Chapter 1

Flow in porous media

$$\partial_t(s_w(p)) - \nabla \cdot (\kappa(s_w(p))(\nabla p - \boldsymbol{g})) = f, \tag{1.1}$$

Function spaces

When discussing PDE's and the numerical schemes to solve them it is important to have a precise notion of what kind of functions we are looking for and their properties. The function spaces discussed here are all normed vector spaces. From now on we assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain.

Definition 1 (L^p space)**.** *Let $L^p(\Omega)$ be the set of all functions u defined on Ω for wich the norm $\|u\|_p = (\int_\Omega u^p dx)^{1/p} < \infty$ is defined, where $p \in [1, \infty]$*

Remark 1. *Note that a L^p space induces equivalence relations on the set of functions. Two functions in L^p is equal if they only differ on a set of meassure zero.*

Theese spaces have the property that they are complete.

Theorem 1.0.1 (Riesz-Fischer Theorem [**?**] chapter 8)**.** *Each L^p space is a Banach space.*

The L^2 space has the property that it is a **Hilbert space**, ie. a complete inner product space.

Definition 2. *Let $u, v \in L^2(\Omega)$, then $\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega uvdx$ defines an inner product.*

Before we continue the study of funtion spaces we develop some convinient notation for derivatives.

Definition 3 (multi index notation)**.** *Let $\overline{\alpha}$ be an ordered n-tuple. We call this a multi-index and denote the length $|\overline{\alpha}| = \sum_{i=1}^n \alpha_i$. Let $\phi \in C^\infty(\Omega)$ we define $D^{\overline{\alpha}} = (\frac{\partial}{\partial x_1})^{\alpha_1}(\frac{\partial}{\partial x_2})^{\alpha_2}...(\frac{\partial}{\partial x_n})^{\alpha_n}\phi$*

We would also like a more general notion of derivative than the one presented in the basic calculus books.

Definition 4 (weak derivative). *Let $L^1_{loc}(\Omega) = \{ f \in L^1(K) : \forall K \in \Omega$ where K is compact $\}$. Let $f \in L^1_{loc}(\Omega)$. If there exists $g \in L^1_{loc}(\Omega)$ such that*
$\int_\Omega g\phi dx = (-1)^{|\overline{\alpha}|} \int_\Omega f D^{\overline{\alpha}} \phi dx \quad \forall \phi \in C^\infty$ *with* $\phi = 0$ *on* $\partial\Omega$
we say that g is the weak derivative of f and denote it by $D^{\overline{\alpha}}_w f$.

We can now define a class of subspaces of the L^p spaces known as the **Sobolev spaces**

Definition 5 (Sobolev space). *Let k be a non-negative integer, define the Sobolev norm as*

$$\|u\|_{W^k_p} = (\sum_{|\overline{\alpha}| \leq k} \left\|D^{\overline{\alpha}}_w u\right\|^p_{L^p(\Omega)})^{1/p}.$$

We then define the sovolev spaces as

$$W^k_p(\Omega) = \{ f \in L^1_{loc}(\Omega) : \|f\|_{W^k_p} < \infty \}$$

Theorem 1.0.2. *The sobolev spaces W^p_k are Banach spaces*

Proof. Let $\{u_i\}_{i=0}^\infty \subseteq W^p_k(\Omega)$ be a cauchy sequence. This implies that for all $\overline{\alpha}$, $|\overline{\alpha}| \leq k$ we have a cauchy sequence in $L^p(\Omega)$.

$$\|u_j - u_i\|_{W^k_p} = (\sum_{|\overline{\alpha}| \leq k} \left\|D^{\overline{\alpha}}_w u_j - D^{\overline{\alpha}}_w u_i\right\|^p_{L^p(\Omega)})^{1/p} < \epsilon \ \forall i,j \geq N$$

$$\implies \left\|D^{\overline{\alpha}}_w u_j - D^{\overline{\alpha}}_w u_i\right\|_{L^p(\Omega)} < \epsilon$$

