side in (1.1), the truncation error in the bi-linear form:

$$\begin{aligned}
&\sup_{w \in V_h} \frac{|a(v,w) - a_h(v,w)|}{\|w\|_1} \\
&= \sup_{w \in V_h} \frac{|\langle v, w \rangle + \tau \langle \boldsymbol{K} \nabla v, \nabla w \rangle - \left\langle \hat{I}_h v, \hat{I}_h w \right\rangle - \tau \langle \boldsymbol{K} \nabla v, \nabla w \rangle|}{\|w\|_1} \\
&= \sup_{w \in V_h} \frac{|\langle v, w \rangle - \left\langle \hat{I}_h v, w \right\rangle + \left\langle \hat{I}_h v, w \right\rangle - \left\langle \hat{I}_h v, \hat{I}_h w \right\rangle|}{\|w\|_1} \\
&= \sup_{w \in V_h} \frac{|\left\langle \hat{I}_h v - v, w \right\rangle + \left\langle \hat{I}_h v, \hat{I}_h w - w \right\rangle|}{\|w\|_1}
\end{aligned} \tag{1.7}$$

By Cauchy Schwarz inequality and lemma 1.0.2 we get

$$\begin{aligned}
&\leq \sup_{w\in V_h} \frac{ch\,\|v\|_0\,\|w\|_0 + \left\|\hat{I}_h v\right\|_0 ch\,\|w\|_0}{\|w\|_1} \\
&\leq \sup_{w\in V_h} \frac{ch\,\|w\|_0\,(\|v\|_0 + \left\|\hat{I}_h v\right\|_0)}{\|w\|_1} + \frac{ch|w|_1(\|v\|_0 + \left\|\hat{I}_h v\right\|_0)}{\|w\|_1} \\
&= ch(\|v\|_0 + \left\|\hat{I}_h v\right\|_0)
\end{aligned} \tag{1.8}$$

The third term, the linear form, can be controlled similarly:

$$\begin{aligned}
\sup_{w\in V_h} \frac{l(w) - l_h(w)}{\|w\|_1} &= \sup_{w\in V_h} \frac{\left\langle F, w - \hat{I}_h w\right\rangle + \left\langle g, w - \hat{I}_{\Gamma_N} w\right\rangle_{\Gamma_N}}{\|w\|_1} \\
&\sup_{w\in V_h} \frac{\|F\|_0\, ch\,\|w\|_0 + \|g\|_{\frac{1}{2},\Gamma_N} \left\|\hat{I}_{\Gamma_N} w\right\|_{-\frac{1}{2},\Gamma_N}}{\|w\|_1}
\end{aligned} \tag{1.9}$$

unsure about the argumentation in the Cao Wolmuth paper about neumann boundary

□