Chapter 1

Time dependent equations

In this chapter we discuss numerical solving algorithms for parabolic equations with non-linearities, such as Richards' equation (??). We start by considering the most famous parabolic equation, namely the heat equation:

$$\begin{aligned} \partial_t u - \nabla \cdot \boldsymbol{K} \nabla u &= F \quad && x \in \Omega \quad && t \in (0, T] \\ \psi &= g && x \in \partial\Omega && t \in (0, T] \\ \psi &= u_0 && x \in \Omega && t = 0 \end{aligned} \tag{1.1}$$

We expect low regularity in time, so there is not much gained by using a higher order discretization in time. The two choices we have left is the forward euler(explicit) and the backward euler(implicit). The obvious choice is backward euler, as it is stable for long timesteps. Let $\{t_n\}_n$ be a sequence of $N+1$ evenly spaced numbers from 0 to T and let $\tau = \frac{T}{N}$ be the timestep. Then we state the semidiscrete version of (1.1) by exchanging the time derivative by a difference quotient $(\partial_t u)^n = \frac{u^n - u^{n-1}}{\tau}$. Note that this difference quotient is implicit because u^n is not explicitly given by terms of the previous timestep.

show why?

$$u^n - \tau \nabla \cdot \boldsymbol{K} \nabla u^n = \tau F^n + u^{n-1} \tag{1.2}$$

Now we have an eliptic problem (1.2) for each timestep. This has almost the same structure as the elliptic model problem (??) we solved in the previous chapters. The difference beieng that we have a

explain and possibly prove convergence of L-scheme

The modified finite element method

Let V_h be the finite dimensional test space as defined in the **??** section, definition **??**, with $\{\phi_i\}_i$ beieng the standard nodal basis. To make our finite element method conservative, we define as in (Cao, Y., Helmig, R. and Wohlmuth, B.I. (2011) [?]), another interpolation operator.

Definition 1 (Piecewise global interpolator). *Let $\hat{I}_h$ be an operator that maps from the test space to functions that are piecewise continuous on each control volume.*

$$\hat{I}_h : C(\Omega) \to \left\{ v_h \in L^2(\Omega) : v_h|_{\Omega_i} = K \right\}$$

And

$$\hat{I}_h v = \sum_{i \in \mathcal{N} \setminus \mathcal{N}_D} v(x_i) \hat{I}_h \phi_i(x)$$

Where

$$\hat{I}_h \phi_i(x) = \begin{cases} 1 & \textit{if } x \in \Omega_i \\ 0 & \textit{otherwise} \end{cases} \tag{1.3}$$

Extend this definition with some figures and for the boundary

The linearized system arising from the simplyfied Richards equation (**??**) now becomes:

$$\begin{gathered} \text{find } \psi^{n,j} \in V_n \text{ such that} \\ \left\langle \theta(\psi^{n,j-1}), \hat{I}_h v_h \right\rangle + L \left\langle \psi^{n,j} - \psi^{n,j-1}, \hat{I}_h v_h \right\rangle \\ + \tau \left\langle \boldsymbol{K} \nabla \psi^{n,j}, \nabla v_h \right\rangle = \tau \left\langle F^n, \hat{I}_h v_h \right\rangle + \left\langle \theta(\psi^{n-1}), \hat{I}_h v_h \right\rangle \\ \text{for all } v_h \in V_h \end{gathered} \tag{1.4}$$

Theorem 1.0.1. *Assume a homogenous domain discretized with paralellograms. Then the MPFA-L method with L-scheme linearization for Richards' equation gives a system that is equivalent to the modified Finite element method* (1.4)

Proof. If we in equation (1.4) test with the basis functions ϕ_i and express the solution u_h as $u_h = \sum u_j^* \phi_j$ we end up with the system

$$\begin{aligned} A_{i,j} &= L \left\langle \phi_i, \hat{I}_h \phi_j \right\rangle + \tau \left\langle \boldsymbol{K} \nabla \phi_i, \nabla \phi_j \right\rangle \\ B_i &= \left\langle \tau F^n + \theta(\psi^{n-1}) + L \psi^{n,j-1} - \theta(\psi^{n,j-1}), \hat{I}_h \phi_i \right\rangle \end{aligned} \tag{1.5}$$

1. $L \left\langle \phi_i, \hat{I}_h \phi_j \right\rangle$ is equivalent to $L \int_{\Omega_i} dx \delta_{ij}$

2.

3.

□

Lemma 1.0.2. *The Bi-linear form in the modified finite element method* (1.4)

$$a_h(v,w) = L\left\langle v, \hat{I}_h w\right\rangle + \tau\left\langle \boldsymbol{K}\nabla v, \nabla w\right\rangle \tag{1.6}$$

Is coercive

Proof. content.. □

Lemma 1.0.3 (First Lemma of Strang, page 155 [?])**.** *Suppose there exists some $\alpha > 0$ such that for all $h > 0$ and $v \in V_h$*

$$\alpha \left\|v\right\|_1^2 \leq a_h(v,v)$$

and let a be continuous in $V \times V$. Then there exist some constant C independent of V_h such that

$$\left\|u - u_h\right\|_1 \leq C\left\{\inf_{v\in V_h}\left\{\left\|u-v\right\|_1 + \sup_{w\in V_h}\frac{|a(v,w) - a_h(v,w)|}{\left\|w\right\|_1} + \sup_{w\in V_h}\frac{|l(w) - l_h(w)|}{\left\|w\right\|_1}\right\}\right\}$$