By (1.0.1) we know that $D^{\overline{\alpha}}_w u_i \to u_{\overline{\alpha}}$ as $i \to \infty$. In particular $u_i \to u$, so now we just need to show that $D^{\overline{\alpha}}_w u = u_{\overline{\alpha}}$. By the definition of weak derivative we have:

$$\int_\Omega D^{\overline{\alpha}}_w u_i \phi dx = (-1)^{|\overline{\alpha}} \int_\Omega u_i D^{\overline{\alpha}} \phi dx$$

Now applying Hölder's inequality on both sides we get the two inequalities:

$$\int_\Omega (D^{\overline{\alpha}}_w u_i - u_{\overline{\alpha}}) \phi dx \leq \left\|(D^{\overline{\alpha}}_w u_i - u_{\overline{\alpha}}\right\|_{L_p} \|\phi\|_{L_q}$$

$$\int_\Omega (u_i - u) D^{\overline{\alpha}} \phi dx \leq \|u_i - u\|_{L_p} \left\|D^{\overline{\alpha}} \phi\right\|_{L_q}$$

Taking the limit, the right hand side goes to zero, and we end up with the fact that we can move the limit out of the integral:

$$\lim_{i \to \infty} \int_\Omega D^{\overline{\alpha}}_w u_i \phi dx = \int_\Omega u_{\overline{\alpha}} \phi dx$$

$$\lim_{i \to \infty} \int_\Omega u_i D^{\overline{\alpha}} \phi dx = \int_\Omega u D^{\overline{\alpha}} \phi dx$$

Now we can put the two equations together to obtain $D_w^{\overline{\alpha}} u = u_{\overline{\alpha}}$

$$\int_\Omega u_{\overline{\alpha}} \phi dx = \lim_{i\to\infty} \int_\Omega D_w^{\overline{\alpha}} u_i \phi dx = \lim_{i\to\infty} \int_\Omega u_i D^{\overline{\alpha}} \phi dx = \int_\Omega u D^{\overline{\alpha}} \phi dx$$

□

Definition 6. *We rename the L^2 Sobolev spaces as follows*

$$H^k(\Omega) = W_2^k(\Omega)$$

And note that this is also an inner product space with the scalar product $\langle \cdot, \cdot \rangle_{H^k(\Omega)}$ defined as follows:

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\overline{\alpha}|\leq k} \int_\Omega D_w^{\overline{\alpha}} u, D_w^{\overline{\alpha}} v dx$$

define the trace operator

In Sobolev spaces it is not obvious that a function is well defined on a lower dimensional subset of Ω, because two functions may map elements of this zero meassure subset to different values and still be of the same equivalence class. This is important to settle if we want to solve boundary value problems.

The $H^k(\Omega)$ spaces are **Hilbert spaces**, and in functional analysis this is the "best" vector spaces. We are now analytically well equipped to study PDE's.

The PDE problem

We begin our study with the poisson equation, which arises in many physical applications. In porous media mathematics, it's solution may be the hydraulic potential.

$$-\nabla^2 u = f \ in\ \Omega \tag{1.2}$$

$$u = 0 \ on\ \partial\Omega \tag{1.3}$$

For this equation to be well defined we require that u has double derivatives in Ω, but it is easy to come across physical examples where this does not make sense. This is some of the motivation for recasting the poisson equation in a weak sense called the **variational formulation**. We obtain this by multiplying (1.2) by a sutable function, integrating over Ω and using integration by parts/divergence theorem.

$$find\ u \in H_0^1(\Omega)\ such\ that \tag{1.4}$$

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \ in\ \Omega \tag{1.5}$$

$$\forall v \in H_0^1(\Omega) \tag{1.6}$$

Here the reason for developing the function spaces becomes clear, we have asdfasdf (1.2)

Flow in porous media

In this section we start by introducing the physics of single-phase flow in porous media.

The REV

A porous medium consists of a solid matrix and some void in between the matrix, filled with fluid of one or more phases. In porous media research one has come to the realization that the solid matrix is too complex to model, instead one takes averages of variables over a reasonable length scale, ie. the REV or the *representative elementary volume*. An important characterizition of a porous medium is the *porosity* ϕ is defined as $\phi = \frac{\textit{volume of voids in REV}}{\textit{volume of REV}}$. Another meassure is the *saturation* S_α of phase α, $S_\alpha \equiv \frac{\textit{volume of } \alpha \textit{ in REV}}{\textit{volume of voids in REV}}$. In single phase flow, the saturation is irrelevant as the saturation is always one. Also note that the content of the phase α in the REV, θ_α, is given by $\theta_\alpha = S_\alpha \phi$.

Darcy's law

In 1856 Henri Darcy performed a famous experiment where he studied the flow of water trough sand. To understand his experiment we must first define some variables for meassuring water content. First note that the pressure at height z above datum developed by a water column of height h above datum is given by

$$p_{abs}(z) = p_{atm} + \rho g(h - z)$$

If we define the *gauge pressure* p by $p \equiv p_{abs} - p_{atm}$ we get an expression for p:

$$p = \rho g(h - z)$$

This can be rearranged to give an expression for the *hydraulic head* h:

$$h = \frac{p}{\rho g} + z \tag{1.7}$$

A *manometer* is a tube sticked into the reservoir with one end in open atmosphere, the water level in this tube is then h. The volumetric flow of water is denoted by $\mathbf{q}_d$. Darcy's experiment is shown in figure 1.1, where water is poured trough a cylinder filled with sand. The cylinder has length L and has cross sectional area A. His observations are given by the equation called Darcy's law

$$\mathbf{q}_d = \kappa \frac{A(h_2 - h_1)}{L} \tag{1.8}$$

Where κ is a coefficient of proportionality.

`Darcy experiment.png`

Figure 1.1: The Darcy experiment

Let u denote the volumetric flowrate per area

$$\boldsymbol{u} \equiv \frac{\mathbf{q}_d}{A} = \kappa \frac{h_2 - h_1}{L}$$

This is now the *flux* of hydraulic potential. We can now state the differential version of Darcy's law, taking the limit as $L \to 0$ we get

$$\boldsymbol{u} = \boldsymbol{\kappa} \nabla h \tag{1.9}$$

We call $\boldsymbol{\kappa}$ the *hydraulic conductivity* and note that it is in general a rank two tensor, a matrix. The *hydraulic conductivity* also has the property that it is symmetric positive definite. With further experiments similar to the one described one can understand what makes up $\boldsymbol{\kappa}$, and it turns out that it is a function of viscosity μ, density ρ, gravity g and *permability* $\boldsymbol{k}$.

$$\boldsymbol{\kappa} = \frac{\boldsymbol{k}\rho g}{\mu} \tag{1.10}$$

The *permability*, which is a property of the soil in the reservoir, is also a second rank tensor which is symmetric positive definite and is in general a function of space.
If we define the *pressure head* ψ as $\psi \equiv \frac{p}{\rho g}$ we can combine (1.7), (1.9) and (1.10) to get another variant of Darcy's law which will be usefull later

$$\boldsymbol{u} = \frac{\boldsymbol{k}\rho g}{\mu}\nabla(\psi + z) \tag{1.11}$$

Governing equations

Darcy's law is not enough if we want to determine the pressure or flow in a reservoir, but we can use the principle of *mass conservation* to add one more equation. The idea is that for every enclosed region in the reservoir, the change of mass inside the region is balanced by the mass flux into the region and the production of mass inside the region.
We end up with the mass balance equation

$$\int_\Omega \frac{\partial(\rho\phi)}{\partial t} dV = -\int_{\partial\Omega} \boldsymbol{n}\cdot\rho\boldsymbol{u}\; dS + \int_\Omega f dV$$

Where $\boldsymbol{n}$ is an outward pointing normal vector to Ω and f is a source or a sink. We can use the divergence theorem on the surface integral to get

$$\int_\Omega \frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) - f dV = 0$$

Since this is true for all enclosed regions Ω, it also holds for the expressions inside the integral yielding the mass conservation PDE

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = f$$

This, together with Darcy's law(1.9) and appropriate boundary and initial conditions closes the system

$$\begin{cases} \boldsymbol{u} = \boldsymbol{\kappa}\nabla h & \\ \frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = f & \\ h = g(\boldsymbol{x}) & \boldsymbol{x}\in\partial\Omega \\ h = f(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \quad t = 0 \end{cases} \tag{1.12}$$

Now we have a model for single phase flow. As it is stated now it is a linear parabolic equation, but for incompressible fluid and matrix it becomes an elliptic equation. One often writes the density as a function of pressure, it then becomes non-linear. See chapter two of [**?**] for a more detailed discussion of (1.12) and modelling options,

Twophase flow and Richards' equation

We restrict our discussion to two phases for simplicity, but the theory can be extended to more phases. In two phase systems one has a *wetting phase* and a *non-wetting phase*. Denoted by the subscripts w and n respectivly.
When we introduce more phases we continue with the equations we already introduced, we write down Darcy's law (1.11) with a modification.

$$\boldsymbol{u}_\alpha = \frac{\boldsymbol{k}_{r,\alpha}\boldsymbol{k}\rho g}{\mu}\nabla(\psi_\alpha + z) \tag{1.13}$$

The scaling in front of the permability $\boldsymbol{k}_{r,\alpha}$ is known as *relative permability* and it has to be deduced from experimental observation.
We can also write down a mass balance equation for each phase:

$$\frac{\partial(S_\alpha\rho_\alpha\phi)}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}_\alpha) = f_\alpha \tag{1.14}$$

Here we assume there is no mass transfer between the phases. If we combine equations (1.13) and (1.14), they give us 2 equations, but we have four unknows ψ_w, ψ_n, S_w and S_n. We therefore introduce a simple algebraic relation

$$S_w + S_n = 1$$

and a not so simple relation

$$p_n - p_w = p_c \tag{1.15}$$

Where p_c is *capillary pressure* and is also determined experimentally. With initial and boundary conditions we again have a closed system.
A common simplification is to assume that the capillary pressure and the relative permability are functions of the saturation, and that the relative permability is isotropic(a scalar).
Another simplification that is used especially in groundwater hydrology is that the non-wetting phase always have $p_n = p_{atm}$. For this assumption to hold it is important that the air always has some path to the surface. Now equation (1.15) simplifies to

$$-p_w(S_w) = p_c(S_w)$$

Note that we can divide by ρg to get an expression for ψ. Also experiments show that the capillary pressure is a monotone decreasing function of saturation, we can therefore invert it. Finally, we can multiply by the porosity to get an expression for the *water content* θ_w

$$\theta_w = \theta_w(\psi_w)$$

Combining this with the twophase Darcy law (1.13) and mass balance (1.14) we get **Richards' equation**

$$\frac{\partial \theta(\psi)}{\partial t} - \nabla \cdot (\boldsymbol{\kappa}(\theta(\psi))(\nabla \psi + e_z)) = F \tag{1.16}$$

Where $\theta = \theta_w$. Note that density is completely eliminated because water is assumed to have constant density. The hydraulic conductivity in (1.13) is simply written $\frac{\boldsymbol{k}_{r,\alpha}\boldsymbol{k}\rho g}{\mu} = \boldsymbol{\kappa}(\theta)$.

Richards' equation contains two non-linearities in θ and $\boldsymbol{\kappa}$, this makes the analysis and numerical simulation more interesting and challenging as we will see. They may also cause the equation to degenerate, ie. the parabolic equation may "collapse" into an elliptic PDE(see figure 1.2) or even an ODE.

Richards.png

Figure 1.2: A sketch of the degeneracy of Richards' equation

Notes